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Structured, Gain-Scheduled Control of Wind Turbines

AALBORG UNIVERSITY

Structured, Gain-Scheduled Control of Wind Turbines
Ph.D. thesis

Contents

Preface and Acknowledgments

This thesis is submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical and Electronic Engineering, specialization in Automation and Control, at Aalborg University. The research has been carried out under the scope of Project CASED - Concurrent Aeroservoelastic Analysis and Design of Wind Turbines.

I am very grateful to my supervisors, Professor Jakob Stoustrup and Associate Professor Torben Knudsen. During the course of the studies, they have shared their extensive knowledge on theoretical as well as practical issues on control systems and wind turbines. They have also provided me non-technical guidance such as motivation and support throughout the study.

I would like to express my gratitude to the the Danish Ministry of Science, Technology and Development. Without their financial support, it would not have been possible to perform the study which this thesis is a result.

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Finally and most importantly I would like to thank my spouse Manoela Onofrio for her infinite patience and support. As we recently found out, it seems that I am type B and she is type A under the personality theory. Therefore, guess who was stressed the most for the finalization of this Ph.D. thesis.

Abstract

Improvements in cost-effectiveness and reliability of wind turbines is a constant in the industry. This requires new knowledge and systematic methods for analyzing and designing the interaction of structural dynamics, aerodynamics, and controllers. This thesis presents novel methods and theoretical control developments, which contributes to the analysis and design of wind turbines in an integrated aeroservoelastic process.

From a control point of view, a wind turbine is a challenging system since the wind, which is the energy source driving the machine, is a poorly known disturbance. Additionally, wind turbines inherently exhibit time-varying nonlinear dynamics along their nominal operating trajectory, motivating the use of advanced control techniques such as gain-scheduling, to counteract performance degradation or even instability problems by continuously adapting to the dynamics of the plant. Robustness and fault-tolerance capabilities are also important properties, which should be considered in the design process.

Novel gain-scheduling and robust control methods that adapt to variations in the operational conditions of the wind turbine are proposed under the linear parameter-varying (LPV) control framework. The modeling and design procedures allow gain-scheduling to compensate for plant non-linearities and reconfiguration of the controller in face of faults on sensors and actuators of the system. Stability and performance in closed-loop are measured in terms of induced $\mathcal{L}_2$-norm. The procedures are appealing to solve some of the practical wind turbine control problems because the controller structure can be chosen arbitrarily, and the resulting controllers are simple to implement online, requiring low data storage and simple math operations. The modeling procedures also allow the generation of reduced-order LPV models from high-fidelity aeroelastic tools. Structured controllers, simplicity in the implementation and aeroelastic codes are in line with the current industrial control practice. Simulation results illustrate the effectiveness of the proposed methods.

Tuning a model-based multivariable controller for wind turbines can be a tedious task. This often involves selecting weighting functions in a trial-and-error procedure. Multiobjective control via linear matrix inequalities (LMI) optimization is exploited to ease controller tuning. Regional pole constraints (D-stability) facilitate intuitive and physical specifications for vibration control, such as minimum damping and decay rate of aeroelastic modes. Moreover, the number of weighting functions and consequently the order of the final controller is reduced.

Inspired by this application, theoretical control developments are presented. New LMI conditions for some hard, structured control problems are proposed. Necessary and sufficient conditions for stability and quadratic performance of vector second-order systems are presented, as well as sufficient conditions for the synthesis of vector second-order controllers. New sufficient conditions to the static output stabilization problem are

also presented. A sufficient characterization is given to the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ model reduction problem. The passive plant design and simultaneous plant-controller design are characterized as sufficient LMI conditions. Due to the linear dependence of the proposed LMIs in the Lyapunov matrix, problem such as simultaneous stabilization, robust synthesis and LPV control can be treated naturally by defining the Lyapunov matrix as multiple or parameter-dependent. The effectiveness of the proposed conditions are verified by numerical experiments. Numerical examples also illustrate their application on wind turbine control.

Synopsis

Forbedringer i omkostningseffektivitet og pålidelighed af vindmøller nyder konstant bevågenhed i branchen. Det kræver ny viden og systematiske metoder til analyse og design af samspillet mellem strukturelle dynamik, aerodynamik, og regulatorer. Denne afhandling præsenterer nye metoder og reguleringsteoretiske resultater, som bidrager til analyse og design af vindmøller i en integreret aeroservoelastic proces.

Fra et reguleringssynspunkt er en vindmølle et udfordrende system, da vinden, hvilket er den energikilde, som driver maskinen, er en dårligt kendt forstyrrelse. Derudover, da vindmøller i sagens natur udviser tidsvarierende lineær dynamik langs deres nominelle driftskurve, motiveres anvendelsen af avancerede teknikker såsom gain-scheduling, for at modvirke forringelse af ydeevnen eller endda stabilitetsproblemer ved løbende at tilpasse sig dynamikken i anlægget. Robusthed og fejltolerance er også vigtige egenskaber, som bør overvejes i designprocessen.

Nye gain-scheduling og robuste reguleringsmetoder, der tilpasser sig variationer i de operationelle betingelser for vindmøllen foreslås i form af den såkaldte lineære parametervarierende (LPV) kontrol. Modellering og design procedurer gør det muligt ved hjælp af gain-scheduling at kompensere for system-ulineariteter og at foretage rekonfiguration af regulatoren i lyset af fejl på sensorer og aktuatorer i systemet. Stabilitet og performance i lukkede sløjfe måles i form af induceret $\mathcal{L}_2$-norm. Procedurerne er velegnede til at løse nogle af de praktiske vindmøllereguleringsproblemer, fordi reguleringsstrukturen kan vælges vilkårligt, og de resulterende regulatorer er enkle at implementere online, da de kræver lav datalagring og simple matematiske operationer. Modelleringsteknikkerne giver også mulighed for generering af reduceret ordens LPV modeller fra high-fidelity aeroelastiske værktøjer. Strukturerede regulatorer, enkelhed i gennemførelsen og aeroelastiske koder er i overensstemmelse med den nuværende industrielle reguleringspraksis. Simuleringsresultater illustrerer effektiviteten af de foreslåede metoder.

Indstilling af en modelbaseret flervariabel regulator til vindmøller kan være en kedelig opgave. Dette indebærer ofte valg af vægtningsfunktioner i en trial-and-error procedure. Flerkriterie-regulerign via lineære matrix uligheder (LMI) optimering udnyttes til at lette regulator-tuningen. Regionale polbegrænsninger (D-stabilitet) muliggør intuitive og fysiske specifikationer for vibrationsregulering, såsom minimum dæmpning og indsvingingstid for aeroelastiske modes. Desuden er antallet af vægtningsfunktioner og dermed rækkefølgen af den endelige regulator reduceret.

Inspireret af denne anvendelse, er teoretiske reguleringsmetoder præsenteret. Nye LMI betingelser for nogle vanskelige, strukturerede reguleringsproblemer foreslås. Nødvendige og tilstrækkelige betingelser for stabilitet og kvadratisk ydeevne af vektor anden ordens systemer er vist, såvel som tilstrækkelige betingelser for syntesen af vektor anden ordens regulatorer. Nye tilstrækkelige betingelser til statisk output-stabilisering er

ogsåpræsenteret. En tilstrækkelig karakterisering er givet til $\mathcal{H}_\infty$ og $\mathcal{H}_2$ modelreduktions-problemet. Passivt systemdesign og samtidig system-regulator design er karakteriseret som tilstrækkelige LMI betingelser. Pågrund af den lineære afhængighed af de foreslåede LMI'er i Lyapunov-matricen, kan problemer såsom simultan stabilisering, robust syntese og LPV kontrol behandles naturligt ved at definere Lyapunov-matricen som multipel eller parameter-afhængig. Effektiviteten af de foreslåede betingelser verificeres af numeriske eksperimenter. Numeriske eksempler illustrerer ogsåderes anvendelse påvindmølleregulering.

Mandatory Page

- Thesis title: Structured, Gain-Scheduled Control of Wind Turbines

- Name of PhD student: Fabiano Daher Adegas

- Name and title of supervisor and any other supervisors: Professor Jakob Stoustrup, Ph.D., Associate Professor Torben Knudsen, Ph.D.

- List of papers:

  - Adegas F.D., New sufficient LMI conditions for static output stabilization. Proceedings of the 5th Symposium on System Structure and Control, 2013, Published.

  - Adegas F.D., Stoustrup J. $\mathcal{H}_\infty/\mathcal{H}_2$ model reduction through dilated linear matrix inequalities. Proceedings of the 7th IFAC Symposium on Robust Control Design, Aalborg, Denmark, 2012, Published.

  - Adegas F.D, Stoustrup J. Structured control of LPV systems with application to wind turbines. Proceedings of the 2012 American Control Conference, Montral, Canada, 2012, Published.

  - Adegas F.D., Sloth C.S., Stoustrup J. Structured Linear Parameter Varying Control of Wind Turbines, Chap. 12. Control of Linear Parameter Varying Systems with Applications, Eds: J. Mohammadpour and C. W. Scherer. ISBN: 978-1-4614-1832-0, Springer-Verlag Berlin Heidelberg, 2012; 303-337. Published.

  - Adegas F.D., Sønderby I.B., Hansen M.H., Stoustrup, J. Reduced-order LPV model of flexible wind turbines from high fidelity aeroelastic codes, IEEE Multi-conference on Systems and Control, 2013. Submitted.

  - Adegas F.D., Stoustrup J. Linear Matrix Inequalities for Analysis and Control of Linear Vector Second-Order Systems. Submitted.

  - Adegas F.D. Linear Matrix Inequalities Conditions for Simultaneous Plant-Controller Design. Submitted.

"This thesis has been submitted for assessment in partial fulfillment of the PhD degree. The thesis is based on the submitted or published scientific papers which are listed above. Parts of the papers are used directly or indirectly in the extended summary of the thesis. As part of the assessment, co-author statements have been made available to the assessment committee and are also available at the Faculty. The thesis is not in its present form acceptable for open publication but only in limited and closed circulation as copyright may not be ensured".

1 | Introduction

This chapter begins with an introduction to the notation used throughout the thesis. Next, an exposure of the motivation of the present Ph.D. research is given. Some background material and literature survey on wind turbine modeling and control as well as control theory relevant to this work is presented follows. Lastly, the research objectives are stated and the thesis structure is outlined.

Notation

Due to the wide range of subjects dealt with as well as the "collection of papers" structure, it is a hard task to make notation uniform throughout the manuscript. However, explanations about the notation are embedded in the text of each chapter and should be easily understandable from the context. The symbol $(\cdot)^T$ denotes transposition. The space of square and symmetric real matrices of dimension n is denoted $\mathbb{S}^n$, while $\mathbb{R}^{m \times n}$ denotes the space of real matrices of m rows and n columns. I_n is the identity matrix with dimension n. If the sub-index n is omitted, the dimensions of the identity matrix are taken from the context. The set of n-dimensional real vectors is denoted by $\mathbb{R}^n$. The Kronecker product is denoted by $\otimes$. The symbol $\succ$ ($\prec$), i.e. $P \succ 0$ ($P \prec 0$) is used to denote that the symmetric matrix P is positive (negative) definite, and $\succeq$ ($\preceq$) stands for positive (negative) semi-definite.

1.1 Motivation

Ongoing improvements in the wind turbine efficiency and reliability led the cost of electricity (COE) of wind turbines to become economically competitive against conventional power production. Increase in the annual energy production and lifetime of wind turbines, and decrease of operation & maintenance costs are among the key factors to reduce the cost of energy.

The CASED project with the full title "Concurrent Aero-Servo-Elastic Analysis and Design of Wind Turbines" is a four year project supported in part by the Danish Council for Strategic Research, wherein Aalborg University and the Technical University of Denmark are collaborating with Vestas Wind Systems A/S on further improving the cost-effectiveness and reliability of wind turbines through new knowledge and systematic methods for analyzing and designing the interaction of structural dynamics, aerodynamics, and controllers of a complex aero-servo-elastic system such as a wind turbine. The present thesis is part of this project and brings contributions towards its objectives.
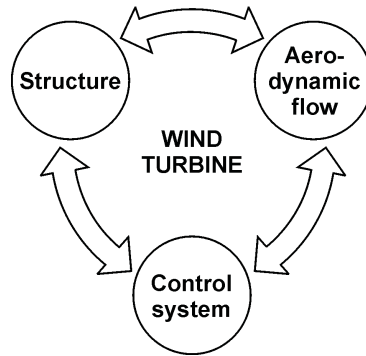
Figure 1.1: Aero-servo-elastic elements of a wind turbine [HJPN].
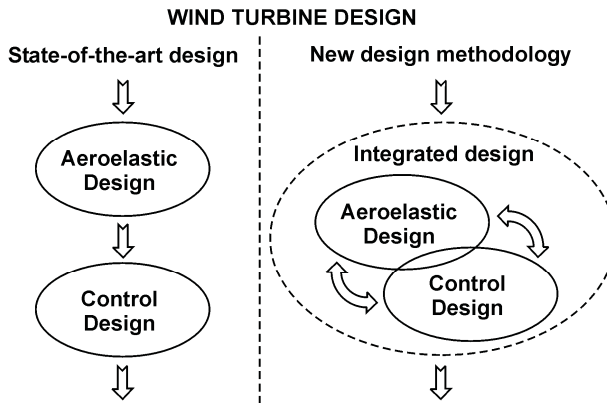
Figure 1.2: Wind turbine design methodologies [HJPN].

Aeroservoelasticity of wind turbines describes the interaction of the aerodynamic flow around the turbine, the servo actions by its controller, and its elastic vibrations. Main aero-servo-elastic elements of a wind turbine are illustrated in Fig.1.1. Fundamental understanding of this interaction is required for the development of future wind turbines that are structurally and aerodynamically optimized. Today, the design of a wind turbine controller is isolated from the structural and aerodynamic design; the controller is considered as an add-on to a pre-determined and fixed aeroelastic design (Fig.1.2). Novel control theories and aero-servo-elastic methods are developed in this thesis, which contributes to the analysis and design of a wind turbine in a concurrent aero-servo-elastic process. Other technical and scientific elements addressed in this thesis are the development of novel gain-scheduling and adaptive control methods that adapt to variations in the operational conditions of the turbine.

Figure 1.3: (left) Windmill in Holland, (middle) Poul La Cour test center in Denmark and (right) American multi-blade model.

Historical Overview of Wind Energy [Joh04, And07]

Conversion of kinetic energy of the wind into useful mechanical energy has been applied by man since ancient times. Wind energy is, along with hydro power/waterwheels, the oldest source of energy used by mankind. Utilization of wind energy has its origin in the eastern civilizations of China, Tibet, India, Afghanistan and Persia, with very uncertain date. It was reported that the Babylonian emperor Hammurabi planned to use wind turbines for irrigation in the seventeenth century BC. Hero of Alexandria, who lived in the third century BC, described a horizontal axis wind turbine with four simple sails to trigger an organ. With more solid evidence, Persians used wind turbines extensively around the seventh century AD; vertical axis machine with a number of radially mounted sails.

From Asia the use of wind energy has spread across Europe. Windmills were used in the XI or XII century in England, taking advantage of lift forces. The first record of an English wind turbine is dated from 1191. The first wind turbine was built for grinding in the Netherlands in 1439. There were a series of technological developments over the centuries, and in 1600 the most common wind turbine was the windmill. The word mill refers to the operation of crushing or grinding grain, so common that all wind turbines were often called windmills, even when applied to some other function. Interestingly, the blades of many Dutch windmills are "twisted" and "tapered" in the same way as modern rotors to optimize the parameters necessary for maximum aerodynamic efficiency, indicating good knowledge of aerodynamics at a much earlier period to present.

By mid-1800, there is a need to develop smaller turbines to pump water. The American West was being populated and there were vast and good grazing areas with no surface water, but with large reserves of groundwater a few feet beneath the surface. In this context, wind turbines known as American multi-blades were developed with high starting torque and efficiency suitable for pumping water. It is estimated that 6.5 million units were built in the United States between 1880 and 1930 by a variety of companies. Many of them still operates satisfactorily.

With the invention of the steam engine in the eighteenth century the world has gradually changed its demand for energy and machinery based on thermodynamic processes, especially with the introduction of fossil fuels (coal, oil and gas). Although the importance of wind power as an energy source declined during the nineteenth century, the research and construction of wind turbines continued on a larger scale. Theorists and practitioners have continued to design and build wind turbines for electricity production.

Figure 1.4: Modern wind turbine (Courtesy of Vestas Wind Systems A/S)

Denmark was the first country to use the wind to generate electricity. In 1890 the Danes utilized a 23 m diameter wind turbine and, in 1910, several hundreds of units with capacity of 5 to 25 kW were in operation. It is worth to highlight the turbines from the Dane P. La Cour (around the turn of the century) and J. Juul (After the second world war). In America, the famous 1250 kW Smith-Putnam also noteworthy.

The development of new materials and technologies marks the beginning of new era of wind energy in 1970. Composite materials such as fiberglass proved very suitable for the blades. Wind turbine were now controlled by electronic systems. These advances led to the modern large wind turbines of nowadays: 3-bladed, variable-speed, pitch controlled. They also reduced the energy generation costs by approximately 80% over the last 20 years [Cha04]. Notably, in late 2012 the global installed capacity (cumulative) of wind turbines connected to the network totals 237,699 MW, and grows exponentially every year [GWE].

1.2   Wind Turbine Control

Control of wind turbines received considerable attention from both academia and industry during the last decade. A number of survey papers can be found on the topic [LC00a, LC00b, Bos00, Bos03b, LPW09, PJ11]. A consensus from all the work done so far is that wind turbine control plays an important role on maximizing the energy generation while alleviating mechanical loads.

The development of a wind turbine controller pass through the definition of control objectives, control strategy, plant modeling, and controller setup. The control objective is a qualitative and quantitative description of the goals which should be achieved by the controller such as increased energy capture, reduced dynamic mechanical loads and power quality. The other three subsequent steps are carried out to ensure that the controller objectives are satisfied. The control strategy is a selection of the operating points that the wind turbine should be regulated around. Plant modeling involves the determination of a wind turbine dynamical model for control design. In the controller setup,

controller structure is determined such as the controlled variables, the performance measures, the reference signals, the switching/gain scheduling procedure between different controllers, as well as the controller tuning.

The main purpose of a wind turbine is to generate energy. Therefore, one would expect a control strategy in which the machine operates at all times at maximum conversion efficiency. However, economical and practical limitations prevent such strategy. Generator speed has to be operated within minimum and maximum limits, in order to comply with centrifugal forces, aerodynamic noise emissions and lubrication of bearings. Produced power should not exceed nominal power, in order to limit reaction torque on the drive train and power flowing on the electrical subsystem. In essence, the electrical power produced by a wind turbine should be maximized for each mean wind speed, but subject to certain operational constraints. The control strategy can be determined by solving a static non-linear optimization problem. We illustrate the problem with the following simplified example

$$(\bar{\Omega}_g^*, \bar{\beta}^*) = \underset{(\bar{\Omega}_g^*, \bar{\beta}^*)}{\arg\max} \bar{P}_g \left(\bar{\Omega}_g, \bar{\beta}, \bar{V} \right) \tag{1.1a}$$

subject to

$$C1 : \bar{\Omega}_{g,min} \leq \bar{\Omega}_g \tag{1.1b}$$

$$C2 : \bar{\Omega}_g \leq \bar{\Omega}_{g,max} \tag{1.1c}$$

$$C3 : \bar{P}_g \leq \bar{P}_N \tag{1.1d}$$

where

$$\bar{P}_g = \frac{1}{2}\rho\pi R^2 \bar{V}^3 C_P \left(\frac{\bar{\Omega}_g R}{N_g \bar{V}}, \bar{\beta} \right) - \bar{\Omega}_g^2 \left(\frac{B_r}{N_g^2} + B_g \right) \tag{1.1e}$$

In the above, $\bar{P}_g$ and $\bar{P}_N$ are the generator and nominal electrical power, respectively, $\bar{\Omega}_g$ is the generator speed, $\bar{\beta}$ is the pitch angle, $\bar{V}$ is the wind speed, ρ is the air density, R is the rotor radius, N_g is the gearbox ratio, B_r and B_g are speed-dependent losses on the generator and rotor side, respectively, and C_P is the power coefficient. The subscripts $(\bar{\cdot})$ and $(\cdot)^*$ stands for steady-state and optimal values, respectively. Other constraints can be included in the above optimization problem, e.g. maximum thrust and noise limits. Instead of the simple equation 1.1e, the generator power P_g and other variables can be a function of a high-complexity aeroelastic model. An implicit dependence of the optimal operating point on the wind speed exists, e.g. $(\bar{\Omega}_g^*(\bar{V}), \bar{\beta}^*(\bar{V}))$. In fact, the collection of operating points can be uniquely parameterized by $\bar{V}$ as depicted in Fig. 1.5.

A wind turbine dynamic model is required to the task of designing controllers, irrespective of whether classical or modern control theories are adopted. Wind turbines are non-linear systems, with aerodynamics being the main source of non-linearities. Three distinct and usual modeling approaches can be highlighted. A quite popular approach relies on first-principles modeling, with few structural degrees of freedom and often based on static aerodynamics. These can also include one or two modes representing unsteady aerodynamics (e.g. dynamic inflow) [Hen11, Hy11] and actuator dynamics (e.g. pitch actuator, power converter). Another way to obtain suitable models for control design is by means of high-fidelity full aeroelastic codes. Linear periodic [Jon] or linear time-invariant [Han11] models are generated from the original non-linear aeroelastic model
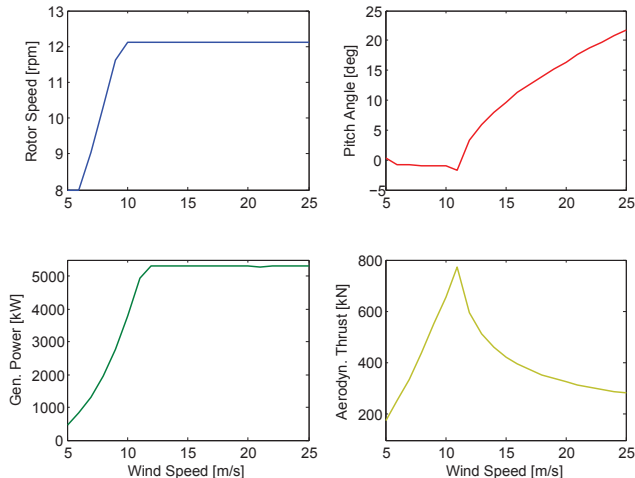
Figure 1.5: Optimal operating points of a 5MW wind turbine.

through numerical differentiation [Jon] or analytical linearization [Han11] around equilibrium points. Lagrangian mechanics is the mathematical machinery to formulate the dynamical equations of motion as vector second-order differential equations

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = Ff(t) \tag{1.2}$$

where $q(t)$ is the vector of positions, M, C and K are the mass, damping and stiffness coefficient matrices, respectively, F is the forcing matrix, $f(t)$ is the force input vector. Depending on the physics chosen to represent the dynamical behavior of the wind turbine, the coefficient matrices have different properties that characterizes the nature of the solution $q(t)$. Simplified first-principles modeling may result in symmetric matrices while full aeroelastic codes usually yields models with non-symmetric matrix coefficients. For control purposes, system (1.2) is often re-written as first-order differential equations

$$\dot{x}(t) = Ax(t) + Bf(t) \tag{1.3a}$$

commonly referred to as state-space form. The relationship between the physical coordinate description (1.2) and the state-space description (1.3) is simply

$$x(t) := \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}, \quad A := \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} \tag{1.4}$$

where a nonsingular matrix M is assumed. The inversion of the mass matrix may bring complicating non-linearities whenever it depends on some parameter, e.g. $M(\theta)$. For example, θ may represent variations of rotor mass due to icing conditions. Wind turbine dynamics change along the range of mean wind speeds $\bar{V}$. Figure 1.6 depicts magnitude plots of a utility-scale wind turbine for mean wind speeds from 14 to 24 m/s obtained with the aeroelastic tool HAWCStab2 [Han11]. The third approach for modeling is through system identification techniques. Although not consolidated in either the wind industry

Figure 1.6: Magnitude plots of a 5MW wind turbine for mean wind speeds from 14 to 24 m/s. Inputs are generator torque, pitch angle and wind speed disturbance. Rotor speed is the ouput.

nor academia, system identification of wind turbines is receiving considerable attention of the research community [vW08, vWHFV09].

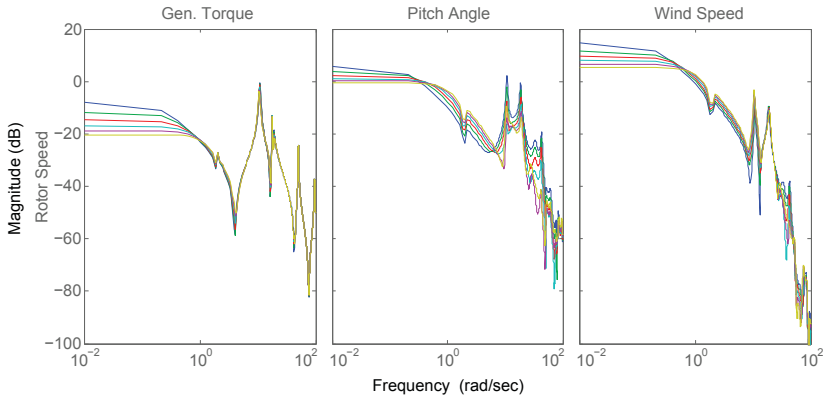A classical structure for the control of wind turbines is illustrated in Fig. 1.7. A couple of control loops with distinct objectives can be noticed. The power and speed controller loop tracks power and generator speed references generated by the reference block. The classical, industry-standard structure of the power and speed controller is build at one controller for partial load and another controller for full load operation. In partial load operation, an usual scheme is a non-linear feedback taking the generator torque as a control signal, proportional to the square of generator speed, e.g. $Q_{g,r} = K\Omega^2$ [Bos00]. Another structure involves a PI controller to track the generator speed reference, $\Omega_{g,r}^*$ with the generator torque, $Q_{g,r}$, as the controlled input. There can also be a combination of the non-linear loop with a PI controller. An open-loop optimal pitch reference is usually supplied to increase energy capture, limit thrust forces and/or avoid stall operation. In full load operation, another PI controller tracks the generator speed reference with pitch reference as a control signal, while the generator torque (or power) are kept at their nominal values. There are systematic methods for tuning the PI gains, including "classical gain-scheduling" to cope with aerodynamic non-linearities [HB02, PLH05].

Modern wind turbines are lightly damped mechanical structures. Resonances can be excited not only by the turbulent wind hitting the rotor, but also by the generator speed and power controllers, increasing fatigue loading of components. Active vibrational control is usually adopted to overcome this. The tower corresponds to a significant amount of the wind turbine cost; therefore mitigation of tower oscillations in fore-aft and side-to-side directions is of major concern. Feedback terms on the tower acceleration in each direction to pitch angle and generator reaction torque can mitigate loads without significant performance losses in terms of power quality and pitch wear [LR05, Bos03b]. Torsional oscillations in the drive-train are also controlled by feedback of generator speed to the generator reaction torque with a band-pass filter suitably tuned at the first drive-train torsional mode [Bos03b]. The wind field is unevenly distributed over the rotor swept area
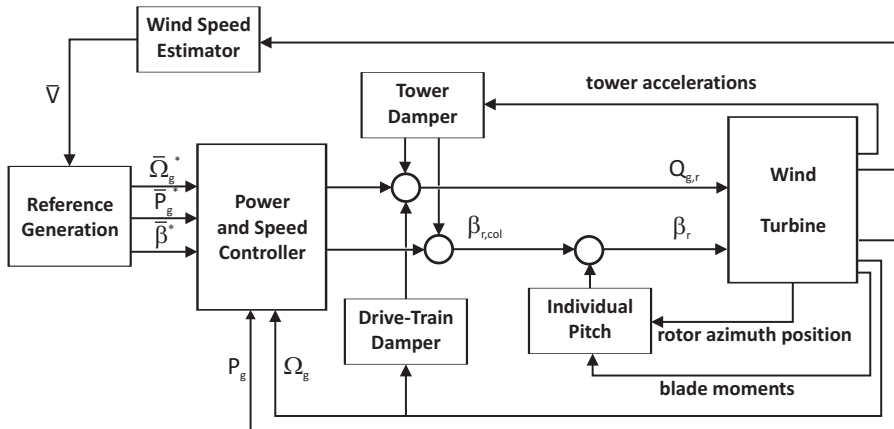
Figure 1.7: Classical structure of a wind turbine control system.

due to wind shear, yaw error or wake from other wind turbines in a wind farm. The resulting asymmetrical loading of the rotor can be mitigated by means of individual pitch control [Bos03a, Bos05, GC07, LMT05]. The principle lies in a feedback scheme that generates zero-mean individual pitch demands for each blade which is superposed on the collective pitch reference from the power and speed controller.

The control scheme can be extended with feed-forward terms from estimates of wind speed disturbances [VRPS99, vdHvE03, SKv+09, DPW+11]. The usual approach of estimating the effective wind speed at hub height is to determine the aerodynamic torque from the dynamical equations govering the inertial of rotor and drive train. The effective wind speed is calculated by an inversion of the aerodynamic model [MPB95, OBS07, OMDJ12]. More sophisticated estimators rely on Kalman filtering techniques assuming more sophisticated models of the wind and the wind turbine [KBS11, Hen11]. A comparison of different wind speed estimators is given in [SKS+12]. Recently, there has been much attention to advanced sensing of wind speed disturbances and feed forward control via laser-based (LIDAR) [WJW11, SBC+11, DSP11] and flow-based instruments [LMT05].

The control structure depicted in Fig. 1.7 requires the power and speed controller to operate in parallel with the active tower dampers, drive train damper, individual pitch controller and possible feed-forward terms. Classical control methods rely on frequency separation and (weak) decoupling arguments for the design of some of the loops, e.g. drive-train damper, individual pitch controller. Since there can be significant interaction between the power and speed controller with the tower dampers, tuning them separately may not result in optimal performance. Using classical methods, an approach which leads to satisfactory closed-loop performance is to tune one loop in isolation as a single-input single-output controller, and augment this loop in the plant model for designing the other loop [Bos00, Bos03b, BW11]. This process is then iterated until both loops are well tuned.

A natural approach to handle the multiple-inputs and multiple-outputs of a wind turbine is by the use of multi-variable controllers. While in classical designs the model is

represented as transfer functions in the Laplace domain, multi-variable control is based on state-space models in time domain. Pole placement is a popular approach of state-space control and has applied in [WB02] to generator speed control at full load using the collective pitch angle as the controlled input. In the single-input case, the pole placement problem has a unique solution [Kai80] facilitating the decision of where to place the closed-loop eigenvalues. This is no longer true for systems with two inputs or more, making the selection of the location of the poles a non-trivial task.

Optimal control might help on the selection of controller gains by attributing a performance index (cost function) to the design problem. In a stochastic setting, the performance is measured in terms of the variance of a specified output assuming that the disturbance input is unit intensity white noise. Such a problem formulation is denoted $\mathcal{H}_2$ control (or LQG control given this interpretation). Several are the applications of this method for LTI wind turbine control [CLG94, Gri92, PLH05, TNP10]. Another famous performance index consist of a measure of the system gain in an energy sense. That is, the disturbance is considered to be an arbitrary l_2 signal and we wish to minimize its worst-case effect on the energy of the performance channel. This problem, denoted $\mathcal{H}_\infty$ control, is particularly useful for guaranteeing closed-loop performance despite uncertainties of the model to be controlled. $\mathcal{H}_\infty$ control has been applied to LTI wind turbine control in several variations [BJC04, BvBD93, CILG92, RF03, RFB05].

LTI design methods can be applied to only a single operating point of the wind turbine. Other techniques should be combined with LTI methods in order to cover the whole range of operating conditions. Online design of controllers is a way to achieve this. Predictive control determines the optimal control policy by solving optimization problems online. The linearized wind turbine prediction model can then be updated according to the current operating condition [CO04, Hen07, Hen11]. Adaptive control of wind turbines was also proposed to cope with changes along the nominal trajectory of operating conditions [SDB00] where specific model parameters are estimated online and subsequently utilized to update the controller parameters. Improving the energy capture in partial load had also been addressed by means of adaptive control [JPBF06]. Instead of an online designing of the controller, the controller gains can be determined offline as a function of the operating condition. What remains to be done online is choosing the controller gains according to the current operating condition. This well known and widely used approach is called gain-scheduling. In classical wind turbine control, gain scheduling is obtained by varying only a few gains in the controller [vEvdHS03, BW11]. Gain scheduling can also be obtained by concatenating optimal LTI controllers. Switching, bumpless transfer, interpolation of the state space matrices, and convex combination of the output of parallel controllers are some of the ways of combining the various controllers [KB93, NES$^+$09, vEvdHS03, BW11].

The above extensions of LTI design methods to nonlinear systems assume variations in the operating conditions to be so slow that they do not contribute to the dynamics of the wind turbine. Time varying changes in the operating point can be systematic addressed by linear parameter varying (LPV) modeling and control methods, in which the nonlinearities of the model are represented by a set of parameters updating the state space formulation. A controller that depends on the same parameters can be formulated by solving a convex optimization problem involving a number of linear matrix inequalities (LMIs). This approach for the control of wind turbines recently received a considerable attention [BJC04, BMC05, BBM07, Øs08, OBS09].

The controller design is isolated from the structural and aerodynamic design. The controller is considered as an add-on to a pre-determined and fixed aeroelastic design. However, these two designs are substantially coupled. A clear example is the alleviation of tower base loads using active tower damping, which leads to the re-design of a lighter tower. The new tower design may have dynamical properties different than the original design, requiring a verification of the closed-loop performance or a re-design of the controller. Research on simultaneous design of controller and aeroelastic parameters of a wind turbine is recent and literature is incipient. The multi-disciplinary optimization approach of [BCC11, BCC12] does not consider controller gains as arguments of the optimization; but for an automation of the aeroelastic design, a gain-scheduled LQR controller was chosen due to its reasonable performance throughout the operating envelope, without the need of manual tuning at each new aeroelastic design. A systems and control point of view is given in [SGV12b, SGV12a]; stiffness and damping parameters of a lumped model and the gains of an LPV controller are simultaneously designed to improve closed-loop $\mathcal{L}_2$-gain performance.

After the exposure of the state-of-the-art and inspired by [BJSB11], a critical analysis of the research and practice on wind turbine control is appropriate at this point. Optimal control methods are attractive as the controller is the minimizer of a reasonably meaningful cost function that is chosen a priori. Once the cost function is determined, one would wonder that controller tuning could be made automatic even for different wind turbines. Classical methods, on the other hand, requires experience and skill of the designer for each new tuning. Optimal methods are also natural for MIMO control design, whereas classical design methods could require an iterative approach of tuning SISO loops. Despite these alleged advantages, controllers designed with classical methods still prevails in commercial wind turbines due to practical considerations.

In practice, building the cost function can be just as difficult as tuning a classical controller. To begin with, the choice of states or outputs composing the cost is not as straightforward as it might appear. Due to mathematical convenience, the cost function is usually defined as a quadratic functional of the states and inputs. The quadratic form may not be appropriate to represent fatigue loads, as fatigue is a highly non-linear process, remaining for the designer to try to indirectly control loads by penalizing positions and velocities of the structural degrees of freedom. Even the variance of stresses on mechanical components is not a precise measure of fatigue, since higher order statistical moments also contributes to mechanical loading [Pn09]. To achieve asymptotic tracking, it is common to include a term to penalize the speed error, and another term for the integral of the speed error. Adjusting the cost weights for these terms is pretty much the same as the selection of proportional and integral gains in a classical PI design.

Simplicity of implementation is an important aspect of the adoption of classical controllers on commercial wind turbines. Non-linearities can be easily compensated with ad-hoc techniques such as gain-scheduling of particular loops. Inclusion of notch filters for small corrections on the controller (e.g. avoid excitation of resonant modes) is straightforwardly done. Numerical conditioning when implementing classical controllers in discrete-time is also more easily handled. Implementation of model-based controllers are often more sophisticated in many aspects. They are usually of full-order (same number of states as the model) requiring non-trivial numerical conditioning for a reliable implementation. Adjustments on model-based controllers requires a complete recalculation of the controller gains. Integration with the supervisory control is also challenging. For

example, a suitable reconfiguration of the controller during a shutdown or faulty scenario might reduce extreme loads. To achieve, this classical controllers can be easily augmented with variable schedules or limits, while it is not so trivial to perform the same changes in model-based controllers.

However, these simple-to-implement but ad-hoc control mechanisms have no theoretical ground on the stability and performance in closed-loop. Such properties are verified in computer simulations, which also serve as basis for re-tuning the controller parameters until a desired performance is achieved. To conclude, the comparison of the academic work with industrial practice highlights the gap and importance of research on model-based control methods for wind turbines with the following properties:

- Controller structure can be chosen a priori. For example PI controllers, fixed-order, decentralized, and others;

- Gain-scheduling to compensate for plant non-linearities and/or to reconfigure the controller in face of system faults. Theoretical aspects of stability and performance should be taken into account;

- Existing controllers can be expanded with new functionalities;

- Cost functions and constraints with intuitive and meaningful design criteria;

- Controller and aeroelastic parameters can be jointly optimized.

Robust and parameter-varying control are branches of control theory applied to systems that have high requirements for robustness/adaptation to parameter variations, and high requirements for disturbance rejection, aligned with what is required on wind turbine control. The controllers that result from these fields are typically of very high order, complicating their implementation. However, if a structural constraint on the controller gains is imposed, the synthesis problem is no longer convex and relatively hard to solve. The term "structured control" refers to controllers with special structure imposed on the gain matrix, and related algorithms to compute them.

1.3   Structured Linear Control

In the control literature, the term "structured control" refers to controllers with special structure imposed on the gain matrix. Structured controllers includes static and fixed-order output feedback control, decentralized control, some model reduction problems, joint plant and control design, and others. Let us introduce some of the problems of our interest. Consider the linear parameter-dependent system described by the state-space equations

$$H(\theta) : \begin{cases} \delta x = A(\theta)x + B_w(\theta)w + B_u(\theta)u, \quad x(0) = 0, \\ z = C_z(\theta)x + D_{zw}(\theta)w + D_{zu}(\theta)u \\ y = C_y(\theta)x + D_{yw}(\theta)w \end{cases} \tag{1.5}$$

where δ represents the time-derivative operator for continuous-time systems or the unitary time-shift operator for discrete-time systems. In this model, the state vector x is assumed

of dimension n, u represents the vector of control inputs, w is a vector of exogenous disturbances, y is the measured output and z is the controlled output. Dependence of these vectors on time t for continuous-time systems, and on time $t := kT$, $k \in \mathbb{N}$ for discrete-time systems, is omitted to save notation. We assume that all matrices are real with dimensions appropriately defined, and continuous functions of some time-varying vector of real valued parameters $\theta = \left[\theta_1, \ldots, \theta_{n_\theta}\right]$. The first contributions dealing with formal aspects of gain-scheduling appeared in the early 90s [Rug91, SA90, SA91, SA92]. Linear parameter varying (LPV) systems were defined in the seminal work [SA91] and is still subject of research by the control community. In the LPV framework, θ is measurable and represent scheduling parameters. The time-varying parameter and its rate of variation are assumed to be bounded as follows. The parameter θ satisfies

$$\theta(t) \in \Theta, \quad \forall t \geq 0 \tag{1.6}$$

where Θ is a compact set. The rate of variation $\delta\theta$ belongs to the hypercube

$$\mathcal{V} := \left\{ |\delta\theta(t)| : |\delta\theta_i(t)| \leq \nu_i, \ i = 1, \ldots, n_\theta, \ \forall t \geq 0 \right\}. \tag{1.7}$$

An LPV system is reduced to an LTI system for a constant parameter trajectory $\theta(t) = \theta_0$, $\forall t \geq 0$. Therefore, the local behavior of the LPV plant can be analyzed from the underlying LTI systems. On the other hand, time-varying properties of the LPV system cannot be inferred from the underlying LTI systems, except if strong assumptions about the evolution of the scheduling parameters (e.g. slowly varying) are done. Consider the following controller

$$C(\theta) : \begin{cases} \delta x_c = A_c(\theta)x_c + B_c(\theta)y \\ u = C_c(\theta)x_c + D_c(\theta)y \end{cases} \tag{1.8}$$

which is also θ-dependent, where the controller state has dimension n_c. Control robust to structured uncertainties on the plant is a particular case of the LPV setup by taking the controller as parameter independent, and the rate bounds $v_i = 0$, $i = 1, \ldots, N$. An LTI controller can also be designed robust to the range of parameter variations and rates $\Theta \times \mathcal{V}$. To the purpose of analysis and synthesis, the dynamical system (1.5)-(1.8) is represented as a closed-loop system expression which derivation follows. Defining the controller matrices in a compact form

$$K(\theta) := \begin{bmatrix} D_c(\theta) & C_c(\theta) \\ B_c(\theta) & A_c(\theta) \end{bmatrix} \tag{1.9}$$

and the augmented state vector as $\tilde{x} := \begin{bmatrix} x^T & x_c^T \end{bmatrix}^T$, the closed-loop system interconnection leads to

$$\delta\tilde{x} = \mathcal{A}(K,\theta)\tilde{x} + \mathcal{B}(K,\theta)w \tag{1.10a}$$

$$z = \mathcal{C}(K,\theta)\tilde{x} + \mathcal{D}(K,\theta)w \tag{1.10b}$$

where the closed-loop system matrices are [SIG98]

$$\mathcal{A}(K,\theta) = \mathbf{A}(\theta) + \mathbf{B}(\theta)\,K(\theta)\,\mathbf{M}(\theta), \quad \mathcal{B}(K,\theta) = \mathbf{D}(\theta) + \mathbf{B}(\theta)\,K(\theta)\,\mathbf{E}(\theta) \tag{1.11a}$$

$$\mathcal{C}(K,\theta) = \mathbf{C}(\theta) + \mathbf{H}(\theta)\,K(\theta)\,\mathbf{M}(\theta), \quad \mathcal{D}(K,\theta) = \mathbf{F}(\theta) + \mathbf{H}(\theta)\,K(\theta)\,\mathbf{E}(\theta) \tag{1.11b}$$
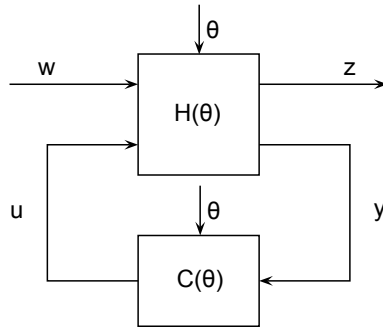
Figure 1.8: Gain-scheduled LPV control of an LPV system

$$\mathbf{A} = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} B_u(\theta) & 0 \\ 0 & I \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} C_y(\theta) & 0 \\ 0 & I \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} D_{yw}(\theta) \\ 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} D_{zu}(\theta) & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} B_w(\theta) \\ 0 \end{bmatrix} \quad (1.12)$$

$$\mathbf{C} = \begin{bmatrix} C_z(\theta) & 0 \end{bmatrix}, \quad \mathbf{F} = D_{zw}(\theta)$$

Let some of the common controller structures be illustrated. This feedback structure becomes a static output feedback (SOF)

$$u = D_c(\theta)y \qquad (1.13)$$

if $n_c = 0$. The full state feedback (SSF), i.e. $u = D_c(\theta)x$, is a particular case of SOF in which the output matrix $C_y(\theta) = I$. The full-order dynamic output feedback arises when $n = n_c$. When $n_c < n$, the resulting structure is the fixed-order dynamic output feedback. Decentralized controllers of arbitrary order are characterized by $K(\theta)$ with structure

$$K(\theta) := \begin{bmatrix} \mathrm{diag}(D_{ci}(\theta)) & \mathrm{diag}(C_{ci}(\theta)) \\ \mathrm{diag}(B_{ci}(\theta)) & \mathrm{diag}(A_{ci}(\theta)) \end{bmatrix} \qquad (1.14)$$

In the simultaneous plant / control design problem [GZS96], matrices $A(\rho)$,$B_w(\rho)$,$C_z(\rho)$ and $D_{zw}(\rho)$ depend on some plant parameters ρ that should be determined. An augmented controller $K_\rho(\theta) := (K(\theta), \rho)$ is structured since K_ρ is not a single unstructured matrix. Optimal model reduction problems seek to provide a reduced-order model that minimizes the error between the full-order and reduced-order model in some norm sense (e.g. $\mathcal{H}_\infty/\mathcal{H}_2$). Consider $K(\theta)$ as a state-space realization of the reduced-order system, and $(A(\theta), B_w(\theta), C_z(\theta), D_{zw}(\theta))$ as a realization of the full-order system. System (1.10) represent the state-space model of the error system when (1.12) is particularized with [SIG98]

$$B_u(\theta) = 0 \quad C_y(\theta) = 0 \quad D_{yw}(\theta) = I \quad D_{zu}(\theta) = -I. \qquad (1.15)$$

Methods for structured control design can be roughly categorized as based on *matrix inequalities* and *nonsmooth* optimization. Bilinear matrix inequalities (BMI) naturally arise when Lyapunov stability theory is applied to control problems. Some BMI synthesis problems can be readily reformulated as linear matrix inequalities (LMI) by the use of

auxiliary parameterizations. That is, controller data does not appear explicitly in the optimization formulation, being obtained a posteriori of the solution via some nonlinear one-to-one transformation of the optimization variables. Examples of problems that can be turned into LMI are the full-state [BGP89, BGFB94] and full-order [SGC97, OGB02] output feedback. To illustrate, let $P \in \mathbb{R}^{n \times n}$, $P \succ 0$ be a Lyapunov variable. The nonlinear change-of-variables

$$\hat{D}_c := D_c P \tag{1.16}$$

where $\hat{D}_c \in \mathbb{R}^{n \times n}$ is an auxiliary matrix, has linearizing properties and turns BMI problems related to full-state feedback into LMIs. Since P is non-singular, the original controller gains can be recovered from the auxiliary ones via the inverse transformation $D_c = \hat{D}_c P^{-1}$. The fact that D_c is an *unstructured* matrix makes this one-to-one transformation possible. LMI reformulations without loss of generality for many other problems such as the design of controllers with arbitrary order and static output [SADG97] or decentralized [Sil90] remain unsolved, and little hope for finding them remains [MSP00]. Of course, with some degree of conservatism, one can also pursue sufficient LMI conditions. The literature is rich in sufficient conditions for static [OGB02, DY08, Tro09], reduced-order [Tro09], decentralized [OGB02], to mention a few.

The BMI problem and the LMI problem with rank constraints are found to be equivalent and NP-hard [GN99]. Therefore, many researchers focused on finding efficient suboptimal methods to solve this hard problem. The DK-iteration procedure [Doy85] became a popular method. A similar method proposed in [Iwa99] solves the BMIs by fixing some variables and optimizing on others in an alternating manner. LMI problems are solved in each step, but convergence is not guaranteed. Another well known method is the alternating projections algorithm [GB99] that seeks an intersection of the convex LMI set and the rank constraint. Other algorithms of this time are the min/max algorithm [GdSS98], XY-centering [IS95], cone complementarity [GOA97], and the convexifying potential function [OCS00]. Global methods to solve the BMI were also proposed, e.g. branch and bound algorithms [Ber97, GSP95, VB00], as well as local methods [LM02, KLSH05].

Nonsmooth approaches let the Lyapunov/LMI framework apart and search directly in the controller parameter space. The nonsmooth algorithms seem more appropriate to handle systems of high-order when compared to LMI methods. The absence of variables related to Lyapunov functions contributes to lower the computational burden. The nonsmooth algorithms for $\mathcal{H}_\infty$ control in [AN06, ANR07] resulted in the code *Hinfstruct* of MATLAB ®Robust Control Toolbox 7.11. At the same time, other approaches [BLO05, GO08, ADGH11] resulted in the code package *HINFOO* for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control.

Due to the possibility of using parameter-dependent Lyapunov functions [FAG96] to verify stability in a gain-scheduling context, this thesis uses methods based on *matrix inequalities*. To the best of our knowledge, nonsmooth techniques for control design cannot yet deal with parameter-dependent certificates of stability.

Up to this point of this introduction, the structured control problem focused on constraining matrix gains of the controller, and assumed a state-space system description. However, the way a system is represented has direct relation to the controller structure and its design. For example, similarity transforms have been used in control theory to ease state-space control design. Canonical forms and pole placement [Kai80] is perhaps the most known and used link between plant representation and controller design. Con-

trol of flexible structures also take advantage of modal state-space realizations to facilitate damping assignment [Gaw98]. Not only similarity transforms, but also the form of the differential equations governing the dynamics of the system, influences control design. Working with differential equations in vector second-order form, instead of first-order state-space form, clearly impacts on the structure of the controller gains. Let an example be given to illustrate this. Consider the full-state feedback

$$f(t) = -G_v\dot{q}(t) - G_p q(t) \tag{1.17a}$$

$$u(t) = -\begin{bmatrix} G_v & G_p \end{bmatrix} \begin{pmatrix} \dot{q}(t) \\ q(t) \end{pmatrix} \tag{1.17b}$$

in vector second-order and state-space form, respectively, where G_p, $G_v \in \mathbb{R}^{n \times n}$. Suppose that only velocities are available for feedback, i.e. $G_p = 0$. While in (1.17b) the gain matrix has a structural constraint, that is, is not a full unstructured matrix, the vector second-order feedback (1.17a) is composed of an unstructured matrix G_v. This simple example suggest that some control structures might benefit from being designed in vector second-order form, as matrix products between the controller matrix and other matrix variables could more easily be handled.

1.4   Research Objectives

A general objective of this study is to develop novel modeling and control methods which enable to analyze and design the dynamics of a wind turbine in a integrated aero-servo-elastic process. There is no ambition of covering all aspects of such a complex problem, but rather give punctual contributions to this process. Another general objective of this research is to pursue theoretical developments on systems and control inspired by the application. Under this general context, some specific objectives can be mentioned.

The development of methods for structured control of wind turbines is one of the specific objectives. These methods should be capable of incorporating gain-scheduling to compensate for plant non-linearities and/or to reconfigure the controller in face of system faults. Theoretical aspects of stability and performance should be taken into account. Aeroservoelastic modeling to achieve these goals is also within the scope of the developments. These methods are an attempt to bring advanced control closer to current practice in the wind industry. Another specific objective within the application is to devise optimal control methods for wind turbines with intuitive and meaningful design criteria.

Theoretical contributions to structured control is also a research goal. We seek LMI conditions for simultaneous plant-controller design which could be extended to uncertain and linear parameter-varying systems. How to analyze and synthesize vector second-order systems under the LMI framework was a late objective of the research.

1.5   Outline of the Thesis

This thesis is written as a collection of the papers, which have been produced during the course of the PhD project. With the state-of-the-art and background now covered, the remainder of the thesis will proceed as follows. The next chapter contains an overview of the content of the papers. Following this, Chapter 3 will provide a conclusion on the

project and give some suggestions to issues which are interesting to address in the future. Lastly, the remainder of the thesis consists of the papers themselves. As such, some repetition of introductory sections should be expected.

2 | Summary of Contributions

A summary of each paper that composes this thesis is presented in this chapter. Firstly, the contributions are summarized in a couple of paragraphs. A more detailed sample of each paper follows, concise enough to avoid excessive repetition of the content.

Short Summary

Papers A and B [AS12a, AS12b] proposes an LPV modeling and control framework that can cope with gain-scheduling towards aerodynamic non-linearities and adaptation toward faults. Nonlinear simulation results of a gain-scheduled pitch controller tolerant to collective blade pitch actuator fault corroborates the effectiveness of the approach. The method is appealing in an industrial context for two main reasons. Firstly, controller structures typically adopted by the wind industry like Proportional-Integral (PI) can be chosen. Secondly, implementation of the resulting controller are considered simpler than other known LPV design techniques. Online matrix inversions and factorizations are common in LPV controller implementations. In the proposed scheme, only products between scalar and matrices and sums of matrices are needed to implement the controller. The drawback of the synthesis procedure is the computational load related to solving several LMI problems sequentially. The technique can be prohibitive for systems with many states and varying parameters.

Paper C [AS12a] brings a theoretical contribution to the $\mathcal{H}_2/\mathcal{H}_\infty$ model reduction problem and application to a wind turbine aeroelastic model. We show that model reduction of linear system with guaranteed norm bounds on the approximation error can be formulated as sufficient LMI conditions extended with multipliers. This is achieved by a proper constraint on the structure of the matrix multiplier and adding "dummy" modes with negligible input-to-output induced norms. The norm bound on the approximation error attained with the sufficient condition may be improved by an algorithm that solves LMI problems sequentially. Numerical experiments compares the proposed sufficient condition with other LMI-based criteria available in the literature, showing that similar or better performance bounds are attained. The iterative LMI algorithm is compared to the $\mathcal{H}_\infty$ model reduction via alternating projections, showing significantly better norm bounds. $\mathcal{H}_2$ model reduction is applied to a flexible wind turbine model and compared with balanced truncation, suggesting a better approximation on low frequencies.

Paper D [ASHS13] presents a procedure to generate reduced-order LPV wind turbine model from a set of high-order LTI models. The reduced-order system depends on a vector of parameter θ which may represent the current operating point; or deviations

on aerodynamics and structural properties for the sake of parametric model uncertainties; or plant parameters to be designed under a simultaneous plant-controller design. Firstly, the high-order LTI models are locally approximated using modal and balanced truncation and residualization. Later, an appropriate manipulation of coordinates yields a consistent state-space representation, which allows interpolation of the model matrices between points of the parameter space. We propose to transform the reduced order LTI systems into a particular realization based on the companion canonical form. A least-squares method is applied to compute parameter-dependent system matrices. The proposed procedure is applied to the NREL 5MW reference wind turbine model. The obtained LPV model encapsulates the wind turbine dynamics throughout the full load region with good approximation bounds.

Paper E [Ade13] presents new sufficient LMI conditions to the static output stabilization problem. These are obtained from a state-space dynamical system where the static feedback controller is not explicitly substituted in the equations of the open-loop system. Different LMI characterizations emerge depending on whether a primal or dual representation of the dynamical system is considered. The drawback of the conditions are two-fold. To reduce conservativeness, a line search in a real scalar is required. Secondly, whenever the solution of the LMI feasibility problem yields a certain multiplier matrix singular, the feedback gains cannot be computed. As the Lyapunov matrix can easily be made parameter-dependent, the proposed LMI is suitable for stabilization of parameter-dependent systems. Numerical experiments with random systems satisfying the generic stabilizability results of Kimura show rates of at least one characterization succeed of over 90%.

Paper F [AS] brings new conditions to the analysis and synthesis of vector-second order systems. These conditions open new directions to verify robust stability and performance of wind turbines and to design robust controllers. We shown that asymptotic stability can be formulated as LMI feasibility problems with explicit dependence on the mass, damping and stiffness matrices. In contrast to other stability results in the literature, the proposed conditions can be applied to vector second-order systems with arbitrary dynamic loading. Moreover, LMI constraints possess linear dependence on the system coefficient matrices, being suitable for parameter-dependent systems. This is the case for standard LMI characterizations in which Lyapunov matrices are decision variables, as well as extended LMI characterizations where multipliers are included in the formulation. Quadratic performance (integral quadratic constraints) conditions based on LMI feasibility problems are also shown to have the same properties. We prove that some of the Lagrange multipliers can be removed from the formulation without loss of generality, leading to formulations computationally less demanding and more revealing to the purpose of synthesis. Synthesis of vector second-order controllers with guaranteed stability and quadratic performance are also formulated as LMI problems. Unfortunately, the synthesis conditions are only sufficient to the existence of controllers. This is the major drawback when compared to synthesis in state-space first-order form, to which necessary and sufficient LMI conditions are available in the literature. A numerical example brings a different perspective to model-based control of wind turbines by considering the design model in its natural, vector second-order form.

Paper G [Ade] presents new sufficient LMI conditions to the simultaneous plant-controller design problem. Formulations for plant design (only plant parameters) are also given. We introduce the notion of linearizing change-of-variables between the plant pa-

rameters to be optimized and matrix multipliers. Synthesis is subject to integral quadratic constraints on inputs and outputs, offering the possibility of designing linear systems with guaranteed $\mathcal{L}_2$-norm performance, passivity properties, and sector bounds on input/output signals. Due to the linear dependence of the proposed LMIs on the Lyapunov matrix, they can easily be extended to parameter-dependent and linear parameter-varying systems.

Extended Summary

A more detailed exposure of the papers is given now.

Paper A: F.D. Adegas, C.S. Sloth and J. Stoustrup, Structured Linear Parameter Varying Control of Wind Turbines

This manuscript deals with structured LPV control of wind turbines. It brings a wider context and content in terms of modeling and controller design when compared to Paper B. We consider wind turbines with faults on actuators and sensors. The proposed framework allows the inclusion of faults representable as varying-parameters. The controllers are scheduled on an estimated wind speed to manage the parameter-varying nature of the model and on information from a fault diagnosis system (Fig. 2.1).
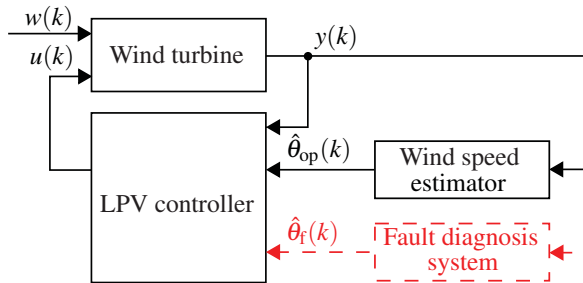
Figure 2.1: Block diagram of the controller structures. The black boxes are common to the LPV controllers, while the red dashed box illustrates the fault diagnosis system required by the AFTC.

An overview of the most common faults that can be modeled as varying parameters is presented, e.g. bias and proportional error in sensors; offset of the generated torque due to an offset in the internal power converter control loops; reduction in conversion efficiency; altered dynamics of pitch system (Pressure drop, pump wear, high air content in the oil). The dynamics of the pitch actuator can be represented as a second-order system. A fault changes the dynamics of the pitch system by varying the damping ratio and natural frequency from their nominal values ζ_0 and $\omega_{n,0}$ to their faulty values ζ_f and $\omega_{n,f}$, with the parameters changing according to a convex combination of the vertices of the parameter sets

$$\omega_n^2(\theta_f) = (1 - \theta_f)\omega_{n,0}^2 + \theta_f\omega_{n,lp}^2 \tag{2.1a}$$

$$-2\zeta(\theta_f)\omega_n(\theta_f) = -2(1 - \theta_f)\zeta_0\omega_{n,0} - 2\theta_f\zeta_{lp}\omega_{n,lp} \tag{2.1b}$$

where $\theta_f \in [0, 1]$ is an indicator function for the fault with $\theta_f = 0$ and $\theta_f = 1$ corresponding to nominal and faulty conditions, respectively.

System and controller matrices are continuous functions of some time-varying parameter vector $\theta = [\theta_1, \ldots, \theta_{n_\theta}]$. Values of the scheduling parameter θ and its rate of variation $\Delta\theta = \theta(k+1) - \theta(k)$ are assumed bounded and belonging to a hyperrectangle and a hypercube, respectively. The plant is parameterized by scalar functions $\rho_i(\theta)$ known as basis functions, that encapsulate possible system's nonlinearities, and the indicator function for the fault θ_f,

$$
\begin{bmatrix} A(\theta) & B_w(\theta) & B_u(\theta) \\ C_z(\theta) & D_{zw}(\theta) & D_{zu}(\theta) \\ C_y(\theta) & D_{yw}(\theta) & D_{yu}(\theta) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_0 + \sum_i \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_i \rho_i(\theta),
$$
$$
+ \sum_m \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_m \theta_{f,m}, \quad i = 1, \ldots, n_\rho, \quad m = 1, \ldots, n_{\theta_f}.
$$

(2.2)

where n_ρ is the number of basis functions. Wind turbine aerodynamics is the main source of nonlinearities. A linearization-based LPV model is adopted. Partial derivatives of aerodynamic torque (Q) and thrust (T) forces with respect to rotor speed (Ω_r), wind speed (V) and pitch angle are natural candidates for the basis functions [BBM07]

$$
\rho_1 := \frac{1}{J_e} \frac{\partial Q}{\partial \Omega}\bigg|_{\theta_{op}}, \quad \rho_2 := \frac{1}{J_e} \frac{\partial Q}{\partial V}\bigg|_{\theta_{op}}, \quad \rho_3 := \frac{1}{J_e} \frac{\partial Q}{\partial \beta}\bigg|_{\theta_{op}},
$$
$$
\rho_4 := \frac{1}{M_t} \frac{\partial T}{\partial \Omega}\bigg|_{\theta_{op}}, \quad \rho_5 := \frac{1}{M_t} \frac{\partial T}{\partial V}\bigg|_{\theta_{op}}, \quad \rho_6 := \frac{1}{M_t} \frac{\partial T}{\partial \beta}\bigg|_{\theta_{op}},
$$

(2.3)

where the scheduling variables representing the operating point is the effective wind speed driving the turbine (i.e. $\theta_{op} := \bar{V}$). Controller and Lyapunov matrices are also made affine dependent in the basis functions and in the fault scheduling variable.

$$
\mathcal{P}(\theta) = P_0 + \sum_{i=1}^{n_\rho} \rho_i(\theta_k)P_i + \sum_{i=1}^{n_{\theta_f}} \theta_{f,i}P_{n_\rho+i},
$$

(2.4a)

$$
K(\theta) = K_0 + \sum_{i=1}^{n_\rho} \rho_i(\theta)K_i + \sum_{i=1}^{n_{\theta_f}} \theta_{f,i}K_{n_\rho+i}.
$$

(2.4b)

Controller synthesis is solved by an iterative LMI-based algorithm which minimizes the $\mathcal{L}_2$-gain from disturbances to performance channels. Due to its similarities with the algorithm proposed in Paper B, details will be omitted here for brevity.

The numerical example addresses the design of a gain-scheduled LPV controller for a pitch controlled wind turbine operating at full power. Controller structure is a static output feedback, taking generator speed error, integral of the generator speed error, and tower velocity as measurements. Collective blade pitch angle (β) is the controlled input. This structure represents a proportional-integral (PI) controller regulating generator speed (Ω_g), while an integral controller increases damping of the tower fore-aft displacement. Controller tuning follows a procedure similar to $\mathcal{H}_\infty$ mixed sensitivities with loop-shaping characteristics which tries to enforce a chosen second-order response from wind disturbances to generator speed as well as increase damping of the tower fore-aft mode.
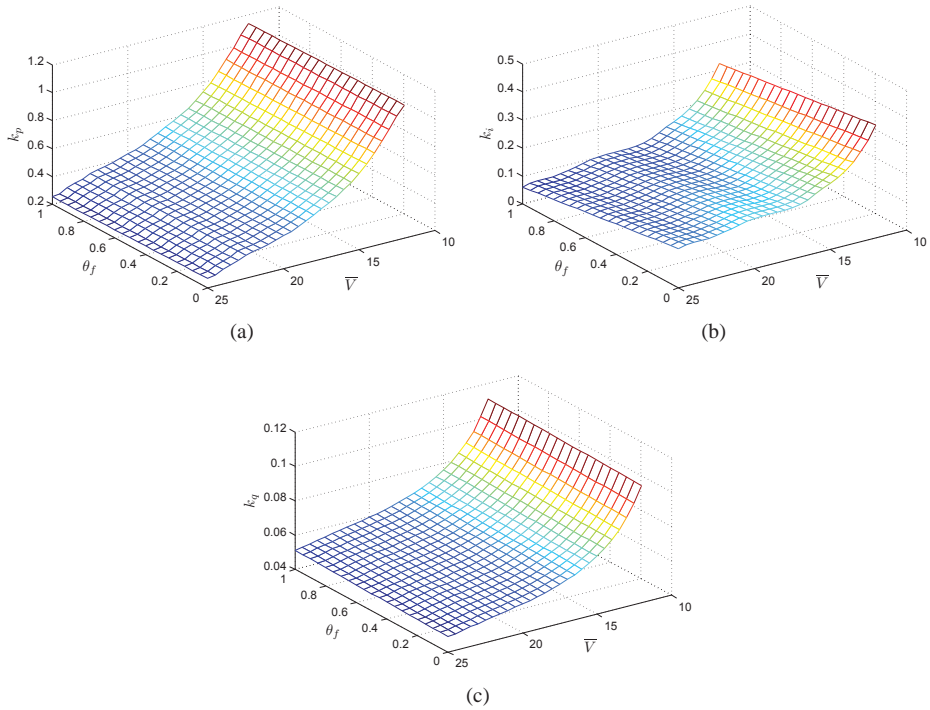
Figure 2.2: Proportional, integral and tower feedback gains as functions of the operating point and fault scheduling variables.

The proportional, integral and tower feedback gains as three-dimensional surfaces of the scheduling parameters $\overline{V}$ and θ_f are illustrated in Fig. 2.2a to 2.2c. The controller gains capture the dependence of the LPV system on the the wind speed given by the basis functions. Also notice the slight changes in k_p and $k_{\dot{q}}$ and the changes in k_i scheduled by the fault scheduling variable θ_f.

The performance of the LPV controllers are accessed in a nonlinear wind turbine simulation environment. Figures 2.3a to 2.3d depict time series of a few variables of interest resulted from a 600 s simulation. At time $t = 200$ s, the pitch system experiences a fault with θ_f increasing from 0 to 1 (Fig. 2.3b). At $t = 430$ s, the pitch system comes to normality with θ_f decreasing from 1 to 0. Both variations on the fault scheduling variable are made with maximum rate of variation. Results of LPV controllers intolerant and tolerant to pitch actuator faults are compared to support a discussion of the consequences of the fault on the closed-loop system as well as fault accommodation. The FT-LPV PI controller successfully accommodates the fault, maintaining rotor speed, generated power and tower oscillations properly regulated.
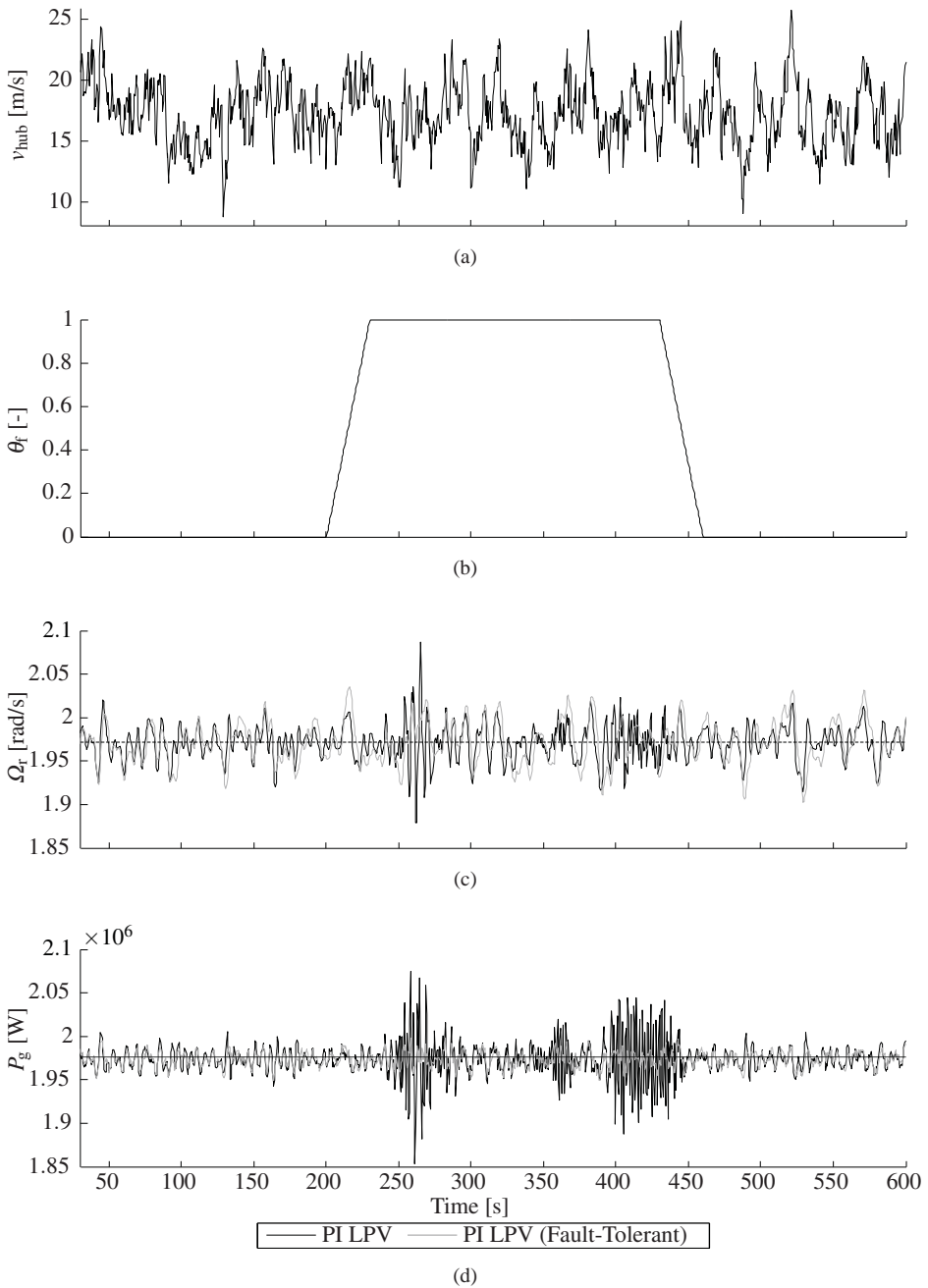
Figure 2.3: Time series of (a) hub height wind speed, (b) fault scheduling variable, (c) rotor speed and (d) electrical power. Simulation results of a 2MW wind turbine controlled by a fault-intolerant and a fault-tolerant LPV PI controller.

Paper B: F.D. Adegas and J. Stoustrup, Structured Control of LPV Systems with Application to Wind Turbines

This paper presents a method for designing structured linear parameter varying controllers (LPV) for wind turbines. System and controller matrices are continuous functions of some time-varying parameter vector $\theta = [\theta_1, \ldots, \theta_{n_\theta}]$. Values of the scheduling parameter θ and its rate of variation $\Delta\theta = \theta(k+1) - \theta(k)$ are assumed bounded and belonging to a hyperrectangle and a hypercube, respectively. Controllers are synthesized via an LMI-based iterative algorithm, which sequentially solves an extension of the bounded real lemma (BRL) for parameter-dependent systems [dSBN06].

Lemma 1 (Extended $\mathcal{L}_2$ Performance). *For a given controller $K(\theta)$, if there exist $\mathcal{P}(\theta) = \mathcal{P}(\theta)^T$ and $\mathcal{Q}(\theta)$ satisfying (2.5) with $r = 1$ for all $(\theta, \Delta\theta) \in \Theta \times \mathcal{V}$, then the system S_{cl} is exponentially stabilizable by the controller $K(\theta)$ and $\|T_{zw}(\theta)\|_{\mathcal{L}_2} < \gamma$.*

$$\begin{bmatrix} r^2\mathcal{P}(\theta^+) & \mathcal{A}(\theta, K(\theta))\mathcal{Q}(\theta) & \mathcal{B}(\theta, K(\theta)) & 0 \\ \star & -\mathcal{P}(\theta) + \mathcal{Q}(\theta)^T + \mathcal{Q}(\theta) & 0 & \mathcal{Q}(\theta)^T\mathcal{C}(\theta, K(\theta))^T \\ \star & \star & \gamma I & \mathcal{D}(\theta, K(\theta))^T \\ \star & \star & \star & \gamma I \end{bmatrix} \succ 0.$$

(2.5)

We propose a modification of the BRL condition suitable for finding feasible controllers in an iterative LMI scheme. The term r^2 multiplying the Lyapunov matrix at the (1,1) entry of (2.5) is artificially inserted into the formulation. BRL condition arises when $r = 1$. When $r > 1$ the set of controllers satisfying the matrix inequality is larger; even systems that are not exponentially stabilizable may satisfy the LMI.

Plant and controller matrices are parameterized by scalar functions $\rho_i(\theta)$ known as basis functions, that encapsulate possible system's nonlinearities. For example, controller matrices are

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}(\theta) = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}_0 + \sum_{i=1}^{n_\theta} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}_i \rho_i(\theta), \quad i = 1, \ldots, n_\theta \qquad (2.6)$$

where n_ρ is the number of basis functions. Open-loop system matrices, multipliers $\mathcal{Q}(\theta)$ and the parameter-dependent Lyapunov function $\mathcal{P}(\theta)$ have similar parametrization. Wind turbine aerodynamics is the main source of nonlinearities. Similarly to Paper A, a linearization-based LPV model is adopted, where $\rho_i(\theta)$ are taken as partial derivatives of aerodynamic torque (Q) and thrust (T) forces with respect to rotor speed (Ω_r), wind speed (V) and pitch angle. The scheduling variable is the effective wind speed driving the turbine (i.e. $\theta := \bar{V}$).

The optimization algorithm iterates between LMI problems by fixing the controller variables and the slack variable alternatively. In order to obtain a finite-dimensional LMI problem, a gridding procedure of the parameter space is proposed. Algorithms for the computation of feasible as well as controllers with sub-optimal performance are presented. The algorithm for optimization of control performance is shown to generate a sequence of solutions such that the cost is non-increasing, that is, $\gamma^{(1)} \geq \gamma^{(j)} \geq \gamma^{(*)}$.

(a) $T_{\hat{V} \to \Omega_r}$, Open-loop

(b) $T_{\hat{V} \to \Omega_r}$, Closed-loop

(c) $T_{\hat{V} \to \dot{q}}$, Open-loop

(d) $T_{\hat{V} \to \dot{q}}$, Closed-loop
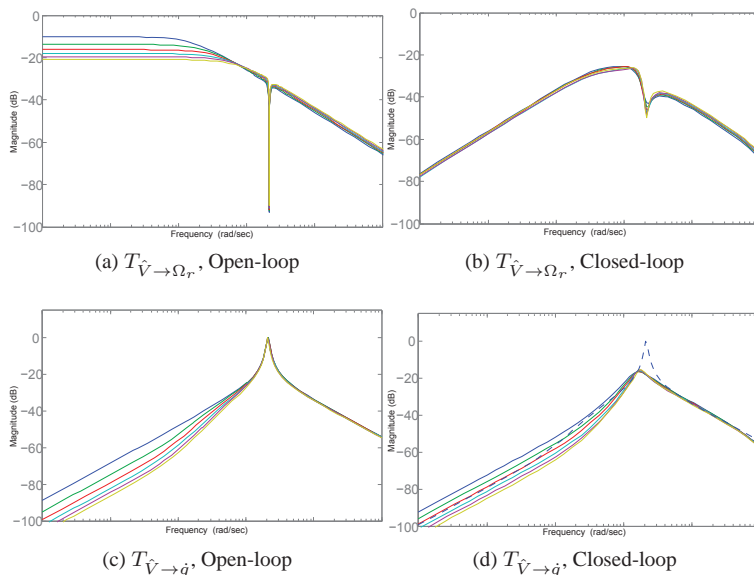
Figure 2.4: Magnitude of transfer functions from wind disturbance to rotor speed and tower velocity.

Controllers resulting from the proposed procedure could be easily implemented in practice due to low data storage and simple math operations involving products between scalar and matrix and sums of matrices.

The numerical example addresses the design of a gain-scheduled LPV controller for a pitch controlled wind turbine operating at full power. The controller structure is composed of two separate loops acting on the collective blade pitch angle (β). A proportional-integral (PI) controller in series with a filter regulates the generator speed (Ω_g), while an integral controller in series with a filter increases damping of the tower fore-aft displacement by measuring tower acceleration $\ddot{q}$. Controller tunning follows an $\mathcal{H}_\infty$ mixed-sensitivity approach.

Magnitude plots of transfer functions from wind disturbance to rotor speed and tower velocity, for the open-loop and closed-loop systems. The increased damping of the tower fore-aft motion is noticeable in Fig.2.4d where the magnitude of the open-loop system (dashed line) is plotted for comparison. A similar response irrespective of the operating point is noticeable, meaning that the controller is gain-scheduling to adapt to the nonlinearities of the plant.

Paper C: F.D. Adegas and J. Stoustrup, $\mathcal{H}_\infty/\mathcal{H}_2$ Model Reduction Through Dilated Linear Matrix Inequalities

This paper proposes sufficient linear matrix inequalities (LMI) conditions to the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ model reduction. To the best knowledge of the authors, LMI extended with multipliers had never been used to address this problem. This summary shows the results for $\mathcal{H}_\infty$ case

only. Consider a stable MIMO LTI dynamical system of order n in state-space form with matrices (A, B, C, D). We seek another model (A_r, B_r, C_r, D_r) with the same number of inputs and outputs and of order $r < n$. The input-output difference between the original system and the reduced system represented by the following state-space description

$$\Delta S : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + \begin{bmatrix} B \\ B_r \end{bmatrix} u(t) \\ y_\Delta(t) = \begin{bmatrix} C & -C_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + \left(D - D_r \right) u(t) \end{cases} \quad (2.7)$$

should be small in an $\mathcal{H}_\infty$ or $\mathcal{H}_2$-norm sense. That is

$$\|\Delta S\|_{\infty \text{ or } 2} \le \gamma \quad (2.8)$$

where γ represents the upper bound on $\mathcal{H}_\infty$ or $\mathcal{H}_2$, depending on the context. A sufficient condition to the existence of reduced-order matrices satisfying an $\mathcal{H}_\infty$ upper bound is stated next.

Theorem 1. $\|S - S_r\|_\infty \le \gamma$ *holds if there exist general auxiliary matrices Q_k, $k = 1, \dots, s + 1$ and H, symmetric matrix X, general matrices $\hat{A}_r$, B_r, $\hat{C}_r$, D_r and a scalar $\mu > 0$ such that*

$$\begin{bmatrix} \hat{A}_\Delta + \hat{A}_\Delta^T & \star & \star & \star \\ \mu\, \hat{A}_\Delta^T - Q + X & -\mu\,(Q + Q^T) & \star & \star \\ \hat{C}_\Delta & \mu\,\hat{C}_\Delta & -\gamma I & \star \\ B_\Delta^T & 0 & D_\Delta^T & -\gamma I \end{bmatrix} \prec 0, \quad (2.9a)$$

$$\hat{A}_\Delta := \begin{bmatrix} AQ_1 & AQ_2 & \dots & AQ_{s+1} \\ \hat{A}_r & \hat{A}_r & \dots & \hat{A}_r \end{bmatrix},$$

$$B_\Delta := \begin{bmatrix} B \\ B_r \end{bmatrix}, \quad D_\Delta := D - D_r \quad (2.9b)$$

$$\hat{C}_\Delta := \begin{bmatrix} CQ_1 - \hat{C}_r & CQ_2 - \hat{C}_r & \dots & CQ_{s+1} - \hat{C}_r \end{bmatrix},$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to

$$A_r = \hat{A}_r H^{-1}, \quad C_r = \hat{C}_r H^{-1}. \quad (2.10)$$

The condition just presented are trivially extended to cope with model reduction of parameter-dependent (PD) systems. We consider system matrices with polytopic dependence on the parameter α

$$\begin{array}{lll} A(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i A_i, & B(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i B_i, & \Lambda := \left\{ \alpha : \sum_{i=1}^{N_\alpha} \alpha_i = 1, \ \alpha_i \ge 0 \right\}. \\ C(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i C_i, & D(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i D_i, & \end{array} \quad (2.11)$$

The aim is to find a reduced system $S_r(\alpha)$ with order $r < n$ and system matrices similar to (2.11) such that $\|S(\alpha) - S_r(\alpha)\| \le \gamma$ for all $\alpha \in \Lambda$.

Theorem 2. $\|S(\alpha) - S_r(\alpha)\|_\infty \leq \gamma$ *holds if there exist general auxiliary matrices* Q_k, $k = 1, \ldots, s+1$ *and* H, *symmetric matrices* X_i, *general matrices* $\hat{A}_{r,i}$, $B_{r,i}$, $\hat{C}_{r,i}$, $D_{r,i}$ *and a scalar* $\mu > 0$ *such that*

$$
\begin{bmatrix}
\hat{A}_{\Delta i} + \hat{A}_{\Delta i}^T & \star & \star & \star \\
\mu\,\hat{A}_{\Delta i}^T - Q + X_i & -\mu\,(Q + Q^T) & \star & \star \\
\hat{C}_{\Delta i} & \mu\,\hat{C}_{\Delta,i} & -\gamma I & \star \\
B_{\Delta i}^T & 0 & D_{\Delta i}^T & -\gamma I
\end{bmatrix} \prec 0, \tag{2.12a}
$$

$$
\hat{A}_{\Delta i} := \begin{bmatrix} A_i Q_1 & A_i Q_2 & \ldots & A_i Q_{s+1} \\ \hat{A}_{r,i} & \hat{A}_{r,i} & \ldots & \hat{A}_{r,i} \end{bmatrix},
$$

$$
B_{\Delta i} := \begin{bmatrix} B_i \\ B_{r,i} \end{bmatrix}, \quad D_{\Delta i} := D_i - D_{r,i} \tag{2.12b}
$$

$$
\hat{C}_{\Delta i} := \begin{bmatrix} C_i Q_1 - \hat{C}_{r,i} & C_i Q_2 - \hat{C}_{r,i} & \ldots & C_i Q_{s+1} - \hat{C}_{r,i} \end{bmatrix},
$$

$$
i = 1, \ldots, N_\alpha,
$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to

$$
A_{r,i} = \hat{A}_{r,i} H^{-1}, \quad C_{r,i} = \hat{C}_{r,i} H^{-1}. \tag{2.13}
$$

The manuscript also proposed an iterative LMI algorithm to improve the result found by the sufficient conditions. This algorithm will not be described here for brevity.

Numerical experiments benchmark the methods proposed in this manuscript with other LMI-based methods found in the literature. Norm bounds on the approximation error obtained with the proposed method were close or better than the ones found with other methods. The order reduction conditions are also applied to an aeroelastic wind turbine model. The objective here is to reduce from 20 states to 10 states without compromising the quality of the model in an $\mathcal{H}_2$ sense. Magnitude plots in frequency domain of the original and reduced models are depicted in Fig. 2.5a. Another reduced model were derived based on the well known balanced truncation model reduction scheme. The comparison with the original model, in this case, is depicted in Fig. 2.5b. When compared to balanced truncation, the $\mathcal{H}_2$ measure seems to be more appropriate in approximating the low frequency range of the model.

Paper D: F.D. Adegas et al., Reduced-Order LPV Model of Flexible Wind Turbines from High Fidelity Aeroelastic Codes

This paper presents a procedure to generate reduced-order LPV wind turbine model from a set of high-order LTI models. We consider N_s stable multiple-input multiple-output (MIMO) LTI dynamical systems of order n corresponding to parameter values $\theta^{(i)}$, $i = 1, 2, \ldots, N_s$,

$$
S_i : \begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i u(t) \\ y(t) = C_i x_i(t) + D_i u(t) \end{cases}, \quad i = 1, \ldots, N_s. \tag{2.14}
$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times n_u}$, $C_i \in \mathbb{R}^{n_y \times n}$, $D_i \in \mathbb{R}^{n_y \times n_u}$. We seek a reduced-order parameterized model $S(\theta)$ of order $r < n$ which approximates S_i,
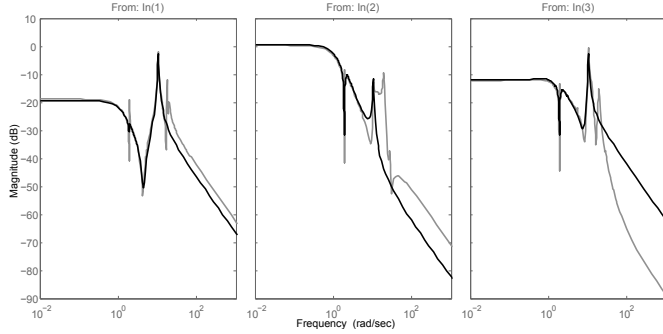
(a) H_2-norm
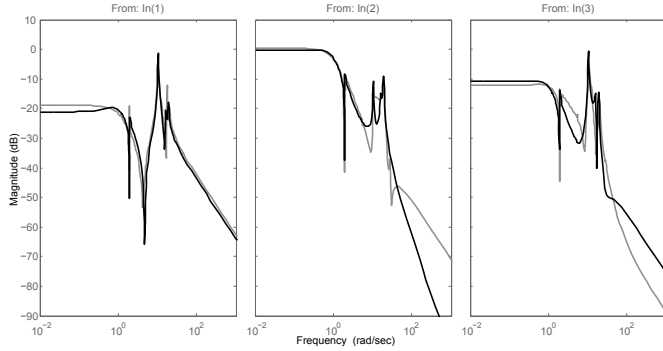
(b) Balanced Truncation

Figure 2.5: Magnitude diagrams of a flexible wind turbine model with 20 states (gray) and reduced 10 states (dark).

$$S(\theta) : \begin{cases} \dot{x} = A(\theta)x(t) + B(\theta)u(t) \\ y(t) = C(\theta)x(t) + D(\theta)u(t) \end{cases} \tag{2.15}$$

where $A(\theta) \in \mathbb{R}^{r \times r}$, $B(\theta) \in \mathbb{R}^{r \times n_u}$, $C(\theta) \in \mathbb{R}^{n_y \times r}$, $D(\theta) \in \mathbb{R}^{n_y \times n_u}$ are continuous functions of a vector of varying parameters $\theta := [\theta_1, \theta_2, \ldots, \theta_{N_\theta}]^T$. The dynamics of the original system S_i and the approximated system $S(\theta)$ are assumed to evolve smoothly with respect to $\theta^{(i)}$ and θ, respectively. The parameter θ may represent the current operating point. It also may describe deviations on aerodynamics and structural properties for the sake of parametric model uncertainties. Plant parameters to be designed under an integrated plant-controller synthesis scheme could also be parameterized.

A flowchart containing the steps of the proposed procedure is depicted in Fig. 2.6. Firstly, the high-order LTI models are locally approximated using modal and balanced truncation and residualization. Then, an appropriate manipulation of the coordinate system is applied to allow interpolation of the model matrices between points of the parameter space. A consistent state-space representation should be found before interpolation methods are applied. We propose to transform the reduced order LTI systems into a representation based on the companion canonical form. There exist algorithms which, for a

system under arbitrary similarity transformation, find a unique companion form [Kai80]. In order to avoid known numerical issues of the companion form, each mode k of the reduced system in modal coordinates is individually transformed into a companion realization. The system matrices of this particular realization are given by (2.16).

$$A_{c,i} = \text{diag}(A_{c,k,i}), \quad B_{c,i} = \begin{bmatrix} B_{c,1,i} \\ B_{c,2,i} \\ \vdots \\ B_{c,k,i} \end{bmatrix},$$

$$C_{c,i} = \begin{bmatrix} C_{c,1,i} & C_{c,2,i} & \cdots & C_{c,k,i} \end{bmatrix},$$

$$\begin{cases} A_{c,k,i} = -a_{k,i} \\ B_{c,k,i} = \begin{bmatrix} 1 & b_{1,k,i} & \cdots & b_{n_u-1,k,i} \end{bmatrix} & \text{for real eigenvalues,} \\ C_{c,k,i} = \begin{bmatrix} c_{1,k,i} \\ \cdots \\ c_{n_y-1,k,i} \\ 0 \end{bmatrix} \end{cases}$$

(2.16)

$$\begin{cases} A_{c,k,i} = \begin{bmatrix} 0 & -a_{k,i,1} \\ 1 & -a_{k,i,2} \end{bmatrix}, \\ B_{c,k,i} = \begin{bmatrix} 0 & b_{11,k,i} & \cdots & b_{1n_u-1,k,i} \\ 1 & b_{21,k,i} & \cdots & b_{2n_u,k,i} \end{bmatrix} & \text{for complex eigen.} \\ C_{c,k,i} = \begin{bmatrix} c_{11,k,i} & \cdots & c_{1r,k,i} \\ \vdots & \ddots & \vdots \\ c_{n_y1,k,i} & \cdots & c_{n_yr,k,i} \end{bmatrix} \end{cases}$$

$$i = 1, \ldots, N_s, \quad k = 1, \ldots, N_m.$$

The characteristic polynomial of each mode appears in the rightmost column of the matrix $A_{c,k,i}$. With the state-space matrices now at a realization suitable for interpolation, a least-squares method is applied to compute parameter-dependent system matrices. The proposed procedure is applied to the NREL 5MW reference wind turbine model [JBMS09] where we aim to find an LPV model encapsulating the wind turbine dynamics operating at the full load region. Large scale MIMO LTI models with 877 states are computed by the aeroelastic code HAWCStab2 for wind speeds equidistant 1 m/s ($\theta^{(i)} \in \{12, 13, \ldots, 25\}$). The model is parameterized by the mean wind speed $\theta := \bar{V}$. A fifth-order polynomial dependence of the LPV system matrices

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \sum_{d=1}^{5} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \theta^d$$

(2.17)

gives a fair trade-off between interpolation accuracy and polynomial order. A comparison of the minimum and maximum singular values for a wind speed of $\bar{V} = 15$ m/s is depicted in Fig. 2.7 and shows an excellent agreement. The location of the poles of the LPV system
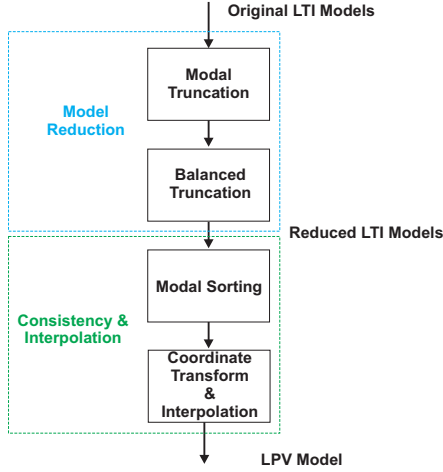
Figure 2.6: Scheme overview.
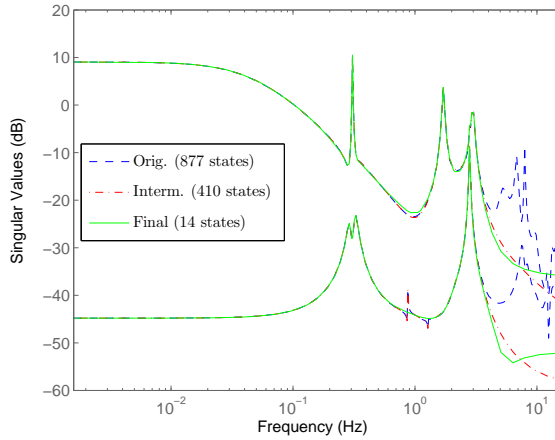
Figure 2.7: Singular values of the original, intermediate and final reduced order models for a mean wind speed of 15 m/s.

for a $2N_s$ grid of equidistant operating points is illustrated in Fig. 2.8. The arrows indicate how the poles move for increasing mean wind speeds. A smooth evolution of the poles along the full load region is noticeable.

The relative difference of the Hankel singular values of the interpolated LPV system and the reduced order system defined as

$$\sigma_{rel,r,i} = \frac{\sigma_{int,r,i} - \sigma_{r,i}}{\sigma_{r,i}} \times 100, \quad i = 1, \ldots, N_s \tag{2.18}$$

serves as a measure of the quality of the interpolation. A good fit can be corroborated by
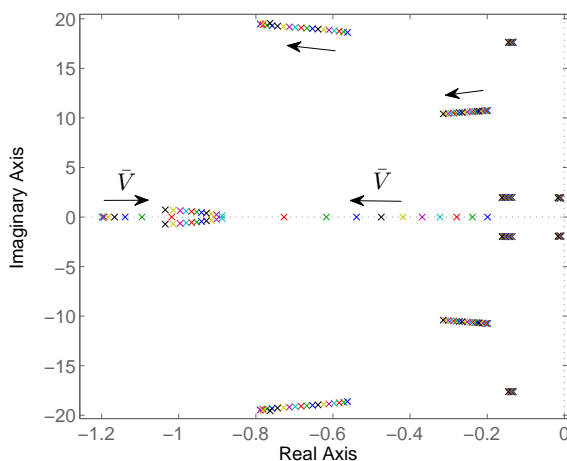
(a) Pole Map

Figure 2.8: Pole location of the LPV model for frozen values of the varying parameter θ.

some metrics of $\sigma_{rel,r,i}$ given in Tab. 2.1. The mean difference in the Hankel singular values is only 0.27% and the maximum difference just 2.75%.

Table 2.1: Difference in the Hankel singular values between the LPV and reduced order system for frozen values of θ.

Max	Min	Mean	Std. dev
2.75	0.001	0.27	0.57

The obtained LPV model is of suitable size for synthesizing modern gain-scheduling controllers based on the recent advances on LPV control design. Time propagation of the varying parameter is not explicitly utilized. Therefore, the procedure assumes that the varying parameter do not vary excessively fast in time, in line with common practices in gain-scheduling control.

Paper E: F.D. Adegas, New Sufficient LMI Conditions for Static Output Stabilization

New LMI conditions to the static output feedback stabilization (SOFS) problem are presented in this paper. We propose to work on enlarged spaces composed by the state $x(t)$ and its time derivative $\dot{x}(t)$, the control input $u(t)$ or the measured output $y(t)$. Different LMI characterizations emerge depending on the considered spaces $(x(t), \dot{x}(t), u(t))$ or $(x(t), \dot{x}(t), y(t))$. This summary presents conditions for the former space only. Let the

dynamical equations be written as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.19a}$$
$$u(t) = KCx(t). \tag{2.19b}$$

Stability of the above dynamical system is characterized by the existence of the set

$$\dot{V}(x(t), \dot{x}(t)) < 0, \quad \forall (x(t), \dot{x}(t), u(t)) \neq 0 \quad \text{satisfying (2.19).} \tag{2.20}$$

where the quadratic form $\dot{V} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\dot{V}(t) := \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \tag{2.21}$$

Let an extended input matrix $\tilde{B} \in \mathbb{R}^{n \times n}$ be the result of augmenting redundant inputs to the system, such that the number of inputs are equal to the number of states. That is

$$\tilde{B} := \begin{bmatrix} B_{u_1} & B_{u_2} & \dots & B_{u_{n_u}} & B_{u_1} & B_{u_2} & \dots \end{bmatrix} \tag{2.22}$$

where B_{u_i} stands for the i-th column of matrix B related to the i-th input. Let also define an augmented feedback gain $\tilde{K} \in \mathbb{R}^{n \times n_y}$ partitioned accordingly

$$\tilde{K} := \begin{bmatrix} K_{1,u_1}^T & K_{1,u_2}^T & \dots & K_{1,u_{n_u}}^T & K_{2,u_1}^T & \dots & K_{2,u_2}^T & \dots \end{bmatrix}^T \tag{2.23}$$

where $K_{j,u_i} \in \mathbb{R}^{1 \times n_y}$ are j-th feedback gain from $y(t)$ to the i-th input u_i. The contribution of the different gains K_{j,u_i} to a particular input is just their sum

$$K_{u_i} := \sum_{j=1}^{N} K_{j,u_i} \tag{2.24}$$

The following theorem presents a new sufficient condition for the existence of a static feedback gain satisfying the set (2.20).

Theorem 3. *(Stabilizability) There exists a static output feedback that renders $A + BKC$ Hurwitz if $\exists\, P \in \mathbb{S}^n, \hat{K} \in \mathbb{R}^{n \times n_y}, \Lambda_1, \Lambda_2, \Gamma_1 \in \mathbb{R}^{n \times n}, \mu \in \mathbb{R}$:*

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad P \succ 0, \quad \mathcal{J} := \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.25a}$$

$$\mathcal{H} := \begin{bmatrix} \Lambda_1 A + \hat{K}C & -\Lambda_1 & \Lambda_1 \tilde{B} - \Lambda_2 \\ \alpha(\Lambda_1 A + \hat{K}C) & -\alpha\Lambda_1 & \alpha(\Lambda_1 \tilde{B} - \Lambda_2) \\ \Gamma_1 A + \mu\hat{K}C & -\Gamma_1 & \Gamma_1 \tilde{B} - \mu\Lambda_2 \end{bmatrix} \tag{2.25b}$$

for an arbitrary scalar $\alpha > 0$ and if the solution yields Λ_2 non-singular.

Whenever Λ_2 is non-singular, the original controller gains can be reconstructed according to $\tilde{K} = \Lambda_2^{-1}\hat{K}$.

Numerical experiments show good success rates in finding a stabilizing controller. They also suggest that the assumption of a solution rendering a non-singular multiplier Λ_2 is not strong. Controllers were computed for sets of 1.000 random triplices (A, B, C)

satisfying the generic stabilizability results of Kimura. Table 2.2 summarizes the success of finding valid solutions for systems with different dimensions (n, n_u, n_y). The number of successes related to the primal dynamical system (Theorem presented above), dual dynamical system (see Paper E), and either one of these two are depicted. The rates of at least one characterization succeed are over 90%, remarkably high. Out of all valid solutions, the smallest absolute eigenvalue of the multiplier Λ_2 was smaller than 10^{-3} only at two occasions (0.03%).

(n_u, n_y)	Primal System	Dual System	Primal or Dual System
		n=4	
(3,2)	924 (92.4%)	859 (85.9%)	983 (98.3%)
(2,3)	847 (84.7%)	943 (93.4%)	972 (97.2%)
		n=6	
(3,4)	796 (79.6%)	856 (85.6%)	943 (94.3%)
(4,3)	881 (88.1%)	819 (81.9%)	954 (95.4%)

Table 2.2: Success rate on the SOFS problem.

The proposed conditions can be exploited in the well known simultaneous and robust stabilization problems. The Lyapunov variable P can be made multiple or parameter-dependent due to its appearance free from products with system matrices. The simultaneous stabilization problem is treated in this summary. Given a family of open-loop plants,

$$G_i := \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \qquad i = 1, 2, \ldots, N_p$$

find a controller K that simultaneously stabilizes the plants, that is,

$$\mathrm{Re}\left(\lambda\left(A_i + B_i K C_i\right)\right) < 0, \qquad i = 1, 2, \ldots, N_p.$$

Considering the enlarged space $(x(t), \dot{x}(t), u(t))$, this problem can be cast as the sufficient LMI feasibility condition: $\exists P_i \in \mathbb{S}^n, \hat{K} \in \mathbb{R}^{n \times n_y}, \Lambda_{1,i}, \Lambda_2, \Gamma_{1,i} \in \mathbb{R}^{n \times n}, \mu \in \mathbb{R}$:

$$\mathcal{J}_i + \mathcal{H}_i + \mathcal{H}_i^T \prec 0, \quad P_i \succ 0, \quad \mathcal{J}_i := \begin{bmatrix} 0 & P_i & 0 \\ P_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.26a}$$

$$\mathcal{H} := \begin{bmatrix} \Lambda_{1,i} A_i + \hat{K} C_i & -\Lambda_{1,i} & \Lambda_{1,i} \tilde{B}_i - \Lambda_2 \\ \alpha(\Lambda_{1,i} A_i + \hat{K} C_i) & -\alpha \Lambda_{1,i} & \alpha(\Lambda_{1,i} \tilde{B}_i - \Lambda_2) \\ \Gamma_{1,i} A_i + \mu \hat{K} C_i & -\Gamma_{1,i} & \Gamma_{1,i} \tilde{B}_i - \mu \Lambda_2 \end{bmatrix} \tag{2.26b}$$

for $i = 1, \ldots, N_p$ where $\alpha > 0$ is an arbitrary scalar, and if the solution yields Λ_2 non-singular. A single Λ_2 is adopted for the family of open-loop plants in order to facilitate change-of-variables involving the controller gain and multiplier, while multiple $\Lambda_{1,i}$ and $\Gamma_{1,i}$ help to reduce conservativeness.

Paper F: F.D. Adegas and J. Stoustrup, Linear Matrix Inequalities for Analysis and Control of Linear Vector Second-Order Systems

This paper contributes to the analysis and synthesis of systems represented in vector second-order. In order to limit the content of this summary, we consider the particular controlled system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F_w w(t) + F_u u(t) \tag{2.27a}$$

$$z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw}w(t) + D_{zu}u(t) \tag{2.27b}$$

$$u(t) = -G_v\dot{q}(t) - G_p q(t) \tag{2.27c}$$

where $q(t) \in \mathbb{R}^n$ is the position vector, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance input, $u(t) \in \mathbb{R}^{n_u}$ the controlled input, M, C, K, F_w, F_u, G_v, G_p are real matrices with appropriate dimensions. The only assumption on the system matrices is non-singularity of M and K. Denote $\mathbf{H}_{zw}$ the input-output operator of system (2.27). The novel necessary and sufficient conditions for asymptotic stability and quadratic performance are given in terms of LMIs with explicit dependence in the system coefficient matrices. The following standard Lyapunov stability condition is a major contribution of this paper. A linear dependence on the system coefficient matrices can be noticed, in contrast to many known stability conditions dependent on inverses of coefficient matrices.

Theorem 4. *System (2.27) is asymptotically stable if, and only if,* $\exists P_1 \in \mathbb{S}^n$, W_2, $W_3 \in \mathbb{R}^{n \times n}$:

$$\begin{bmatrix} -(W_2 K + K^T W_2^T) & P_1 - W_2 C - K^T W_3^T \\ P_1 - W_3 K - C^T W_2^T & P_2 + P_2^T - (W_3 C + C^T W_3^T) \end{bmatrix} \prec 0, \tag{2.28a}$$

$$\begin{bmatrix} P_1 & W_2 M \\ M^T W_2^T & W_3 M \end{bmatrix} + \begin{bmatrix} P_1 & W_2 M \\ M^T W_2^T & W_3 M \end{bmatrix}^T \succ 0, \tag{2.28b}$$

$$W_3 M = M^T W_3^T \tag{2.28c}$$

New LMI conditions extended with multipliers are also proposed. Elimination of multipliers is investigated, to determine in which circumstances multipliers can be removed without loss of generality, leading to LMI with fewer variables and better suited to synthesis. We illustrate this in the context of integral quadratic constraints on the input and output signals. Consider the constrained Lyapunov problem

$$\dot{V}(q(t), \dot{q}(t), \ddot{q}(t)) < - \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Z^T Q Z & Z^T(S + Q D_{zw}) \\ (S + Q D_{zw})^T Z & \bar{R} \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ w(t) \end{pmatrix},$$

$$\bar{R} := R + D_{zw}^T Q D_{zw} + D_{zw}^T S + S^T D_{zw} \tag{2.29a}$$

$$\forall (q(t), \dot{q}(t), \ddot{q}(t), w(t)) \text{ satisfying } M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F_w w(t), \tag{2.29b}$$

$$(q(t), \dot{q}(t), \ddot{q}(t), w(t)) \neq 0, \tag{2.29c}$$

Sufficient conditions with reduced number of multipliers can be derived from the above Lyapunov problem by applying the Elimination Lemma. They become also necessary if

the acceleration vector (or position vector) is absent in the performance vector $z(t)$ i.e. $U = 0$.

Theorem 5. *The set of solutions of the Lyapunov problem* (2.29) *with* $P \succ 0$ *is not empty if* $\exists\, P_1,\, P_3 \in \mathbb{S}^n,\, P_2,\, \Phi,\, \Lambda \in \mathbb{R}^{n \times n},\, \alpha \in \mathbb{R}$:

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad where \tag{2.30a}$$

$$\mathcal{J} := \begin{bmatrix} 0 & P_1 & P_2 & 0 \\ P_1 & P_2 + P_2^T & P_3 & 0 \\ P_2^T & P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} X^T Q X & X^T Q V & X^T Q U & X^T(S + Q D_{zw}) \\ \star & V^T Q V & V^T Q U & V^T(S + Q D_{zw}) \\ \star & \star & U^T Q U & U^T(S + Q D_{zw}) \\ \star & \star & \star & \bar{R} \end{bmatrix}, \tag{2.30b}$$

$$\mathcal{H} := \begin{bmatrix} \Phi K & \Phi C & \Phi M & -\Phi F_w \\ (\alpha\Phi + \Lambda)K & (\alpha\Phi + \Lambda)C & (\alpha\Phi + \Lambda)M & -(\alpha\Phi + \Lambda)F_w \\ \alpha\Lambda K & \alpha\Lambda C & \alpha\Lambda M & -\alpha\Lambda F_w \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha > 0, \tag{2.30c}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \tag{2.30d}$$

This is necessary and sufficient whenever $U = 0$ *in* (2.29).

Regarding synthesis, the manuscript has focused on stabilizing and $\mathcal{L}_2$-gain controllers in vector second-order form. The $\mathcal{L}_2$ performance criteria is a particularization of the above by setting

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad \leftarrow \quad \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

The next theorem states the existence of a static controller of the form (2.27c) with guaranteed $\mathcal{L}_2$ performance.

Theorem 6. *There exists a controller of the form* (2.27c) *such that* $\|\mathbf{H}_{zw}\|_{\mathcal{L}_2} < \gamma^2$ *if* $\exists\, \hat{P}_1, \hat{P}_3 \in \mathbb{S}^n,\, \hat{P}_2,\, \Gamma \in \mathbb{R}^n,\, \hat{G}_v,\, \hat{G}_p \in \mathbb{R}^{n_u \times n},\, \alpha, \mu \in \mathbb{R}$:

$$\begin{bmatrix} \mu(K\Gamma + F_u \hat{G}_p) + \star & \hat{P}_1 + \mu(C\Gamma + F_u \hat{G}_v) + (1 + \alpha\mu)(K\Gamma + F_u \hat{G}_p)^T & \hat{P}_2 + \mu M\Gamma + \alpha(K\Gamma + F_u \hat{G}_p)^T & -\mu F_w & \Gamma^T X^T - \hat{K}_p^T D_{zu}^T \\ \star & \hat{P}_2 + (1 + \alpha\mu)(C\Gamma + F_u \hat{G}_v) + \star & \hat{P}_3 + (1 + \alpha\mu)M\Gamma + \alpha(C\Gamma + F_u \hat{G}_v)^T & -(1 + \alpha\mu)F_w & \Gamma^T V^T - \hat{K}_v^T D_{zu}^T \\ \star & \star & \alpha(M\Gamma + \Gamma^T M^T) & -\alpha F_w & \Gamma^T U^T \\ \star & \star & \star & -\gamma^2 I & D_{zw}^T \\ \star & \star & \star & \star & -I \end{bmatrix} \prec 0, \tag{2.31a}$$

$$\alpha > 0, \quad \mu > 0, \quad \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} \succ 0 \tag{2.31b}$$

.

If a solution is found, the controller gains can always be reconstructed from the auxiliary ones according to $G_v = \hat{G}_v \Gamma^{-1}$ and $G_p = \hat{G}_p \Gamma^{-1}$.

The inherent decoupling of the Lyapunov and system matrices facilitates parameter-dependent Lyapunov functions as certificates of stability. Assume uncertain system coefficient matrices taking values in a domain defined as a polytopic combination of N given matrices. We consider the robust stability problem as an example. Such system is robustly stabilizable by a static feedback law of the form (2.27c) if $\exists \Lambda$, $P_{2,i} \in \mathbb{R}^{n \times n}$, $P_{1,i}$, $P_{3,i} \in \mathbb{S}^n$, $\hat{G}_v$, $\hat{G}_p \in \mathbb{R}^{n_u \times n}$, α, $\mu \in \mathbb{R}$:

$$\mathcal{J}_i + \mathcal{H}_i + \mathcal{H}_i^T \prec 0, \quad \mathcal{J}_i := \begin{bmatrix} 0 & P_{1,i} & P_{2,i} \\ P_{1,i} & P_{2,i} + P_{2,i}^T & P_{3,i} \\ P_{2,i}^T & P_{3,i} & 0 \end{bmatrix}, \tag{2.32a}$$

$$\mathcal{H}_i := \begin{bmatrix} \mu(K_i \Lambda + F_{u,i} \hat{G}_p) & \mu(C_i \Lambda + F_{u,i} \hat{G}_v) & \mu(M_i \Lambda) \\ (1 + \alpha\mu)(K_i \Lambda + F_{u,i} \hat{G}_p) & (1 + \alpha\mu)(C_i \Lambda + F_{u,i} \hat{G}_v) & (1 + \alpha\mu)(M_i \Lambda) \\ \alpha(K_i \Lambda + F_{u,i} \hat{G}_p) & \alpha(C_i \Lambda + F_{u,i} \hat{G}_v) & \alpha(M_i \Lambda) \end{bmatrix} \tag{2.32b}$$

$$\begin{bmatrix} P_{1,i} & P_{2,i} \\ P_{2,i}^T & P_{3,i} \end{bmatrix} \succ 0, \quad \alpha > 0, \mu > 0, \tag{2.32c}$$

for $i = 1, \ldots, N$.

Numerical examples illustrate the design of controllers. One of the examples brings a different perspective to modern control of wind turbines by considering the design model in its natural, vector second-order form. For clarity, the turbine model contains only the two structural degrees of freedom with lowest frequency contents: rigid body rotation of the rotor and fore-aft tower bending described by the axial nacelle displacement. Controller design follows a $\mathcal{H}_\infty$ model matching criteria, which has an elegant structure when considered in vector second-order form. The performance of the system in closed-loop should approximate a given a reference model

$$M_r \ddot{q}_r(t) + C_r \dot{q}_r(t) + K_r \ddot{q}_r(t) = F_{wr} w(t) \tag{2.33a}$$
$$z_r(t) = U_r \ddot{q}_r(t) + V_r \dot{q}_r(t) + X_r \ddot{q}_r(t) \tag{2.33b}$$

in an $\mathcal{H}_\infty$-norm sense. The matrices of the reference model are chosen to enforce a desired second-order closed-loop sensitivity function from wind speed disturbance $v(t)$ to rotor speed $\dot{\psi}(t)$. Bode plots of the closed-loop, open-loop and reference systems are depicted in Fig.2.9a. A good agreement between the closed-loop and reference model is noticeable. The chosen reference model indirectly impose some damping of the tower fore-aft displacement by trying to reduce the difference in magnitude between open-loop and reference model at the tower natural frequency. Step responses of the controlled and reference systems are compared in Fig.2.9b, showing a good correspondence.
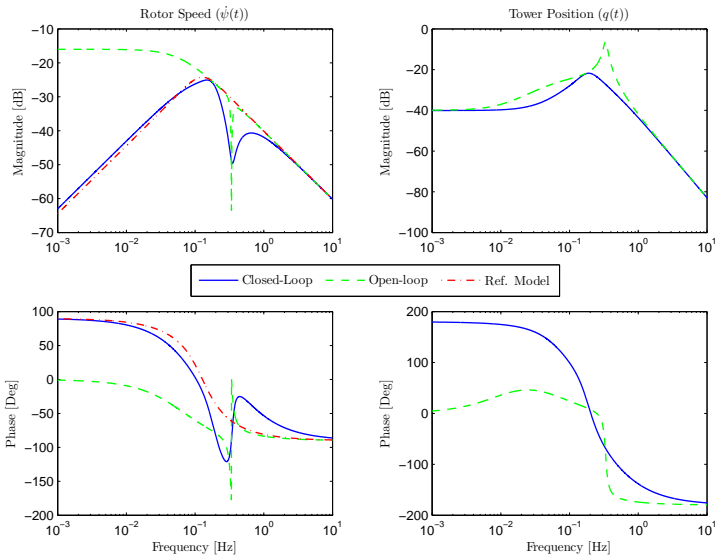
(a) Bode Plots
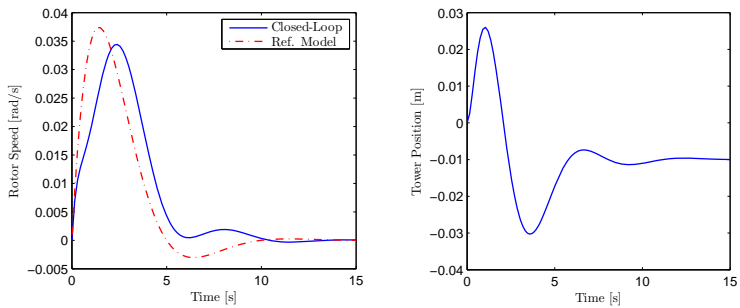
(b) Step Response

Figure 2.9: $\mathcal{H}_\infty$ model matching control of a simplified wind turbine model.

Paper G: F.D. Adegas, Linear Matrix Inequalities Conditions for Simultaneous Plant-Controller Design

In this paper, LMI conditions to the simultaneous plant-controller design are derived. We consider the closed-loop system

$$\dot{x}(t) = (A(\rho) + B_u K)x(t) + B_w(\rho)w(t) \tag{2.34a}$$

$$z(t) = (C_z(\rho) + D_{zu}K)x(t) + D_{zw}(\rho)w(t) \tag{2.34b}$$

where $x \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector, $z(t) \in \mathbb{R}^{n_z}$ is the performance channel, $K \in \mathbb{R}^{n_u \times n}$ is the full-state feedback gain, and systems matrices are real valued with appropriate dimensions, and defined as

$$A(\rho) := A_0 + A_\rho \rho, \quad B_w(\rho) := B_{w0} + B_{w\rho}\rho,$$
$$C_z(\rho) := C_{z0} + C_{z\rho}\rho, \quad D_{zw}(\rho) := D_{zw0} + D_{zw\rho}\rho,$$
$$A_\rho := \begin{bmatrix} A_{\rho_1} & A_{\rho_2} & \dots & A_{\rho_N} \end{bmatrix}, \quad B_{w\rho} := \begin{bmatrix} B_{w\rho_1} & B_{w\rho_2} & \dots & B_{w\rho_N} \end{bmatrix},$$
$$C_{z\rho} := \begin{bmatrix} C_{z\rho_1} & C_{z\rho_2} & \dots & C_{z\rho_N} \end{bmatrix}, \quad D_{zw\rho} := \begin{bmatrix} D_{zw\rho_1} & D_{zw\rho_2} & \dots & D_{zw\rho_N} \end{bmatrix},$$
$$\rho := \begin{bmatrix} \rho_1 I_n & \rho_2 I_n & \dots & \rho_N I_n \end{bmatrix}^T, \quad \underline{\rho}_i \le \rho_i \le \overline{\rho}_i, \quad i = 1, \dots, N.$$

The above matrices are affine dependent on ρ representing deviations of the plant parameters from the nominal ones. Matrices A_ρ, $B_{w\rho}$, $C_{z\rho}$, $D_{zw\rho}$ define how parameter deviations affect the nominal plant matrices A_0, B_{w0}, C_{z0}, D_{zw0}. These parameters are assumed bounded by a hypercube with lower limit $\underline{\rho}_i$ and upper limit $\overline{\rho}_i$.

The simultaneous plant-controller design is subject to integral quadratic constraints (ICQ) on the input and output signals

$$\int_0^\infty \begin{pmatrix} z'(t) \\ w'(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z'(t) \\ w'(t) \end{pmatrix} \ge 0 \tag{2.36}$$

where $Q \in \mathbb{S}^{n_z}$, $R \in \mathbb{S}^{n_w}$, $S \in \mathbb{R}^{n_z \times n_w}$, $R \succ 0$. Considering the algebraic dual of system (2.34)-(2.35), and resorting to Lyapunov theory and the S-procedure [OS01], it can be shown that system (2.34)-(2.35) is asymptotically stable and satisfy the ICQ constraint (2.36) if, and only if, there exists $V(x(t)) > 0$, $\forall x(t) \ne 0$ such that

$$\dot{V}(x^{'}(t), \dot{x}^{'}(t)) < - \begin{pmatrix} x^{'}(t) \\ p^{'}_x(t) \\ p^{'}_w(t) \\ w^{'}(t) \end{pmatrix}^T M \begin{pmatrix} x^{'}(t) \\ p^{'}_x(t) \\ p^{'}_w(t) \\ w^{'}(t) \end{pmatrix}, \tag{2.37a}$$

$$M := \begin{bmatrix} B_{w0}QB^T_{w0} & B_{w0}QB^T_{w\rho} & B_{w0}QD^T_{zw\rho} & B_{w0}SD_{zw0} \\ \star & B_{w\rho}QB^T_{w\rho} & B_{w\rho}QD^T_{zw\rho} & B_{w\rho}SD_{zw0} \\ \star & \star & D_{zw\rho}QD^T_{zw\rho} & D_{zw\rho}SD^T_{zw0} \\ \star & \star & \star & \bar{R} \end{bmatrix} \tag{2.37b}$$

$$\bar{R} := R + D_{zw0}QD^T_{zw0} + D_{zw0}S + S^T D^T_{zw0} \tag{2.37c}$$

$$\forall (x^{'}(t), \dot{x}^{'}(t), p^{'}_x(t), p^{'}_w(t), w(t)) \neq 0 \quad \text{satisfying} \tag{2.37d}$$

$$\dot{x}^{'}(t) = (A_0 + B_u K)^T x^{'}(t) + (C_{z0} + D_{zu}K)w^{'}(t) + p^{'}_x(t) + p^{'}_w(t) \tag{2.37e}$$

$$p^{'}_x(t) := \rho^T A^T_\rho x^{'}(t), \quad p^{'}_w(t) := \rho^T C^T_{z\rho} w^{'}(t). \tag{2.37f}$$

where $(\cdot)^{'}$ represents the dualized system variables. The nonlinear change-of-variables

$$\hat{\rho} := \rho \psi I_n, \quad \hat{\rho} = \begin{bmatrix} \hat{\rho}_1 I_n & \hat{\rho}_2 I_n & \dots & \hat{\rho}_N I_n \end{bmatrix}^T \tag{2.38}$$

involving the plant parameters ρ_i and a scalar multiplier ψ is an original contribution and facilitates the derivations that follows. Whenever $\psi \neq 0$, the original plant parameters can be reconstructed according to $\rho_i = \hat{\rho}_i \psi^{-1}$. LMI conditions arises by applying Finsler's lemma to (2.37). Let synthesis conditions for the plant design problem (controller gains $K = 0$) be presented first.

Theorem 7 (Plant Synthesis). *There exists plant parameters ρ_i, $i = 1, \dots, N$ such that the set (2.37) is not empty if $\exists P \in \mathbb{S}^n$, $\hat{\rho}_i$, $i = 1, \dots, N$, ψ, $\alpha \in \mathbb{R}$, Φ_1, Γ_1, Λ_1, Π_1, $\in \mathbb{R}^{n \times n}$:*

$$\mathcal{J} + \mathcal{H}^T + \mathcal{H} \prec 0, \qquad P \succ 0, \tag{2.39}$$

$$\mathcal{J} := \begin{bmatrix} B_{w0}QB^T_{w0} & P & B_{w0}QB^T_{w\rho} & B_{w0}QD^T_{zw\rho} & B_{w0}SD_{zw0} \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & B_{w\rho}QB^T_{w\rho} & B_{w\rho}QD^T_{zw\rho} & B_{w\rho}SD_{zw0} \\ \star & \star & \star & D_{zw\rho}QD^T_{zw\rho} & D_{zw\rho}SD^T_{zw0} \\ \star & \star & \star & \star & \bar{R} \end{bmatrix}, \tag{2.40}$$

$$\mathcal{H} := \begin{bmatrix} A_0 \Phi_1 + A_\rho \hat{\rho} & \alpha(A_0 \Phi_1 + A_\rho \hat{\rho}) & A_0 \Lambda_1 + A_\rho \hat{\rho} & A_0 \Pi_1 + A_\rho \hat{\rho} & 0 \\ -\Phi_1 & -\alpha \Phi_1 & -\Lambda_1 & -\Pi_1 & 0 \\ \Phi_1 - \psi I_n & \alpha(\Phi_1 - \psi I_n) & \Lambda_1 - \psi I_n & \Pi_1 - \psi I_n & 0 \\ \Phi_1 - \psi I_n & \alpha(\Phi_1 - \psi I_n) & \Lambda_1 - \psi I_n & \Pi_1 - \psi I_n & 0 \\ C_{z0} \Phi_1 + C_{z\rho} \hat{\rho} & \alpha(C_{z0} \Phi_1 + C_{z\rho} \hat{\rho}) & C_{z0} \Lambda_1 + C_{z\rho} \hat{\rho} & C_{z0} \Pi_1 + C_{z\rho} \hat{\rho} & 0 \end{bmatrix}, \tag{2.41}$$

$$\underline{\rho}_i \psi \geq \hat{\rho}_i \geq \overline{\rho}_i \psi. \tag{2.42}$$

and if the solution yields ψ non-singular.

In this case, the original plant parameters can be recovered from the auxiliary ones according to $\rho_i = \hat{\rho}_i \psi^{-1}$. The results from the plant design are extended with the state-feedback controller yielding the main contribution of this paper.

Theorem 8 (Simultaneous Plant-Controller Synthesis). *There exist plant parameters ρ_i, $i = 1, \ldots, N$ and a controller gain K such that the set (2.37) is not empty if $\exists\ P \in \mathbb{S}^n$, $\hat{\rho}_i$, $i = 1, \ldots, N$, ψ, $\alpha \in \mathbb{R}$, Φ_1, Γ_1, Λ_1, Π_1, $\in \mathbb{R}^{n \times n}$:*

$$\mathcal{J} + \mathcal{H}^T + \mathcal{H} \prec 0, \qquad P \succ 0, \tag{2.43}$$

$$\mathcal{J} := \begin{bmatrix} B_{w0}QB_{w0}^T & P & B_{w0}QB_{w\rho}^T & B_{w0}QD_{zw\rho}^T & B_{w0}SD_{zw0}^T \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & B_{w\rho}QB_{w\rho}^T & B_{w\rho}QD_{zw\rho}^T & B_{w\rho}SD_{zw0}^T \\ \star & \star & \star & D_{zw\rho}QD_{zw\rho}^T & D_{zw\rho}SD_{zw0}^T \\ \star & \star & \star & \star & \bar{R} \end{bmatrix}, \tag{2.44}$$

$$\mathcal{H} := \begin{bmatrix} A_0\Phi + B_u\hat{K} + A_\rho\hat{\rho} & \alpha(A_0\Phi + B_u\hat{K} + A_\rho\hat{\rho}) & A_0\Phi + B_u\hat{K} + A_\rho\hat{\rho} \\ -\Phi & -\alpha\Phi & -\Phi \\ \Phi - \psi I_n & \alpha(\Phi - \psi I_n) & \Phi - \psi I_n \\ \Phi - \psi I_n & \alpha(\Phi - \psi I_n) & \Phi - \psi I_n \\ C_{z0}\Phi + D_{zu}\hat{K} + C_{z\rho}\hat{\rho} & \alpha(C_{z0}\Phi + D_{zu}\hat{K} + C_{z\rho}\hat{\rho}) & C_{z0}\Phi + D_{zu}\hat{K} + C_{z\rho}\hat{\rho} \end{bmatrix}$$

$$\begin{bmatrix} A_0\Phi + B_u\hat{K} + A_\rho\hat{\rho} & 0 \\ -\Phi & 0 \\ \Phi - \psi I_n & 0 \\ \Phi - \psi I_n & 0 \\ C_{z0}\Phi + D_{zu}\hat{K} + C_{z\rho}\hat{\rho} & 0 \end{bmatrix}, \tag{2.45}$$

$$\underline{\rho}_i \psi \geq \hat{\rho}_i \geq \overline{\rho}_i \psi. \tag{2.46}$$

The linear dependence of the LMIs on the Lyapunov matrix facilitates the usage of parameter-dependent Lyapunov functions as certificates of stability of uncertain and time-varying systems. Results on simultaneous plant-controller design for uncertain and linear-parameter varying systems have already been derived and will be presented in a future manuscript. Results for plants with polynomial dependence on the parameters to be designed were also developed and will be published shortly.

Report H: F.D. Adegas, D-Stability Control of Wind Turbines

In this report, multiobjective output-feedback control via LMI optimization is explored for wind turbine control design. In particular, we use D-stability constraints to increase damping and decay rate of resonant structural modes. In this way, the number of weighting functions and consequently the order of the final controller is reduced, and the control design process is physically more intuitive and meaningful.

Our interest lies in the design of a full-order dynamic output feedback controller for a wind turbine, such that satisfactory generator speed (and power) regulation as well as mitigation of mechanical vibrations are attained. We denote input-output operators (transfer matrices) of the augmented system for controller synthesis as $T_{w \to z}$. We adopt time and frequency domain specifications to the closed-loop system including induced norms such as

- $\mathcal{L}_2$-norm ($\|T_{w\to z}\|_{i,2}^2 < \gamma$): "energy gain" of a system to a worst-case disturbance, e.g. a constant relating the energy of the worst-case wind (wind gust) to the energy on the gen. speed;

- Generalized $\mathcal{H}_2$-norm ($\|T_{w\to z}\|_g < \varrho$): energy-to-peak gain of a system, e.g. a constant relating wind gust with maximum pitch angle rates;

and regional pole placement constraints, such as the ones illustrated in Fig. 2.10.

- Conic region: minimum damping for all modes;

- Decay region: minimum decay rate for all modes;

- Circle region: a circle with radius r and center $-c$.
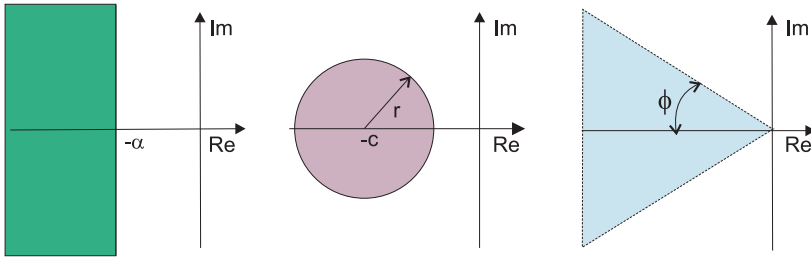
Figure 2.10: Regions of the complex plane for pole placement constraints.

The numerical example brings a generic utility-scale wind turbine model with 9-states composed of the following modes/states:two-mass flexible drive-train, tower fore-aft displacement, second-order pitch actuator, first-order generator lag and integral of generator speed. The chosen design criteria is to minimize the performance level γ subject to

- $\|T_{v\to z}\|_{i,2}^2 < \gamma, \qquad z(t) = [\Omega_i(t) \quad Q_g(t) \quad \beta(t)]^T$;

- $\left\|T_{v\to\dot{\beta}}\right\|_g < 8$ deg/s;

- $\arg(\lambda(\mathcal{A})) < \phi, \operatorname{Re}(\lambda(\mathcal{A})) < \alpha$.

where Ω_i is the integral of speed error, Q_g is the generator torque and β is the pitch angle. In words, this performance criteria states that the generator speed "integral square error (ISE)" should be small with little control effort while respecting the pitch rate limit, for the worst-case wind disturbance. The augmented system for synthesis purposes is depicted in Fig. 2.11. The weighting functions were chosen frequency independent on the form $G_1(s) := k_1$ and $G_2(s) := \operatorname{diag}(k_2, k_3)$. All modes should be contained in a conic region with internal angle 2ϕ and decay rate of at least α. The adopted multiobjective LMI formulation follows [SGC97]. The LMIs were solved with the semidefinite programming code SeDuMi [Stu99] and parser YALMIP [LÖ4].

The drive-train torsional mode is the least damped mode in open-loop. Active vibration control can be obtained by a D-stability constraint on the minimum damping of the
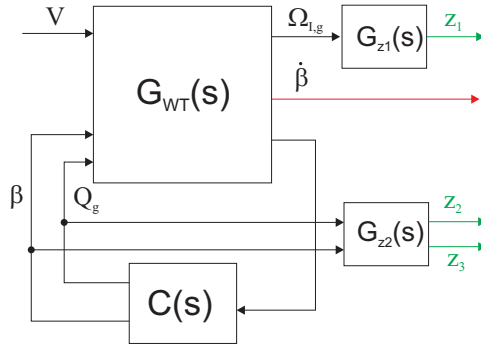
Figure 2.11: Augmented system for synthesis.
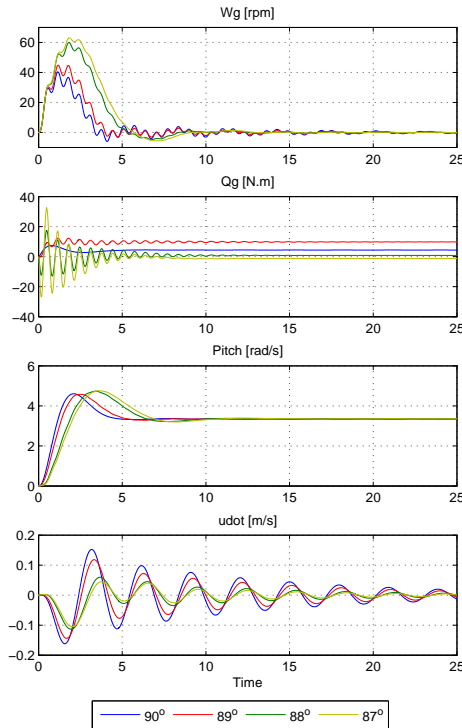
Figure 2.12: Closed-loop response for a unit step on wind speed disturbance for different angles of the conic region.

closed-loop system. via the conic region constraint. Figure 2.12 depicts the closed-loop system response for a unit step on wind speed disturbance, under various internal angles of the conic region. The location of the closed-loop poles are illustrated in Fig. 2.13. The effect of the damping constraint on the generator speed Ω_g and generator torque Q_g sig-
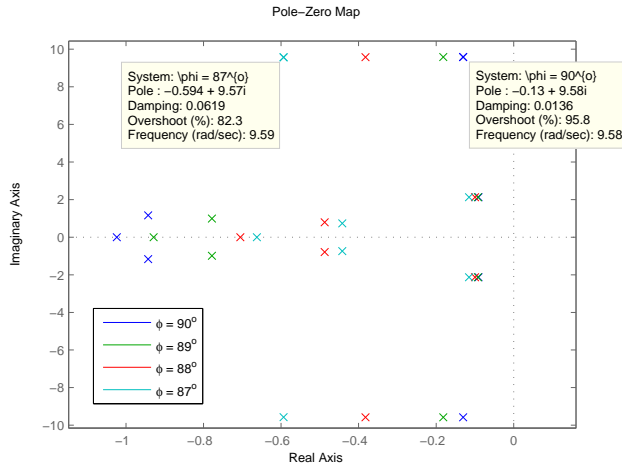
Figure 2.13: Location of the closed-loop poles for different angles of the conic region.

nals is noticeable. The report also shows how to achieve drive-train and tower damping simultaneously, as well as the influence of energy-to-peak constraint in the pitch rate in the system response.

Other Contributed Papers

During the Ph.D. project, four other papers were been written by the author of this thesis and published. Two of the papers addressed theoretical developments in the area of structured control of uncertain and LPV systems,

- Adegas FD, Stoustrup J. Robust structured control design via LMI optimization. Proceedings of the 18th IFAC World Congress, IFAC: Milan, Italy, 2011.

- Adegas FD, Stoustrup J. Structured control of affine linear parameter varying systems. Proceedings of the 2011 American Control Conference, San Francisco, CA, USA, 2011.

and their content are partially related to the iterative LMI algorithm of Paper A. It was decided to let them out to favor conciseness of the thesis. The other two papers were in collaboration with M.Sc. students. Although within the field of wind turbine control and the scope of Project CASED, they have been left out in favor of a more uniform thesis content around structured control of wind turbines and LMI methods. The following paper

- J. Friis, E. Nielsen, J. Bonding, F.D. Adegas, J. Stoustrup, P.F. Odgaard, "Repetitive model predictive approach to individual pitch control of wind turbines," IEEE Conf. on Decision and Control and European Control Conference (CDC-ECC), pp.3664-3670, 12-15 Dec., 2011.

proposes a novel model predictive (MPC) approach for individual pitch control of wind turbines. A repetitive wind disturbance model is incorporated into the MPC prediction. As a consequence, individual pitch feed-forward control action is generated by the controller, taking future wind disturbance into account. Information about the estimated wind spatial distribution one blade experience can be used in the prediction model to better control the next passing blade. A simulation comparison between the proposed controller and an industry-standard PID controller shows better mitigation of drive-train, blade and tower loads.

New analysis tools and an adaptive control law to increase the energy captured by a wind turbine is proposed in

- L. Diaz-Guerra, F.D. Adegas, J. Stoustrup, M. Monros, "Adaptive control algorithm for improving power capture of wind turbines in turbulent winds," American Control Conference (ACC), pp.5807-5812, 27-29 June 2012.

Due to its simplicity, it can be easily added to existing industry-standard controllers. The effectiveness of the proposed algorithm is accessed by simulations on a high-fidelity aeroelastic code.

3 | Conclusion and Future Work

In this chapter, the main conclusions from the work presented in the previous chapters are drawn. Suggestions to possible future research directions follows.

3.1 Conclusions

Aeroservoelastic LPV Modeling of Wind Turbines

The LPV framework was shown suitable for integrating aeroelastic models of wind turbines and controller models. By the use of simple first-principles modeling, aerodynamic non-linearities and common faults on sensors and actuators of a control system can be simultaneously represented in the plant and captured by the gain-scheduled controller.

Wind turbine LPV models are often simple, first-principles based, neglecting dynamics related to aerodynamic phenomena and some structural modes. This in turn restricted LPV control of wind turbines to the academic environment only. On the other hand, the high-fidelity aeroelastic models used by the wind industry for controller design are often linear-time invariant and of high-order. Based on modal and balanced truncation, one of the proposed methods proved very successful for order reduction of high-fidelity wind turbine models. With the assumption that scheduling parameters varies slowly, the method is able to encapsulate the varying dynamics over the operating envelope as a reduced-order LPV system. Modeling methods such as this can motivate the industrial use of LPV control. Parameter-dependent models can also serve to the purpose of simultaneous plant-controller design. Instead of performing linearizations of the non-linear aeroservoelastic model during the course of the optimization, an entire family of designs can be computed off-line and encapsulated as a reduced-order parameter-dependent system. This system can later be utilized for aeroservoelastic optimization.

Synthesis of Wind Turbine LPV Controllers

This thesis has demonstrated the factibility of including structure on gain-scheduled controllers for wind turbines. The linear parameter-varying modeling and control framework using parameter-dependent Lyapunov functions and linear matrix inequalities was shown appropriate to address some of the control problems commonly faced by the wind industry with theoretical soundness on stability and performance of the system in closed-loop.

The numerics play an important role when implementing LPV methods in practice. The methods proposed in this thesis sacrifices elegance in the offline synthesis in favor of

a simple online implementation. Matrix inversions and factorizations are common operations carried online when the LPV controllers are computed from LMI conditions with auxiliary variables. Iterative LMI algorithms may be computationally demanding, but due to the absence of auxiliary variables, the resulting LPV controllers does not demand online operations as such. Sums of matrices and multiplication between matrix and a scalar are the only required operations. Ease of implementation can also encourage the immersion of LPV control of wind turbines in an industrial setting.

Intuitive Performance Specifications

In the design of LPV controllers, the $\mathcal{H}_\infty$ mixed-sensitivities loop-shaping approach tries to enforce a second-order behavior from wind disturbances to rotor speed. The desired second-order behavior is specified in terms of natural frequency and damping ratio, usual requirements in the design of PI controllers for wind turbines.

D-Stability was shown to be an interesting approach to active vibration control. Pole placement constraints have direct physical meaning, such as minimum damping and decay rate of the closed-loop poles, which facilitates controller design with high-level specifications. Specification of these constraints are also independent of the aeroelastic model being used for control design. An integrated aeroservoelastic optimization can take advantage of this fact to guarantee a certain level of vibration attenuation despite re-design of the plant and controller.

Theoretical Developments

The thesis also brought new conditions and perspectives to some of the important control problems which remains open to date.

The analysis and synthesis conditions of vector second-order systems obtained during our studies have the potential to increase the practice of working with systems directly in vector second-order form. LMI conditions for verifying asymptotic stability and quadratic performance were shown to be necessary and sufficient, irrespective of the type of dynamic loading. Due to their linear dependence in the coefficient matrices and the inclusion of multipliers on the formulation, the conditions are appropriate to robust analysis of systems with structured uncertainty. Synthesis of vector second-order controllers with guaranteed stability and quadratic performance are also formulated as LMI problems. Unfortunately, the synthesis conditions are only sufficient to the existence of full state-feedbacks. This is the major drawback when compared to synthesis in state-space first-order form, to which necessary and sufficient LMI conditions are available in the literature. However, when structural constraints are imposed on the controller gains, the design in vector second-order form may render less conservative results.

The novel $\mathcal{H}_\infty/\mathcal{H}_2$ model reduction conditions and procedures based on LMIs extended with multipliers shown to be efficient when compared to other works on the literature. The proposed conditions are particularly useful for reducing the order of parameter-dependent systems, limited to original models with small to moderate number of states. These could be readily applied to model reduction of gain-scheduled controllers where the rate of variation of the scheduling variable is assumed null.

The proposed sufficient static output stabilization conditions based on LMIs extended with multipliers also shown to be efficient. A success rate of finding stabilizing controllers

of over 90% were achieved in numerical experiments with randomly generated state-space models. In a wind turbine context, since PI controllers can be formulated as static output feedback (by augmenting the integrator on the plant), the proposed simultaneous stabilization conditions can be useful to design passive fault-tolerant PI controllers.

A different perspective to the simultaneous plant-controller design problem was given by proposing convex, sufficient LMI conditions to this problem. The extra degrees of freedom introduced by the matrix multipliers facilitated a novel change-of-variables involving the plant parameters to be designed. The linear dependence of the conditions on the Lyapunov matrix ease the usage of parameter dependent Lyapunov functions as certificates of stability to the simultaneous plant-controller design of parameter-dependent and linear parameter-varying systems. This is a direct contribution to concurrent aeroservoelastic design of wind turbines. Suboptimal designs with quadratic performance can be computed efficiently using modern convex optimization tools.

3.2   Future Work

Suggestions to future work are provided in this section, based on limitations of some of the presented results, issues that were not addressed due to lack of time, and on more general impressions gathered during the research.

Aeroservoelastic LPV Modeling of Wind Turbines

Other methods for encapsulating the varying dynamics of the wind turbine as an LPV system is a subject of further research. Our approach relied on reducing the order of the collection of original LTI models and subsequent interpolation of the state-space matrices. An alternative approach could fit a high-order LPV model from the original LTI systems, and later apply model reduction techniques for parameter-varying systems. This approach would benefit of interpolating the system matrices in the original and consistent coordinates. The drawback is the large least-squares problem required for interpolation of the large matrices which can be numerically challenging.

When only LTI models are available, it is usual to resort to the strong assumption that scheduling variables varies slowly in time. An issue of future research is to find an LPV representation from a high-fidelity non-linear aeroelastic model which does not assume slow parameter variations.

Expand the LPV modeling and control framework of wind turbines to deal with de-rated power operation is of interest. Changes in power reference result in changes of the operating point of the machine, which in turn leads to varying dynamics. A scheduling parameter can be add to the LPV formulation allowing the controller to adapt to the envelope of varying dynamics.

Synthesis of Wind Turbine LPV Controllers

LMI conditions can also be explored to the design of wind turbine LPV controllers. Full-state and full-order dynamic output feedback can be turned into LMI conditions, without loss of generality, by performing linearizing change-of-variables between parameter-dependent controller data and parameter-dependent multiplier. Other structures such as static output feedback can also be handled with some conservatism. Our experience says

that iterative LMI algorithms are usually less conservative than sufficient LMI conditions with multipliers. A comparison between sufficient LMI and iterative LMI methods for LPV control of wind turbines would give an empirical indication of the relative degree of conservatism of each approach.

Intuitive Performance Specifications

Two research lines on this subject were initiated during the Ph.D. studies. The first one is to tailor LMI-based optimal control to consider fatigue explicitly in the cost function. To accomplish this, the formulation should include in the objective function not only variance, but also higher-order statistical moments of the signals of interest. A state-space formulation of this problem would be valuable.

During the Ph.D. studies, research on modal control of wind turbines achieved a certain degree of maturity, and some results are expected to be published soon. Modal control ease the task of damping assignment to lightly damped structures. To understand the limitations and develop methods of modal control of wind turbines is an interesting subject of future research.

Theoretical Developments

To seek necessary and sufficient LMI criteria to the synthesis of vector second-order controllers is of utmost importance. In this thesis, the Lyapunov function was defined similarly to the first-order state-space case. That is, quadratic function of velocities and positions. Other formulations of the Lyapunov function might result in necessary and sufficient or less conservative conditions. $\mathcal{H}_2$ performance specifications can be derived from the presented derivations with little effort. During the Ph.D. studies, an attempt to find a suitable linearizing change-of-variables in the dynamic output vector second-order feedback case was not successful. This remains as future work.

A more detailed comparison of the proposed sufficient LMI condition for static output stabilization with others found on the literature is required. Numerical experiments can give an empirical measure of the conservativeness relative to each of the existing conditions.

An extension of the proposed $\mathcal{H}_\infty/\mathcal{H}_2$ model reduction conditions to cope with LPV systems is straightforward, and useful to reduce the order of wind turbine LPV controllers.

Results on simultaneous plant-controller design for linear-parameter varying systems have already been derived and will be presented in a future manuscript. Results for plants with polynomial dependence on the parameters to be designed were also developed and will be published shortly.

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Contributions

Paper A

Structured Linear Parameter Varying Control of Wind Turbines

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Abstract

High performance and reliability are required for wind turbines to be competitive within the energy market. To capture their nonlinear behavior, wind turbines are often modeled using parameter-varying models. In this chapter, a framework for modelling and controller design of wind turbines is presented. We specifically consider variable-speed, variable-pitch wind turbines with faults on actuators and sensors. Linear parameter-varying (LPV) controllers can be designed by a proposed method that allows the inclusion of faults in the LPV controller design. Moreover, the controller structure can be arbitrarily chosen: static output feedback, dynamic (reduced order) output feedback, decentralized, among others. The controllers are scheduled on an estimated wind speed to manage the parameter-varying nature of the model and on information from a fault diagnosis system. The optimization problems involved in the controller synthesis are solved by an iterative LMI-based algorithm. The resulting controllers can also be easily implemented in practice due to low data storage and simple math operations.

1   Introduction

Motivated by environmental concerns and the depletion of fossil fuels, as well its mature technological status, wind energy consolidate as a viable sustainable energy source for the decades to come. Over the past 20 years, the global installed capacity of wind power increased at an average annual growth of more than 25% from around 2.5 GW in 1992 to just under 200 GW at the end of 2010 [1]. Due to ongoing improvements in the wind turbine efficiency and reliability, and higher fuel prices, the cost of electricity produced (COE), which, roughly speaking, takes into account the annual energy production, lifetime of wind turbines, and Operation & Maintenance costs, is becoming economically competitive with conventional power production.

Automatic control is one of the engineering areas that significantly contributed to reduce the cost of wind generated electricity. In order to reduce COE, a modern wind turbine is not only controlled to maximize energy production, but also to minimize mechanical loads. The controlled system also has to comply with external requirements, such as acoustic noise emissions and power quality grid codes. Since many wind turbines are installed at remote locations, the introduction of fault-tolerant control is considered a suitable way of improving reliability/availability and lowering costs of repairs. Finally, the lack of accurate models must be alleviated by robust control strategies capable of securing stability and satisfactory performance despite model uncertainties [2].

From a control point of view, a wind turbine is a challenging system since the wind, which is the energy source driving the machine, is a poorly known stochastic disturbance. Add to that wind turbines inherently exhibit time-varying nonlinear dynamics along their nominal operating trajectory, motivating the use of advanced control techniques such as gain-scheduling, to counteract performance degradation or even instability problems by continuously adapting to the dynamics of the plant. Wind turbine controllers typically consist of multiple gain-scheduled controllers, which are designed to operate in the proximity of a certain operating point. The gain-scheduling approach for industry-standard classical controllers can be either based on switching or interpolation of controller gains [3], [4]. Controller structure may also change by either switching [3] or bumpless transfer [5, 6] according to the wind speed experienced by the wind turbine. The underlying

assumption for such control schemes is that parameters only change slowly compared to the system dynamics, which is generally not satisfied in turbulent winds. Additionally, classical gain-scheduling controllers only ensure performance guarantees and stability at the operating points where the linear controllers are designed [7].

A systematic way of designing controllers for systems with linearized dynamics that vary significantly with the operating point is within the framework of linear parameter-varying (LPV) control. An LPV controller can be synthesized after solving an optimization problem subject to linear matrix inequality (LMI) constraints. In control systems for wind turbines, robustness and fault-tolerance capabilities are important properties, which should be considered in the design process, calling for a generic and powerful tool to manage parameter-variations and model uncertainties. In this chapter, design procedures for nominal controllers for parameter-varying models as well as active/passive fault-tolerance, are provided. The framework can be trivially extended to design controllers robust to uncertainties in the model [8], e.g., aerodynamic uncertainties [9]. Indeed, handling known parameter-dependencies, unknown parameter variations, and faults, constitute the main challenges for the application of wind turbine control.

An overview of the proposed control structure is illustrated by the block diagram depicted in Fig. 4.1, where $u(k)$ is the control signal and $w(k)$ is the disturbance. The LPV controllers depend on the measurements $y(k)$ and an estimate of the current operating point, $\hat{\theta}_{op}(k)$, which is used as scheduling parameter. Additionally, a fault diagnosis system provides the scheduling parameter $\hat{\theta}_f(k)$ for the active fault-tolerant controller (AFTC). The extra degree of freedom added by allowing the AFTC to adapt in case of a fault may introduce less conservatism than for the passive fault-tolerant controller. The AFTC is a conventional LPV controller scheduled on $\theta_{op}(t)$ and $\theta_f(t)$; the reason for denoting it an active fault-tolerant controller arises from the origin of the scheduling parameters.
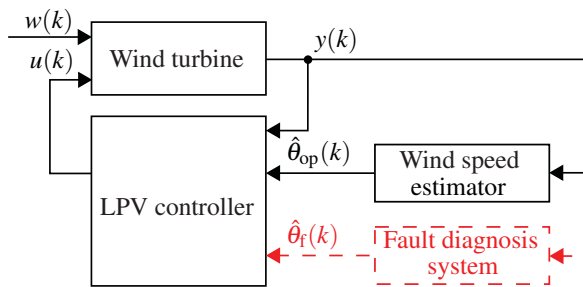
Figure 4.1: Block diagram of the controller structures. The black boxes are common to the LPV controllers, while the red dashed box illustrates the fault diagnosis system required by the AFTC.

The list of faults occurring in wind turbines is extensive, reflecting the complexity of the machines. On system level, faults occur in sensors, actuators, and system components, ranging from slow gradual faults to abrupt component failures. The occurrence of faults may change the system behavior dramatically. This motivates us to develop methods for fault diagnosis and fault-tolerant control, offering several benefits:

- Prevent catastrophic failures and faults from deteriorating other parts of the wind turbine, by early fault detection and accommodation.

- Reduce maintenance costs by providing remote diagnostic details and avoiding replacement of functional parts, by applying condition-based maintenance instead of time-based maintenance.

- Increase energy production when a fault has occurred by means of fault-tolerant control.

This chapter gives an overview of the most common faults that can be modelled as varying parameters. For a clear exposure, the fault-tolerant controller is designed to cope with the simple case of a single fault: altered dynamics of the hydraulic pitch system due to low hydraulic pressure. The fault is a gradual fault affecting the control actions of the turbine. The method used also applies to fast parameter variations, i.e. abrupt faults in the extreme case [10].

Realizing advanced gain-scheduled controllers can in practice be difficult and may lead to numerical challenges [2, 11]. Several plant and controller matrices must be stored on the controller memory. Moreover, matrix factorizations and inversions are among the operations that must be done online by the controller at each sampling time [12, 13].

The synthesis procedures presented in this chapter are serious candidates for solving a majority of practical wind turbine control problems, provided a sufficiently good model of the wind turbine is available. We believe that the resulting controller can also be easily implemented in practice due to the following reasons:

1. **Structured controller:** the controller structure can be chosen arbitrarily. Decentralized of any order, dynamic (full or reduced-order) output feedback, static output, and full state feedback are among the possible structures. This is in line with the current control philosophy within wind industry.

2. **Low data storage:** the required data to be stored in the control computer memory is only the controller matrices, and scalar functions of the scheduling variables representing plant nonlinearities (basis functions).

3. **Simple math operations:** the mathematical operations needed to compute the controller gains at each sampling time are look-up tables with interpolation, products between a scalar and a matrix, and sums of matrices.

The versatile controller structure and facilitated implementation comes with a price. Due to the (possible) nonconvex characteristics of the synthesis problem, the controller design is solved by an iterative LMI optimization algorithm that may be demanding from a computational point of view. However, the authors consider that it is worth to transfer the computational burden from the controller implementation to the controller design.

The chapter is organized as follows. Section 2 describes the LPV wind turbine plant modeling including typical faults and uncertainties. The LPV controller design procedure, based on an iterative LMI optimization algorithm, is presented in Section 3. Section 4 contains a design example on how state of the art industrial controllers can be designed within the LPV framework. A fault-tolerant gain-scheduled PI pitch controller

for the full load region is designed and compared to a gain-scheduled controller without fault accommodation capabilities. Simulation results presented in the same section compares the performance of both LPV controllers to show that pitch actuator faults due to low pressure can be accommodated by the fault-tolerant LPV controller, avoiding the shutdown of the wind turbine. Section 5 concludes the paper.

2 Wind Turbine LPV model

In this section, an LPV model is derived from a nonlinear time-varying wind turbine model. The nonlinear model consists of several sub-systems, namely aerodynamics, the tower, the drive train, the generator, the pitch system, and the converter actuator. The interconnection of the wind turbine sub-models is illustrated in Fig. 4.3. For simulation purposes, the wind disturbance input, $V(t)$, is provided by a wind model which includes both tower shadow and wind shear [14] together with a turbulence model [15]. The a detailed description of the model is provided in [10]. The sub-models are individually described in the sequel.

Wind Model

The driving force of a wind turbine is generated by the wind. Therefore, a model of this external input to the wind turbine, $V_w(t)$, has to be provided.

Generally, the wind speed is influenced by several components, which depend on the environment where the wind turbine is located; however, we restrict our model to include only three effects: wind shear, tower shadow, and turbulence. A more thorough wind model can be found in [10]. We will not provide a detailed description of the wind model, but only explain its three components briefly.

Wind shear is caused by the ground and other obstacles in the path of the wind, which cause frictional forces to act on the wind. The frictional forces imply that the mean wind speed becomes dependent on the height above ground level. Therefore, the mean wind speed depends on the location of the three blades. When a blade is located in front of the tower, the lift on that blade decreases because the tower reduces the effective wind speed. This phenomenon is called tower shadow and implies that the force acting on each blade decreases every time a blade is located in front of the tower. Finally, the variations in the wind speed, which are not included in the mean wind speed, are called turbulence and are caused by multiple factors. The utilized turbulence model is based on the Kaimal spectrum that describes the turbulence of a point wind. Since the wind model describes the wind speed averaged over the entire rotor plane, a low-pass filter is applied to smooth the wind speed signal. Fig. 4.2 shows an output of the wind model $V_w(t)$. Note that a detailed description of the wind model can be found in [10].

Nonlinear Model

The rotor of a wind turbine converts kinetic energy of the wind into rotational energy of the rotor blades and shaft. Aerodynamic forces over the rotor blades are often determined with the assumptions of Blade Element Momentum (BEM) theory [16]. Fig. 4.4 illustrates the forces and velocity vectors on a blade element.
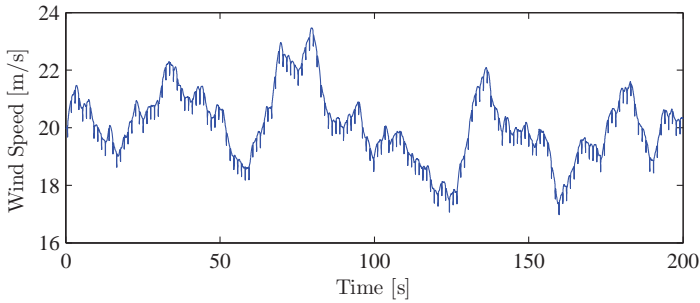
Figure 4.2: Output of the wind model at a constant rotor speed. The periodic decrease of the wind speed is caused by tower shadow.
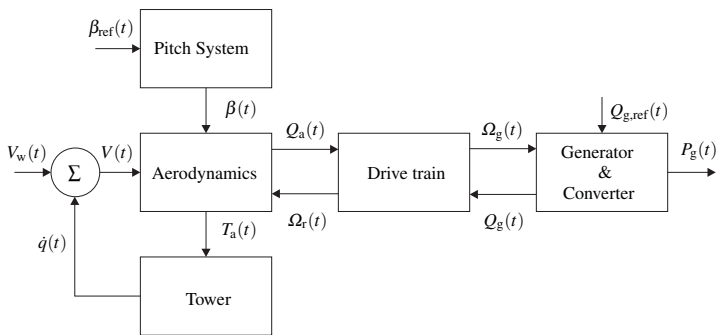
Figure 4.3: Sub-model-level block diagram of a variable-speed variable-pitch WT.
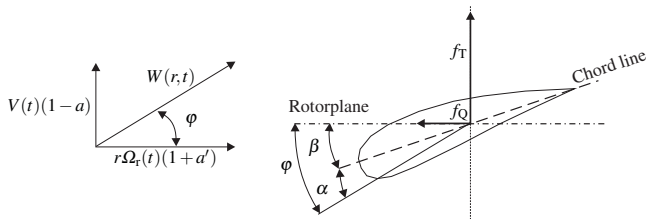
Figure 4.4: Forces on a blade element.

Assuming a symmetric aerodynamic rotor driven by a uniform inflow, and neglecting unsteady aerodynamic effects, the local tangential f_Q and axial f_T forces along the local blade radius r are given by,

$$f_Q = \frac{1}{2}\rho c(r)W^2(r,t)\Big(C_L(r,\alpha(r,t))\sin\varphi(r,t) - C_D(r,\alpha(r,t))\cos\varphi(r,t)\Big) \quad \text{[N]}$$

$$f_T = \frac{1}{2}\rho c(r)W^2(r,t)\Big(C_L(r,\alpha(r,t))\sin\varphi(r,t) + C_D(r,\alpha(r,t))\cos\varphi(r,t)\Big) \quad \text{[N]}$$

with the squared local inflow velocity $W^2(r, t)$, local angle of attack $\alpha(r, t)$ and local inflow angle $\varphi(r, t)$ described as,

$$W^2(r, t) = (V(t)(1 - a(r)))^2 + (r\Omega_r(t)(1 + a'(r)))^2, \quad [\mathrm{m}^2/\mathrm{s}^2]$$
$$\alpha(r, t) = \varphi(r, t) - \phi(r) - \beta(t), \quad [^\circ]$$
$$\varphi(r, t) = \tan^{-1}\left(V(t)(1 - a(r))(r\Omega_r(t)(1 + a'(r)))^{-1}\right) \quad [^\circ].$$

In the above expressions, ρ is the air density, $c(r)$ is the local chord length, $C_L(r, \alpha)$ and $C_D(r, \alpha)$ are the local steady-state lift and drag coefficients, $V(t)$ is a mean wind speed over the rotor disk, $\Omega_r(t)$ is the rotor speed, $a(r)$ and $a'(r)$ are the axial and tangential flow induction factors, respectively, $\phi(r)$ is the local chord twist angle along the blade, and $\beta(t)$ is the blade pitch angle.

In the aerodynamic model, we assume that a yawing system exists, which always keeps the rotor plane perpendicular to the direction of the wind; hence, $V(t)$ is always perpendicular to the rotor plane. However, as the rotor rotates the resulting wind speed at a blade, called the inflow velocity $W(r, t)$, has an angle φ with respect to the rotor plane. The drag force given by $1/2\rho c W^2 C_D$ is defined to point in the opposite direction as $W(r, t)$ and the lift force given by $1/2\rho c W^2 C_L$ is perpendicular to drag force. Via projections of these forces we obtain f_Q and f_T.

The aerodynamic torque Q_a and thrust force T_a produced by the rotor can be expressed as the summation of the forces over the B number of rotor blades,

$$Q_a(V, \Omega_r, \beta, a, a') = B \int_0^R f_Q(r, V, \Omega_r, \beta, a(r), a'(r))\, r\, dr \quad [\mathrm{Nm}], \qquad (4.1\mathrm{a})$$

$$T_a(V, \Omega_r, \beta, a, a') = B \int_0^R f_T(r, V, \Omega_r, \beta, a(r), a'(r))\, dr \quad [\mathrm{N}]. \qquad (4.1\mathrm{b})$$

After integration, the aerodynamic torque and thrust are represented as,

$$Q_a(t) = \frac{1}{2\Omega_r(t)} \rho A V^3(t) C_P(\lambda(t), \beta(t)) \quad [\mathrm{Nm}] \qquad (4.2\mathrm{a})$$

$$T_a(t) = \frac{1}{2} \rho A V^2(t) C_T(\lambda(t), \beta(t)) \quad [\mathrm{N}] \qquad (4.2\mathrm{b})$$

with the tip-speed ratio denoting the ratio between the blade tip and the mean wind speed,

$$\lambda(t) = \frac{\Omega_r(t)\, R}{V(t)} \quad [\cdot]$$

where R is the rotor radius and $A = \pi R^2$ is the rotor swept area. The power coefficient $C_P(\lambda, \beta)$ and thrust coefficient $C_T(\lambda, \beta)$ are smooth surfaces usually given in tabular form. Fig. 4.5 depicts C_P and C_T surfaces of a typical 2MW wind turbine.

Aerodynamic torque Q_a drives a drive train model consisting of a low-speed shaft and a high-speed shaft having inertias J_r and J_g, and friction coefficients B_r and B_g. The shafts are interconnected by a transmission having gear ratio N_g, combined with torsion stiffness K_{dt}, and torsion damping B_{dt}. This results in a torsion angle, $\theta_\Delta(t)$, and a torque

(a) Power coefficient (b) Thrust coefficient
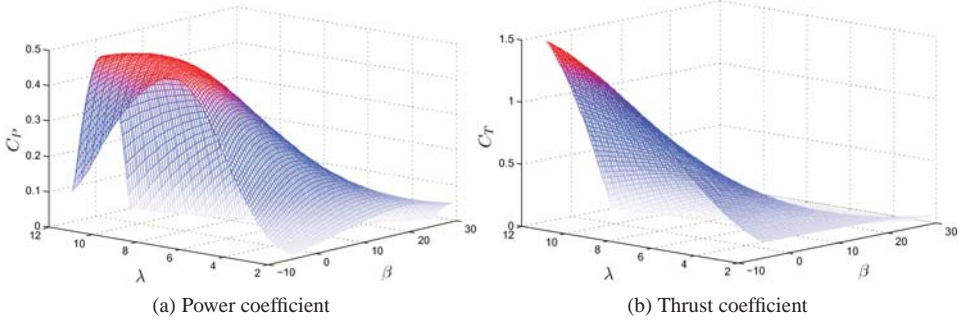
Figure 4.5: Power and thrust coefficients of a typical utility-scale wind turbine.

applied to the generator, $Q_g(t)$, at a speed $\Omega_g(t)$. The model of the drive train is shown in Fig. 4.6 and given by,

$$J_r\dot{\Omega}_r(t) = Q_a(t) + \frac{B_{dt}}{N_g}\Omega_g(t) - K_{dt}\theta_\Delta(t) - (B_{dt} + B_r)\Omega_r(t) \qquad [\text{Nm}] \qquad (4.3a)$$

$$J_g\dot{\Omega}_g(t) = \frac{K_{dt}}{N_g}\theta_\Delta(t) + \frac{B_{dt}}{N_g}\Omega_r(t) - \left(\frac{B_{dt}}{N_g^2} + B_g\right)\Omega_g(t) - Q_g(t) \qquad [\text{Nm}] \qquad (4.3b)$$

$$\dot{\theta}_\Delta(t) = \Omega_r(t) - \frac{1}{N_g}\Omega_g(t) \qquad [\text{rad/s}]. \qquad (4.3c)$$
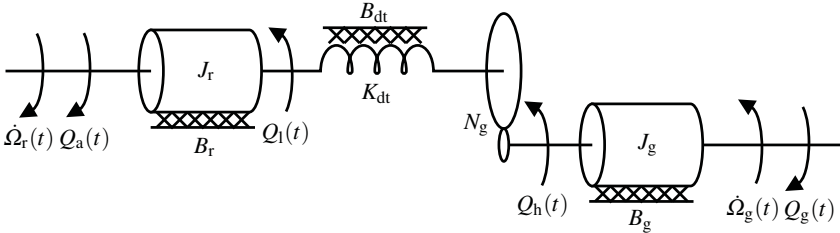
Figure 4.6: Diagram of the drive train of the wind turbine.

The thrust T_a acting on the rotor introduces fore-aft tower bending described by the axial nacelle linear translation $q(t)$. Sideward movements are ignored by neglecting yawing and drive train reaction torque on the tower. The tower translates in the same direction as the wind, therefore aerodynamic torque and thrust are in fact driven by the relative wind speed $V(t) = V_w(t) - \dot{q}(t)$. The tower dynamics is modeled as a mass-spring-damper system,

$$M_t\ddot{q}_t(t) = T_a(t) - B_t\dot{q}_t(t) - K_tq_t(t) \qquad (4.4)$$

where M_t is the modal mass of the first fore-aft tower bending mode, B_t is structural damping coefficient, and K_t is the modal stiffness for axial nacelle motion due to fore-aft tower bending.

Hydraulic pitch systems are satisfactorily modeled as a second order system with a time delay, t_d, and reference angle $\beta_{ref}(t)$,

$$\ddot{\beta}(t) = -2\zeta\omega_n\dot{\beta}(t) - \omega_n^2\beta(t) + \omega_n^2\beta_{ref}(t - t_d) \tag{4.5}$$

where the natural frequency, ω_n, and damping ratio, ζ, specify the dynamics of the model. To represent the limitations of the pitch actuators, for simulation purposes the model includes constraints on the slew rate and the range of the pitch angle.

Electric power is generated by the generator, while a power converter interfaces the wind turbine generator output with the utility grid and controls the currents in the generator. The generator torque in (4.6) is controlled by the reference $Q_{g,ref}(t)$. The converter dynamics are approximated by a first order system with time constant τ_g and time delay $t_{g,d}$.

$$\dot{Q}_g(t) = -\frac{1}{\tau_g}Q_g(t) + \frac{1}{\tau_g}Q_{g,ref}(t - t_{g,d}). \tag{4.6}$$

Just as for the model of the pitch system, the slew rate and the operating range of the generator torque are both bounded to match the limitations of the real system. The power produced by the generator can be approximated from the mechanical power calculated in (4.7), where η_g denotes the efficiency of the generator, which is assumed constant.

$$P_g(t) = \eta_g\Omega_g(t)Q_g(t). \tag{4.7}$$

Linear Varying Parameters

From the model description, is clear that a wind turbine is a nonlinear, time-varying system. What is not apparent is how to find an LPV description that captures this dynamic behavior. Wind turbines can be represented as Quasi-LPV models [2, 17] and Linear Fractional Transformation models [2], but the most common approach relies on the classical linearization around equilibrium or operating points resulting in a linearized LPV model [2, 11, 13]. The latter approach is adopted in this chapter.

Aerodynamics

The underlying assumption of a wind turbine LPV model based on linearization is that wind speed, rotor speed and pitch angle can be described by slow and fast components,

$$V(t) = \overline{V}(t) + \hat{V}(t), \quad \Omega_r = \overline{\Omega}(t) + \hat{\Omega}_r(t), \quad \beta(t) = \overline{\beta}(t) + \hat{\beta}(t).$$

The collection of operating points $(\overline{V}, \overline{\Omega}, \overline{\beta})$ is what defines the control strategy of a wind turbine, selected to match steady-state requirements such as maximize energy capture, minimize static loads, and limit noise emissions.

A typical control strategy of a generic 2MW wind turbine is depicted in Fig. 4.7. A more detailed treatment of different operating strategies for wind turbines [3, 13] is outside the scope of this chapter. Three subareas on a typical control strategy can be distinguished:
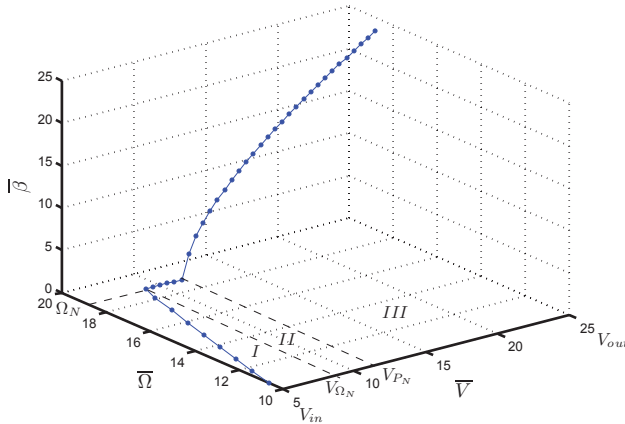
Figure 4.7: Operating locus of a typical utility-scale wind turbine.

1. On Region I (V_{in} to $V_{\Omega_{\text{N}}}$) the energy capture is maximized by keeping the aerodynamic efficiency as high as possible. This can be accomplished by tracking a rotational speed set-point using generator torque as the control input variable. Pitch actuation is not utilized for tracking purposes; the pitch angle remains at the value of maximum aerodynamic efficiency. With only one input and one output to be controlled, a multivariable controller is not necessary on this region. Notice that $\overline{\Omega}$ is proportional to $\overline{V}$ as a consequence of optimal aerodynamic efficiency.

2. On Region II ($V_{\Omega_{\text{N}}}$ to $V_{P_{\text{N}}}$) the wind turbine maintains constant rotational speed at a nominal value Ω_{N}, by acting on the generator torque. The rotational speed is limited due to acoustic noise emission limits. Pitch actuation remains unused. A multivariable controller is still not needed.

3. On Region III ($V_{P_{\text{N}}}$ to V_{out}) rated power P_{N} is reached and the main goal is to minimize power fluctuations. Small fluctuations on the generator torque around rated value add damping to the drive train torsional mode and fine control the electrical power. Therefore, pitch angle should be gradually increased as wind speed rises to limit generated power by lowering the rotor aerodynamic efficiency. In some wind turbines, active tower damping is also implemented on this region.

A linearization-based LPV model is obtained by classical linearization around the operating points given by the control strategy. The aerodynamic model is exclusively the source of time-varying nonlinearities. A first order Taylor series expansion of (4.2) leads to the following linearized representations of torque and thrust,

$$Q_{\text{a}} \approx \overline{Q}_{\text{a}(\overline{V},\overline{\Omega},\overline{\beta})} + \left.\frac{\partial Q_{\text{a}}}{\partial V}\right|_{(\overline{V},\overline{\Omega},\overline{\beta})} \hat{V} + \left.\frac{\partial Q_{\text{a}}}{\partial \Omega_{\text{r}}}\right|_{(\overline{V},\overline{\Omega},\overline{\beta})} \hat{\Omega}_{\text{r}} + \left.\frac{\partial Q_{\text{a}}}{\partial \beta}\right|_{(\overline{V},\overline{\Omega},\overline{\beta})} \hat{\beta} \qquad (4.8\text{a})$$

$$T_{\text{a}} \approx \overline{T}_{\text{a}(\overline{V},\overline{\Omega},\overline{\beta})} + \left.\frac{\partial T_{\text{a}}}{\partial V}\right|_{(\overline{V},\overline{\Omega},\overline{\beta})} \hat{V} + \left.\frac{\partial T_{\text{a}}}{\partial \Omega_{\text{r}}}\right|_{(\overline{V},\overline{\Omega},\overline{\beta})} \hat{\Omega}_{\text{r}} + \left.\frac{\partial T_{\text{a}}}{\partial \beta}\right|_{(\overline{V},\overline{\Omega},\overline{\beta})} \hat{\beta} \qquad (4.8\text{b})$$

where $\overline{Q}_{a(\overline{V},\overline{\Omega},\overline{\beta})}$ and $\overline{T}_{a(\overline{V},\overline{\Omega},\overline{\beta})}$ are equilibrium components of the aerodynamic torque and thrust, respectively. The partial derivatives of Q_a and T_a are given by,

$$\frac{\partial Q_a}{\partial V} = \frac{\rho A V^2}{2\Omega_r}\left(3C_P + V\frac{\partial C_P}{\partial \lambda}\frac{\partial \lambda}{\partial V}\right), \qquad \frac{\partial T_a}{\partial V} = \frac{\rho A V}{2}\left(2C_T + V\frac{\partial C_T}{\partial \lambda}\frac{\partial \lambda}{\partial V}\right),$$

$$\frac{\partial Q_a}{\partial \Omega_r} = \frac{\rho A V^3}{2\Omega_r}\left(\frac{\partial C_P}{\partial \lambda}\frac{\partial \lambda}{\partial \Omega_r} - \frac{C_P}{\Omega_r}\right), \qquad \frac{\partial T_a}{\partial \Omega_r} = \frac{\rho A V^2}{2}\frac{\partial C_T}{\partial \lambda}\frac{\partial \lambda}{\partial \Omega_r},$$

$$\frac{\partial Q_a}{\partial \beta} = \frac{\rho A V^3}{2\Omega_r}\frac{\partial C_P}{\partial \beta}, \qquad \frac{\partial T_a}{\partial \beta} = \frac{\rho A V^2}{2}\frac{\partial C_T}{\partial \beta},$$

$$\tag{4.9}$$

and must be evaluated at the time-varying equilibrium point $(\overline{V},\overline{\Omega},\overline{\beta})$. The partial derivatives of a typical 2MW wind turbine for the whole operational envelope are depicted in Fig. 4.8. The aerodynamic partial derivatives given by (4.9), hereafter also referred to as aerodynamic gains, are varying parameters in an LPV wind turbine model.

With the assumption that the wind turbine is operating on the nominal trajectory specified in Fig. 4.7, the equilibrium values for pitch angle and rotor/generator speed can be described uniquely by the wind speed, e.g. $\overline{\Omega}\left(\overline{V}\right)$, $\overline{\beta}\left(\overline{V}\right)$. This means that the wind turbine can be described by an LPV model scheduled on wind speed only,

$$\theta_{op}(t) := \overline{V}(t). \tag{4.10}$$

Depending on the region of interest in the control strategy and the model complexity, the varying parameters can be approximated as an explicit function of the scheduling variable. An affine representation is always preferable to diminish the computational cost of solving a LMI constrained optimization. If tower dynamics are omitted and the aim is to design a controller for Region III, the aerodynamic torque gains can be fairly well approximated by a linear function of the wind speed. In this case, the parameter variations in the nominal LPV plant model are approximated using an affine description in the wind speed [9]. If tower dynamics are taken into account, the aerodynamic gains can be fairly approximated by polynomial functions in Region III. For the most general case, which is the design of a single LPV controller covering the full control strategy locus, representing the aerodynamic gains by polynomials is difficult and one has to resort to grid-based methods at high computational cost [11, 13].

On most wind turbines, the wind speed is measured by an anemometer on the nacelle, which only measures the wind speed at a single point in space and is affected by the presence of the rotor. Therefore, this measurement is not a good estimate of (4.10). To obtain the wind speed for scheduling purposes, an effective wind speed estimator must be designed [18]. The effective wind speed is defined as the spatial average of the wind field over the rotor plane with the wind stream being unaffected by the wind turbine. By inspecting the output of wind models and real field measurements, we determine the rate bounds on the effective wind speed $\hat{\theta}_{op}(t)$ to be -2 m/s^2 and 2 m/s^2.

Faults

Faults in a wind turbine have different degrees of severity and accommodation requirements. A safe and fast shut down of the wind turbine is necessary to some of them, while to others the system can be reconfigured to continue power generation. Linear parameter
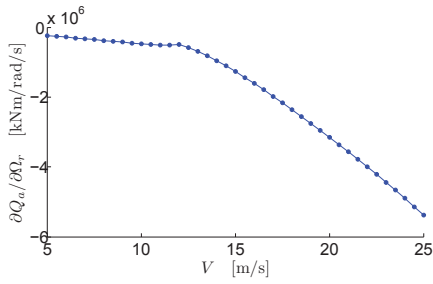
(a) From rotor speed to torque.
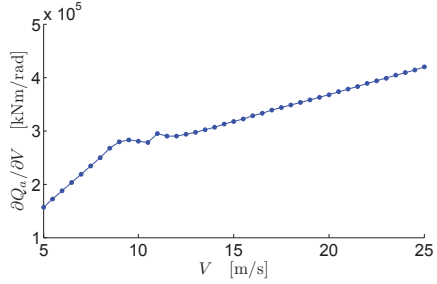
(b) From wind speed to torque.
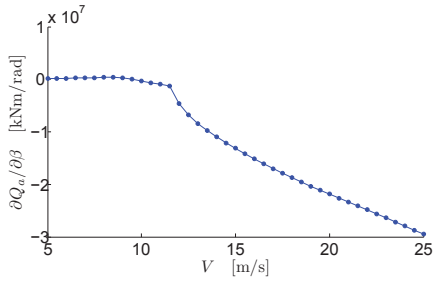
(c) From pitch angle to torque.
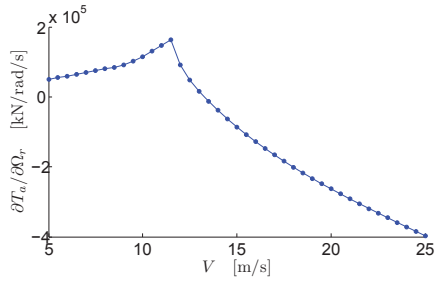
(d) From rotor speed to thrust.
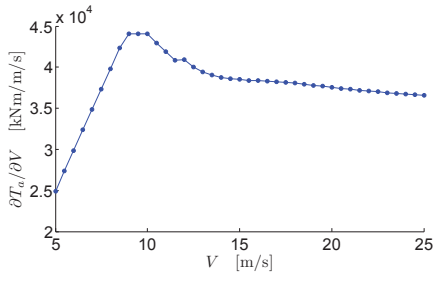
(e) From wind speed to thrust.
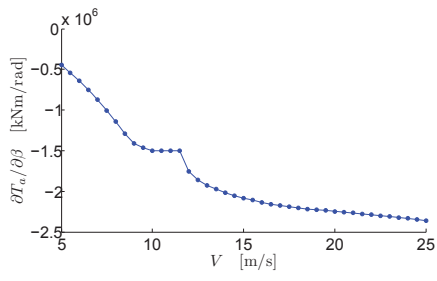
(f) From pitch angle to thrust.
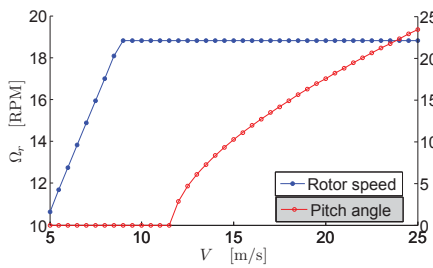
(g) Rotor speed and pitch angle.
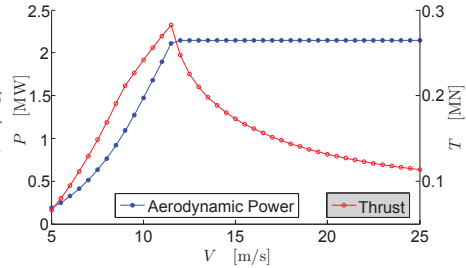
(h) Power and thrust.

Figure 4.8: Aerodynamic parameters of a typical 2MW wind turbine.

varying control can be applied in the case of failures that gradually change system's dynamics. The most common faults along with their magnitude and the rate at which they can be introduced are summarized in Tab. 4.1.

Fault	Specification
Pitch Sensor	
Bias	$\dot{\beta}_{\text{bias}}(t) \in$ [-1°/month, 1°/month]
	$\beta_{\text{bias}}(t) \in$ [-7°, 7°]
Pitch Actuator	
High Air Content	$\dot{\theta}_{\text{ha}}(t) \in$ [-1/month, 1/month]
	$\theta_{\text{ha}}(t) \in$ [0, 1]
Pump Wear	$\dot{\theta}_{\text{pw}}(t) \in$ [0, 1/(20 years)]
	$\theta_{\text{pw}}(t) \in$ [0, 1]
Hydraulic Leakage	$\dot{\theta}_{\text{hl}}(t) \in$ [0, 1/(100 s)]
	$\theta_{\text{hl}}(t) \in$ [0, 1]
Pressure Drop	$\dot{\theta}_{\text{pd}}(t) \in$ [−0.033/s, 0.033/s]
	$\theta_{\text{pd}}(t) \in$ [0, 1]
Generator Speed Sensor	
Proportional Error	$\dot{\theta}_{\text{pe}}(t) \in$ [-1/month, 1/month]
	$\theta_{\text{pe}}(t) \in$ [-0.1, 0.1]

Table 4.1: Specification of ranges and rate limits of gradual faults.

Pitch position and generator speed measurements are the sensors most affected by failures. Originated by either electrical or mechanical anomalies, they can result in either a bias or a gain factor on the measurements. A biased pitch sensor measurement affects both the pitch system model and the pitch angle measurement. When the bias is introduced, the pitch actuator model and measurement equation are modified as,

$$\ddot{\beta}(t) = \text{-}2\zeta\omega_{\text{n}}\dot{\beta}(t) - \omega_{\text{n}}^2\left(\beta(t) + \beta_{\text{bias}}(t)\right) + \omega_{\text{n}}^2\beta_{\text{ref}}(t - t_{\text{d}}) \qquad (4.11\text{a})$$

$$\beta_{\text{mes}}(k) = \beta(k) + \beta_{\text{bias}}(k) + v_\beta(k) \qquad (4.11\text{b})$$

where $v_\beta(k)$ is a measurement noise. A bias can either be a result of inaccurate calibration of the pitch system or be an gradual fault.

A proportional error on the generator speed sensor changes the sensor gain. The measurement equation,

$$\Omega_{\text{g,mes}}(k) = \left(1 + \theta_{\text{pe}}(k)\right)\Omega_{\text{g}}(k) + v_{\Omega g}(k), \qquad (4.12)$$

is a linear function of the gain deviation θ_{pe}, where $v_{\Omega g}(k)$ is a measurement noise.

The power converter and pitch systems are the actuators most likely to fail. Power converter faults can result in an offset of the generated torque due to an offset in the internal converter control loops. An offset in the internal converter control loops modifies the generator and converter model as follows,

$$\dot{T}_{\text{g}}(t) = \text{-}\frac{1}{\tau_{\text{g}}}(Q_{\text{g}}(t) + Q_{\text{g,bias}}(t)) + \frac{1}{\tau_{\text{g}}}T_{\text{g,ref}}(t - t_{\text{g,d}}) \qquad (4.13)$$

where $Q_{\text{g,bias}}(t)$ is an offset of the generated torque.

A fault changes the dynamics of the pitch system by varying the damping ratio and natural frequency from their nominal values ζ_0 and $\omega_{\text{n},0}$ to their faulty values ζ_{f} and $\omega_{\text{n,f}}$. The dynamics of the pitch system can then be represented as,

$$\ddot{\beta}(t) = \text{-}2\zeta(\theta_{\text{f}})\omega_{\text{n}}(\theta_{\text{f}})\dot{\beta}(t) - \omega_{\text{n}}^2(\theta_{\text{f}})\beta(t) + \omega_{\text{n}}^2(\theta_{\text{f}})\beta_{\text{ref}}(t - t_{\text{d}}) \quad [°/\text{s}^2] \qquad (4.14)$$

with the parameters changing according to a convex combination of the vertices of the parameter sets [19],

$$\omega_{\text{n}}^2(\theta_{\text{f}}) = (1 - \theta_{\text{f}})\omega_{\text{n},0}^2 + \theta_{\text{f}}\omega_{\text{n,lp}}^2 \qquad (4.15\text{a})$$

$$\text{-}2\zeta(\theta_{\text{f}})\omega_{\text{n}}(\theta_{\text{f}}) = \text{-}2(1 - \theta_{\text{f}})\zeta_0\omega_{\text{n},0} - 2\theta_{\text{f}}\zeta_{\text{lp}}\omega_{\text{n,lp}} \qquad (4.15\text{b})$$

where $\theta_{\text{f}} \in [0,\ 1]$ is an indicator function for the fault with $\theta_{\text{f}} = 0$ and $\theta_{\text{f}} = 1$ corresponding to nominal and faulty conditions, respectively. Pitch system failures are usually occasioned by the following reasons.

- Pump Wear is introduced very slowly and results in low pump pressure. When $\theta_{\text{f}}(t) = 0$ the pump delivers the nominal pressure, but as $\theta_{\text{f}}(t)$ goes to one the pressure drops. Notice that $\dot{\theta}_{\text{f}}(t) \geq 0$ for all t, since the wear is irreversible without replacing the pump. The fault described by $\theta_{\text{f}} = 1$, corresponding to a pressure level of 75%, can emerge after approximately 20 years of operation.

- Hydraulic leakage is introduced considerably faster than pump wear. Again $\dot{\theta}_{\text{f}}(t) \geq 0$ for all t, since a leakage cannot be reversed without repair of the system. Notice that the pressure for $\theta_{\text{f}} = 1$ corresponds to 50% of the nominal pressure.

- High air content in the oil is a fault that, in contrast to pump wear and hydraulic leakage, may disappear; hence, $\dot{\theta}_{\text{f}}(t)$ can be both positive and negative. The extreme values caused by $\theta_{\text{f}} = 0$ and $\theta_{\text{f}} = 1$ correspond to air contents of 7% and 15% in the hydraulic oil.

Values for the natural frequency and damping ratio of the pitch system under faults are desribed in Tab. 4.2. Step responses for the normal and fault conditions in the case of high air content in the oil are illustrated in Fig. 4.9.

Fault	Parameters
No fault	$\omega_{\text{n}} = 11.11$ rad/s, $\zeta = 0.6$
High Air Content in the Oil	$\omega_{\text{n,ha}} = 5.73$ rad/s, $\zeta_{\text{ha}} = 0.45$
Pump Wear	$\omega_{\text{n,pw}} = 7.27$ rad/s, $\zeta_{\text{pw}} = 0.75$
Hydraulic Leakage	$\omega_{\text{n,hl}} = 3.42$ rad/s, $\zeta_{\text{hl}} = 0.9$
Pressure Drop	$\omega_{\text{n,hl}} = 3.42$ rad/s, $\zeta_{\text{hl}} = 0.9$

Table 4.2: Parameters for the pitch system under different conditions. The normal air content in the hydraulic oil is 7%, whereas high air content in the oil corresponds to 15%. Pump wear represents the situation of 75% pressure in the pitch system while the parameters stated for hydraulic leakage corresponds to a pressure of only 50%.
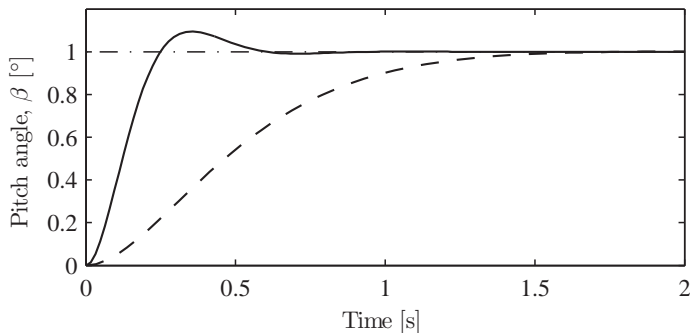
Figure 4.9: Step responses of hydraulic pitch model under normal (solid) and fault (dashed) conditions.

If a number n_{θ_f} of faults are considered on the modeling, θ_f denotes a vector of scheduling parameters,

$$\theta_f = [\theta_{f,1}, \ldots, \theta_{f,m}], \quad m = 1, \ldots, n_{\theta_f}.$$

System Description

The synthesis of LPV controllers are posed similarly to the $\mathcal{H}_\infty$ control of linear systems. The first step is to identify the input variable w known as disturbance or exogenous perturbation, and the fictitious output variable z called performance output or error. Next, weighting functions for these inputs and outputs are chosen, usually rational functions of the Laplace operator s stressing the frequencies of interest. The standard state-space interconnections of the LPV model of the plant and the weighting functions is called augmented plant, given by the general continuous-time LPV system description shown in (4.16),

$$\begin{aligned}
\dot{x}(t) &= A(\theta(t))x(t) + B_w(\theta(t))w(t) + B_u(\theta(t))u(t) \\
z(t) &= C_z(\theta(t))x(t) + D_{zw}(\theta(t))w(t) + D_{zu}(\theta(t))u(t) \\
y(t) &= C_y(\theta(t))x(t) + D_{yw}(\theta(t))w(t)
\end{aligned} \quad (4.16)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{n_w}$ is the vector of exogenous perturbation, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output, and $y(t) \in \mathbb{R}^{n_y}$ is the measured output. $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are continuous functions of the time-varying parameter vector $\theta = [\theta_{op} \quad \theta_f]$.

Possible types of dependency of the aerodynamic gains on the scheduling parameters have already been discussed in the Aerodynamics subsection. The general case where no restrictions are imposed on the parameter dependence is treated here [12, 13]. It is necessary to choose scalar functions of the varying parameters such that the LPV model of the augmented plant (4.16) is affine in these functions. That is,

$$
\begin{bmatrix} A(\theta) & B_w(\theta) & B_u(\theta) \\ C_z(\theta) & D_{zw}(\theta) & D_{zu}(\theta) \\ C_y(\theta) & D_{yw}(\theta) & D_{yu}(\theta) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_0 + \sum_i \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_i \rho_i(\theta),
$$

$$
+ \sum_m \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_m \theta_{f,m}, \quad i = 1, \dots, n_\rho, \quad m = 1, \dots, n_{\theta_f}.
$$

(4.17)

where $\rho_i(\theta)$ are scalar functions known as basis functions. The aerodynamic partial derivatives are natural candidates for basis functions related to plant nonlinearities [13],

$$
\rho_1(\theta) := \frac{1}{J_r} \frac{\partial Q_a}{\partial \Omega}\bigg|_{\overline{V}}, \qquad \rho_2(\theta) := \frac{1}{J_r} \frac{\partial Q_a}{\partial V}\bigg|_{\overline{V}}, \qquad \rho_3(\theta) := \frac{1}{J_r} \frac{\partial Q_a}{\partial \beta}\bigg|_{\overline{V}},
$$

$$
\rho_4(\theta) := \frac{1}{M_t} \frac{\partial T_a}{\partial \Omega}\bigg|_{\overline{V}}, \qquad \rho_5(\theta) := \frac{1}{M_t} \frac{\partial T_a}{\partial V}\bigg|_{\overline{V}}, \qquad \rho_6(\theta) := \frac{1}{M_t} \frac{\partial T_a}{\partial \beta}\bigg|_{\overline{V}},
$$

where the division by J_r and M_t is adopted to improve numerical conditioning.

3 LPV Controller Design Method

In this section an LMI-based optimization procedure for designing structured discrete-time LPV controllers is presented. Decentralized controllers of any order, fixed-order and static output feedback are among the possible control structures. Stability is assessed via a parameter-dependent Lyapunov function with varying parameters having their rates of variation contained in a compact closed convex set. A parameter-varying nonconvex condition for an upper bound on the induced $\mathcal{L}_2$-norm performance is solved via an iterative LMI-based algorithm [20, 8].

An open-loop, discrete-time augmented LPV system with state-space realization of the form,

$$
\begin{aligned}
x(k+1) &= A(\theta)x(k) + B_w(\theta)w(k) + B_u(\theta)u(k) \\
z(k) &= C_z(\theta)x(k) + D_{zw}(\theta)w(k) + D_{zu}(\theta)u(k) \\
y(k) &= C_y(\theta)x(k) + D_{yw}(\theta)w(k),
\end{aligned}
$$

(4.18)

is considered for the purpose of synthesis, where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ is the vector of disturbance, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $z(k) \in \mathbb{R}^{n_z}$ is the controlled output, and $y(k) \in \mathbb{R}^{n_y}$ is the measured output. $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ are continuous functions of some time-varying parameter vector $\theta = [\theta_1, \dots, \theta_{n_\theta}]$. The same matrix notation to both the continuous-time augmented plant (4.16) and the discrete-time counterpart (4.18) have been adopted. Throughout the text, the context makes it clear when a continuous or discrete system is being referred to.

Assume θ ranges over a hyperrectangle denoted Θ,

$$
\Theta = \left\{ \theta : \underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i, \ i = 1, \dots, n_\theta \right\}.
$$

The rate of variation $\Delta\theta = \theta(k+1) - \theta(k)$ belongs to a hypercube denoted $\mathcal{V}$,

$$
\mathcal{V} = \left\{ \Delta\theta : \ |\Delta\theta_i| \leq v_i, \ i = 1, \dots, n_\theta \right\}.
$$

The LPV controller has the form,

$$\begin{aligned}
x_c(k+1) &= A_c(\theta)x_c(k) + B_c(\theta)y(k) \\
u(k) &= C_c(\theta)x_c(k) + D_c(\theta)y(k),
\end{aligned} \tag{4.19}$$

where $x_c(k) \in \mathbb{R}^{n_c}$ and the controller matrices are continuous functions of θ. Note that depending on the controller structure, some of the matrices may be zero. The controller matrices can be represented in a compact way,

$$K(\theta) := \begin{bmatrix} D_c(\theta) & C_c(\theta) \\ B_c(\theta) & A_c(\theta) \end{bmatrix}. \tag{4.20}$$

The interconnection of system (4.18) and controller (4.19) leads to the following closed-loop LPV system denoted S_{cl},

$$S_{cl} : \begin{cases} x(k+1) = \mathcal{A}(\theta, K(\theta))x_{cl}(k) + \mathcal{B}(\theta, K(\theta))w(k) \\ z(k) = \mathcal{C}(\theta, K(\theta))x_{cl}(k) + \mathcal{D}(\theta, K(\theta))w(k), \end{cases} \tag{4.21}$$

where the closed-loop matrices are [21],

$$\begin{aligned}
\mathcal{A}(\theta, K(\theta)) &= \mathbf{A}(\theta) + \mathbf{B}(\theta)K(\theta)\mathbf{M}(\theta), \quad \mathcal{B}(\theta, K(\theta)) = \mathbf{D}(\theta) + \mathbf{B}(\theta)K(\theta)\mathbf{E}(\theta), \\
\mathcal{C}(\theta, K(\theta)) &= \mathbf{C}(\theta) + \mathbf{H}(\theta)K(\theta)\mathbf{M}(\theta), \quad \mathcal{D}(\theta, K(\theta)) = \mathbf{F}(\theta) + \mathbf{H}(\theta)K(\theta)\mathbf{E}(\theta),
\end{aligned}$$

with,

$$\begin{aligned}
\mathbf{A}(\theta) &= \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix}, & \mathbf{M}(\theta) &= \begin{bmatrix} C_y(\theta) & 0 \\ 0 & I \end{bmatrix}, & \\
\mathbf{C}(\theta) &= \begin{bmatrix} C_z(\theta) & 0 \end{bmatrix}, & \mathbf{F}(\theta) &= D_{zw}(\theta), & \mathbf{B}(\theta) &= \begin{bmatrix} B_u(\theta) & 0 \\ 0 & I \end{bmatrix}, \\
\mathbf{E}(\theta) &= \begin{bmatrix} D_{yw}(\theta) \\ 0 \end{bmatrix}, & \mathbf{D}(\theta) &= \begin{bmatrix} B_w(\theta) \\ 0 \end{bmatrix}, & \mathbf{H}(\theta) &= \begin{bmatrix} D_{zu}(\theta) & 0 \end{bmatrix},
\end{aligned}$$

This general system structure can be particularized to some usual control topologies. If $K(\theta)$ is an *unconstrained* matrix and $n_c = 0$, the problem becomes a static output feedback (SOF). The static state feedback (SSF) is a particular case of SOF, when the system output is a full rank linear transformation of the state vector $\forall \theta$. If $n = n_c$, the full-order dynamic output feedback arises. In a structured control context, more elaborate control systems can be designed by constraining $K(\theta)$. A fixed-order dynamic output feedback has $n_c < n$. For decentralized controllers of arbitrary order, the structure of $K(\theta)$ is constrained to be,

$$K(\theta) := \begin{bmatrix} \text{diag}(D_c(\theta)) & \text{diag}(C_c(\theta)) \\ \text{diag}(B_c(\theta)) & \text{diag}(A_c(\theta)) \end{bmatrix}$$

where $\text{diag}(\cdot)$ stands that $(\cdot)$ has a block-diagonal structure.

The design of a closed-loop system usually consider performance specifications that can be characterized in different ways. Define $T_{zw}(\theta)$ as the input-output operator that represents the forced response of (4.21) to an input signal $w(k) \in \mathcal{L}_2$ for zero initial conditions. The induced $\mathcal{L}_2$-norm of a given input-output operator,

$$\|T_{zw}\|_2 := \sup_{(\theta, \Delta\theta) \in \Theta \times \mathcal{V}} \sup_{\|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2}$$

is commonly utilized as a measure of performance of LPV systems and allows formulating the control specification as in $\mathcal{H}_\infty$ control theory. It is of interest to note that an upper bound $\gamma > 0$ on the induced $\mathcal{L}_2$-norm $\|T_{zw}\|_2$ can be interpreted in terms of the upper bound on the system's energy gain,

$$\lim_{h\to\infty} \sum_{k=0}^{h-1} z(k)^T z(k) < \gamma^2 \lim_{h\to\infty} \sum_{k=0}^{h-1} w(k)^T w(k).$$

The LPV system (4.21) is said to have performance level γ when it is exponentially stable and $\|T_{zw}\|_2 < \gamma$ holds. An extension of the Bounded Real Lemma (BRL) for parameter-varying systems provides suficient conditions to analyze the performance level, by solving a constrained LMI optimization problem [22, 23]. For a given scalar γ and a a given LPV controller $K(\theta)$, if there exists a θ-dependent matrix function $\mathcal{P}(\theta) = \mathcal{P}(\theta)^T$ satisfying

$$\begin{bmatrix} \mathcal{P}(\theta + \Delta\theta) & \mathcal{A}(\theta, K(\theta))\mathcal{P}(\theta) & \mathcal{B}(\theta, K(\theta)) & 0 \\ \star & \mathcal{P}(\theta) & 0 & \mathcal{P}(\theta)\mathcal{C}(\theta, K(\theta))^T \\ \star & \star & \gamma I & \mathcal{D}(\theta, K(\theta))^T \\ \star & \star & \star & \gamma I \end{bmatrix} \succ 0 \qquad (4.22)$$

$\forall(\theta, \Delta\theta) \in \Theta \times \mathcal{V}$, then the system S_{cl} is exponentially stable and $\|T_{zw}(\theta)\|_2 < \gamma$. The symbol $\star$ means inferred by symmetry.

The parameter-varying BRL just shown can be also applied to the case where $w(k)$ is not an energy signal ($\|w(k)\|_2$ not finite) but has a nonzero root mean-square (RMS) value,

$$w_{RMS} := \left[\lim_{h\to\infty} \frac{1}{h} \sum_{k=0}^{h-1} w(k)^T w(k) \right]^{1/2} \neq 0.$$

In this context, $\mathcal{L}_2$-norm of a system is given in terms of the RMS values of the signals of interest, instead of $\|\cdot\|_2$. Such a situation is more appropriate to interpret control performance of a wind turbine, since the turbulent wind is a stochastic disturbance that persists for long periods of time, thus $\|w(k)\|_2$ is not a good measure of the signal.

When an LPV controller with performance level γ is not given but should be found (synthesized), the inequality (4.22) is no longer an LMI in the unknown variables due to the product between the variables $K(\theta)$ and $\mathcal{P}(\theta)$. Thus, convex optimization algorithms cannot be applied to the condition *as it is*. Reformulations into sufficient (possibly conservative) LMI constraints are readily available for particular controller structures and type of parameter dependencies [24, 23].

We propose to design the controllers via an iterative algorithm, instead of attempting to reduce the problem to LMIs. The iterative algorithm relies on the following equivalent non-LMI parametrization that is suitable for iterative design [20]. If there exist $K(\theta)$, $\mathcal{P}(\theta) = \mathcal{P}(\theta)^T$, and $\mathcal{G}(\theta)$ satisfying,

$$\begin{bmatrix} \mathcal{P}(\theta + \Delta\theta) & \mathcal{A}(\theta, K(\theta)) & \mathcal{B}(\theta, K(\theta)) & 0 \\ \star & -\mathcal{G}(\theta)^T \mathcal{P}(\theta) \mathcal{G}(\theta) + \mathcal{G}(\theta)^T + \mathcal{G}(\theta) & 0 & \mathcal{C}(\theta, K(\theta))^T \\ \star & \star & \gamma I & \mathcal{D}(\theta, K(\theta))^T \\ \star & \star & \star & \gamma I \end{bmatrix} \succ 0,$$
$$(4.23)$$

$\forall\,(\theta, \Delta\theta) \in \Theta \times \mathcal{V}$, then the system S_{cl} is exponentially stable and $\|T_{zw}(\theta)\|_2 < \gamma$.

The affine dependence of the reformulated condition on $K(\theta)$ allows the controller matrices to be variables, irrespective of the chosen controller structure. The inequality remains nonconvex due to the product between $\mathcal{P}(\theta)$ and the introduced slack variable $\mathcal{G}(\theta)$. Furthermore, it involves the satisfaction of infinitely many inequalities, since (4.23) should hold for all $(\theta, \Delta\theta) \in \Theta \times \mathcal{V}$.

In order to make the problem computationally tractable, the iterative algorithm solves LMI optimization problems with the slack matrix $\mathcal{G}(\theta)$ constant during an iteration. An *iteration* should be understood to be a LMI constrained optimization. The use of $\mathcal{G}(\theta)$ as a parameter-dependent slack variable is facilitated by updating its value at each iteration according to some predefined rule. In particular, the update rule is

$$\mathcal{G}(\theta)^{\{j+1\}} = \left(\mathcal{P}(\theta)^{\{j\}}\right)^{-1} \tag{4.24}$$

where $^{\{\cdot\}}$ is the iteration index and j is the current iteration number.

The iterative algorithm for the design of a structured LPV controller with minimum performance level γ is formulated next.

Algorithm 0. *Set $j = 0$, a convergence tolerance ϵ, an initial $\mathcal{G}(\theta)^{\{0\}}$ and start to iterate:*

1. *For fixed $\mathcal{G}(\theta)^{\{j\}}$, find $\mathcal{P}(\theta)^{\{j\}}$, $\mathcal{P}(\theta + \Delta\theta)^{\{j\}}$, $K(\theta)^{\{j\}}$ and $\gamma^{\{j\}}$ satisfying the LMI constrained problem,*

$$\textit{Minimize } \gamma \textit{ subject to (4.23)}.$$

2. *If $\left|\gamma^{\{j\}} - \gamma^{\{j-1\}}\right| \le \epsilon$, stop. Otherwise, $\mathcal{G}(\theta)^{\{j+1\}} = \left(\mathcal{P}(\theta)^{\{j\}}\right)^{-1}$, set $j = j+1$ and go to step 1.*

Initial Slack Matrix $\mathcal{G}(\theta)^{\{0\}}$

The initial value of $\mathcal{G}(\theta)^{\{0\}}$ required to initialize Algorithm 0 can be obtained in different ways. If a given initial controller $K(\theta)$ satisfies the following optimization problem,

$$\textit{Minimize } \gamma \textit{ subject to (4.22)}, \forall(\theta, \Delta\theta) \in \Theta \times \mathcal{V},$$

then the resulting $\mathcal{P}(\theta)$ can be utilized to derive $\mathcal{G}(\theta)^{\{0\}} = \mathcal{P}(\theta)^{-1}$. The example section shows the usage of this approach.

Alternatively, an iterative feasibility algorithm can be created by relaxing the inequality (4.23). Instead of requiring the inequality to be positive definite ($\succ 0$), a variable term is included to the right hand side ($\succ \text{diag}(\tau I, \tau G^T G, \tau I, \tau I)$), where τ is a scalar variable. The algorithm maximizes τ until the value reaches a certain chosen $\upsilon > 0$.

Algorithm 1. *Set $j = 0$, a convergence tolerance ϵ, a $\upsilon > 0$, an initial $\mathcal{G}(\theta)^{\{0\}} = I$ and start to iterate:*

1. *For fixed $\mathcal{G}(\theta)^{\{j\}}$, find $\mathcal{P}(\theta)^{\{j\}}$, $\mathcal{P}(\theta + \Delta\theta)^{\{j\}}$, $K(\theta)^{\{j\}}$, $\gamma^{\{j\}}$ and scalar τ satisfying the LMI constrained problem,*

$$\textit{Maximize } \tau \textit{ subject to (4.23) with the right hand side changed from } \succ 0 \textit{ to}$$
$$\succ \textit{diag}(\tau I, \tau G^T G, \tau I, \tau I), \textit{ and } \tau < \upsilon.$$

2. If $\left|\tau^{\{j\}} - \tau^{\{j-1\}}\right| \leq \epsilon$, stop. Otherwise, $\mathcal{G}(\theta)^{\{j+1\}} = \left(\mathcal{P}(\theta)^{\{j\}}\right)^{-1}$, set $j = j + 1$ and go to step 1.

The resulting $\mathcal{G}(\theta)^{\{0\}}$ can subsequently be used to initialize Algorithm 0.

From Infinite to Finite Dimensional

The LMI problems of Algorithm 0 involve infinitely many LMIs, as θ and $\Delta\theta$ are defined in a continuous space. When LMIs depend affinely on θ and $\Delta\theta$, the synthesis problem at each iteration is reduced to an optimization problem with a finite number of LMIs checked at $(\theta, \Delta\theta) \in \text{Vert } \Theta \times \text{Vert } \mathcal{V}$. Note that Vert Θ is the set of all vertices of Θ. For LMIs polynomially θ-dependent, relaxations based on multiconvexity arguments also reduce the problem to check LMIs at the vertices of the parameter space [20, 8]. This procedure, based on sufficient conditions, may lead to extra conservatism. In the general case, where no restrictions on the parameter dependence are imposed, one has to resort to ad-hoc gridding methods [12]. The gridding procedure consists of defining a gridded parameter subset denoted $\Theta_g \subset \Theta$, designing a controller that satisfies the LMIs $\forall \theta \in \Theta_g$, and checking the LMI constraints in a denser grid. If the last step fails, the process is repeated with a finer grid.

Due to the assumption of general parameter dependence of the open-loop plant on the scheduling variables (4.17), the gridding approach is used in the controller design. The Lyapunov and the LPV controller matrices are affine in the basis functions,

$$\mathcal{P}(\theta) = P_0 + \sum_{i=1}^{n_\rho} \rho_i(\theta_k)P_i + \sum_{i=1}^{n_{\theta_f}} \theta_{f,i}P_{n_\rho+i}, \tag{4.25a}$$

$$K(\theta) = K_0 + \sum_{i=1}^{n_\rho} \rho_i(\theta)K_i + \sum_{i=1}^{n_{\theta_f}} \theta_{f,i}K_{n_\rho+i}. \tag{4.25b}$$

Due to the bounded parameter rate set $\mathcal{V}$ assumed known, the Lyapunov function at sample $k + 1$ can be described as,

$$\mathcal{P}(\theta + \Delta\theta) = P_0 + \sum_{i=1}^{n_\theta} \rho_i(\theta + \Delta\theta)P_i + \sum_{i=1}^{n_{\theta_f}} \left(\theta_{f,i} + \Delta\theta_{f,i}\right)P_{n_\rho+i}. \tag{4.26}$$

Note the general parameter dependence of (4.26) on $\Delta\theta$ occasioned by $\rho_i(\theta + \Delta\theta)$. Conveniently, the basis functions at sample $k + 1$ are represented as a linear function of $\rho(\theta)$ and $\Delta\theta$,

$$\rho_i(\theta + \Delta\theta) := \rho_i(\theta) + \frac{\partial \rho_i(\theta)}{\partial \theta}\Delta\theta, \tag{4.27}$$

thereby turning inequality (4.23) affine dependent on the rate of variation $\Delta\theta$. Thus, it is sufficient to verify (4.23) with (4.26)-(4.27) only at Vert $\mathcal{V}$.

The iterative algorithm for a chosen grid $\Theta_g \subset \Theta$ is presented in the sequel.

Algorithm 2. *Set $j = 0$, a convergence tolerance ϵ, initialize $\mathcal{G}(\theta)^{\{0\}} \forall \theta \in \Theta_g$, and start to iterate:*

1. *For fixed $\mathcal{G}(\theta)^{\{j\}}$, and $i = 0, 1, \ldots, n_\rho + n_{\theta_f}$, find $P_i^{\{j\}} > 0$, $K_i^{\{j\}}$, and $\gamma^{\{j\}}$ satisfying the LMI constrained problem,*

$$\text{Minimize } \gamma \text{ subject to (4.23), } \forall\, (\theta, \Delta\theta) \in \Theta_g \times \text{ Vert}\mathcal{V}.$$

2. *If $\left|\gamma^{\{j\}} - \gamma^{\{j-1\}}\right| \leq \epsilon$, stop. Otherwise, $\mathcal{G}(\theta)^{\{j+1\}} = \mathcal{P}(\theta)^{\{j\}-1}$, $\forall\, \theta \in \Theta_g$. Set $j = j + 1$ and go to step 1.*

The Lyapunov variable $\mathcal{P}(\theta)^{\{j\}} \succ 0$ may be close to singular at each iteration, making the inversion required to compute $\mathcal{G}(\theta)$ possibly ill-conditioned. To alleviate this issue, an additional LMI constraint

$$\mathcal{P}(\theta)^{\{j\}} \succ \mu I,$$

improves numerical condition of the inversion by imposing a lower bound on the eigenvalues of $\mathcal{P}(\theta)^{\{j\}}$, where $\mu > 0$ is a chosen scalar. There exists a tradeoff between the value of μ and the attained value of γ. Higher values of μ may lead to more conservative controllers, although from our experience, the small value of μ required to better condition the inversion does not influence significantly on the performance level γ.

The gridding procedure for controller synthesis can be summarized by the following steps.

1. Define a grid Θ_g for the compact set Θ.

2. Find initials $\mathcal{G}(\theta)^{\{0\}}$, $\forall \theta \in \Theta_g$.

3. Solve Algorithm 2.

4. Define a denser grid.

5. Verify the feasibility of the LMI (4.22) with the computed controller $K(\theta)$, in each point of the new grid. If it is infeasible, choose a denser grid and go to step 2.

Controller Implementation

The iterative LMI optimization algorithm provides the controller matrices $A_{c,i}$, $B_{c,i}$, $C_{c,i}$, $D_{c,i}$, for $i = 0, 1, \ldots, n_\rho + n_{\theta_f}$. These matrices, the basis functions, and the value of the scheduling variables are the only required information to determine the control signal u. At each sample time k, the scheduling variable θ is measured (or estimated) and a control signal is obtained as follows.

1. Compute the value of the basis functions $\rho_i(\theta)$, for $i = 0, 1, \ldots, n_\rho$. The basis functions may be stored in a lookup table that takes θ as an input and outputs an interpolated value of $\rho(\theta)$.

2. With the value of the basis functions in hand, determine the controller matrices $A_c(\theta)$, $B_c(\theta)$, $C_c(\theta)$, $D_c(\theta)$ according to,

$$A_c(\theta) = A_{c,0} + \sum_{i=1}^{n_\rho} \rho_i(\theta) A_{c,i} + \sum_{i=1}^{n_{\theta_f}} \theta_{f,i} A_{c,n_\rho+i},$$

$$B_c(\theta) = B_{c,0} + \sum_{i=1}^{n_\rho} \rho_i(\theta) B_{c,i} + \sum_{i=1}^{n_{\theta_f}} \theta_{f,i} B_{c,n_\rho+i},$$

$$C_c(\theta) = C_{c,0} + \sum_{i=1}^{n_\rho} \rho_i(\theta) C_{c,i} + \sum_{i=1}^{n_{\theta_f}} \theta_{f,i} C_{c,n_\rho+i},$$

$$D_c(\theta) = D_{c,0} + \sum_{i=1}^{n_\rho} \rho_i(\theta) D_{c,i} + \sum_{i=1}^{n_{\theta_f}} \theta_{f,i} D_{c,n_\rho+i}.$$

3. Once the controller matrices have been found, the control signal $u(k)$ can be obtained by the dynamic equation (4.19) of the LPV controller, which only involves multiplications and additions.

4 Example: LPV PI Controller Tolerant to Pitch Actuator Faults

The proportional and integral (PI) is the most utilized controller by the wind energy industry. At low wind speeds, the PI speed control using generator torque as controlled input can be quite slow, thus tuning is not significantly challenging. However, at high wind speeds, the PI speed control using pitch angle as controlled input strongly couples with the tower dynamics, denoting a multivariable problem, and should be properly designed. Inappropriate gain selection can make rotational speed regulation "loose" around the set-point or make the system unstable, as well as excite poorly damped structural modes [3].

The concepts seen throughout this chapter are here applied to the state-of-the-art controller structure of the wind turbine industry [4]. The present example intends to show that theoretical rigorousness on the design of gain-scheduled controllers may bring advantages in terms of performance and reliability of wind turbines in closed-loop.

Controller Design

For a clear and didactic exposure, the adopted control structure depicted in Fig. 4.10 is simpler than a industry standard Region III controller [4], but includes the most common control loops.

The generator speed is regulated by a PI-controller of the form,

$$G_{PI} := k_p(\theta) + k_i(\theta) \frac{(s + z_I)}{s}$$

where s denotes the Laplace operator. Instead of a pure integrator, the PI controller is composed by an integrator filter,

$$G_I(s) := \frac{s + z_I}{s},$$
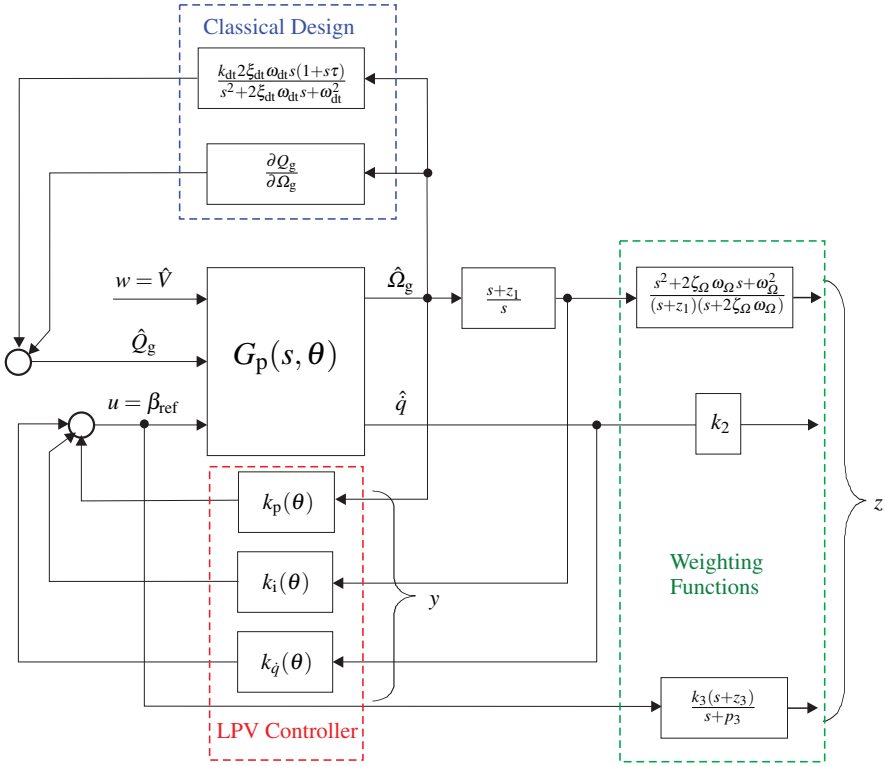
Figure 4.10: Schematic block diagram of a controlled wind turbine in Region III.

for reasons to be explained later, where the filter zero z_I is a design parameter.

It is possible to provide an extra signal by using an accelerometer mounted in the nacelle, allowing the controller to better recognize between the effect of wind speed disturbances and tower motion on the measured power or generator speed. With this extra feedback signal, tower bending moments loads can be reduced without significantly affecting speed or power regulation [3]. Therefore, it is assumed that tower velocity $\dot{q}$ is available for measurement, by integrating tower acceleration $\ddot{q}$, and is multiplied by a parameter-dependent constant $k_{\dot{q}}(\theta)$ for feedback.

Additionally, active drive train damping is deployed by adding a signal to the generator torque to compensate for the oscillations in the drive train. This signal should have a frequency, ω_{dt}, equal to the eigenfrequency of the drive train, which is obtained by filtering the measurement of the generator speed using a bandpass filter of the form,

$$G_{dt} := K_{dt} \frac{2\zeta_{dt}\omega_{dt}s(1 + \tau_{dt}s)}{s^2 + 2\zeta_{dt}\omega_{dt}s + \omega_{dt}^2}$$

The time constant, τ_{dt}, introduces a zero in the filter, and can be used to compensate for time lags in the converter system. The filter gain k_{dt} and the damping ratio ζ_{dt} are selected based on classical design techniques.

A power controller for reducing fast power variations is treated simplistically as a proportional feedback from generator speed to generator torque. Considering a constant power control scheme, the generator torque can be represented as a function of the generator speed. The proportional feedback is nothing but the partial derivative of generator torque with respect to generator speed,

$$\frac{\partial Q_\mathrm{g}(\Omega_\mathrm{g})}{\partial \Omega_\mathrm{g}} = -\frac{P_N}{N_\mathrm{g}\Omega_{\mathrm{g},N}^2}.$$

In real implementations, a slow integral component is added to the loop to include asymptotical power tracking.

Instead of the classical control techniques, the design of PI speed and tower feedback loops are revisited under the LPV framework. For a didactic and clear exposure, the interconnection of the drive train with the damper is now considered as a first order low pass filter from aerodynamic torque to generator speed, and the rotor speed proportional to the generator speed. The LPV controller can now be designed to trade off the tracking of generator speed and tower oscillations with control effort (wear on pitch actuator). The dynamic model of the variable-speed wind turbine can then be expressed as an LPV model of the form,

$$G : \begin{cases} \dot{x} = A(\theta)\,x + B_w(\theta)\,\hat{u} + B_u(\theta)\,\beta_\mathrm{ref} \\ y = C_y\,x \end{cases}$$

where states, controllable input and measurements are,

$$x = \begin{bmatrix} \Omega_\mathrm{r} & \dot{q} & q & \dot{\beta} & \beta & x_{\Omega,i} \end{bmatrix}^T, \quad u = \beta_\mathrm{ref}, \quad y = \begin{bmatrix} \Omega_\mathrm{g} & y_{\Omega,i} & \dot{q} \end{bmatrix}^T.$$

with open-loop system matrices,

$$A(\theta) = \begin{bmatrix} \rho_1(\theta) - \dfrac{1}{J_\mathrm{r}+J_\mathrm{g}N_\mathrm{g}^2}\dfrac{\partial Q_\mathrm{g}}{\partial \Omega} & -\rho_2(\theta) & 0 & 0 & \rho_3(\theta) & 0 \\ \rho_4(\theta) & -\dfrac{1}{M_\mathrm{t}}B_\mathrm{t}-\rho_5(\theta) & -\dfrac{K_\mathrm{t}}{M_\mathrm{t}} & 0 & \rho_6(\theta) & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44}(\theta_\mathrm{f}) & -a_{12}(\theta_\mathrm{f}) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ N_g & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_u = \begin{bmatrix} \rho_2(\theta) & \rho_5(\theta) & 0 & 0 & 0 & 0 \end{bmatrix}^\mathrm{T}, \quad B_w = \begin{bmatrix} 0 & 0 & 0 & b_{4,1}(\theta_\mathrm{f}) & 0 & 0 \end{bmatrix}^\mathrm{T},$$

$$C_y = \begin{bmatrix} N_\mathrm{g} & 0 & 0 & 0 & 0 \\ z_\mathrm{I} & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad a_{12}(\theta_\mathrm{f}) = b_{41}(\theta_\mathrm{f}) = (1-\theta_\mathrm{f}(t))\omega_{\mathrm{n},0}^2 + \theta_\mathrm{f}(t)\omega_{\mathrm{n,lp}}^2,$$

$$a_{44}(\theta_\mathrm{f}) = \text{-}2(1-\theta_\mathrm{f}(t))\zeta_0\omega_{\mathrm{n},0} - 2\theta_\mathrm{f}(t)\zeta_\mathrm{lp}\omega_{\mathrm{n,lp}}.$$

The basis functions $\rho_1(\theta), \dots, \rho_6(\theta)$ related to the parameter-varying aerodynamic gains are selected as,

$$\rho_1 := \frac{1}{J_\mathrm{r}+J_\mathrm{g}N_\mathrm{g}^2}\frac{\partial Q}{\partial \Omega}\bigg|_{\hat{V}}, \quad \rho_2 := \frac{1}{J_\mathrm{r}+J_\mathrm{g}N_\mathrm{g}^2}\frac{\partial Q}{\partial V}\bigg|_{\hat{V}}, \quad \rho_3 := \frac{1}{J_\mathrm{r}+J_\mathrm{g}N_\mathrm{g}^2}\frac{\partial Q}{\partial \beta}\bigg|_{\hat{V}},$$

$$\rho_4 := \frac{1}{M_\mathrm{t}}\frac{\partial T}{\partial \Omega}\bigg|_{\hat{V}}, \quad \rho_5 := \frac{1}{M_\mathrm{t}}\frac{\partial T}{\partial V}\bigg|_{\hat{V}}, \quad \rho_6 := \frac{1}{M_\mathrm{t}}\frac{\partial T}{\partial \beta}\bigg|_{\hat{V}}.$$

Notice the PI controller integrator filter G_I conveniently augmented into the state-space of G, represented by the state $x_{\Omega,i}$ and the output $y_{\Omega,i}$. The plant G_p is defined as the wind turbine model solely (plant G without the augmentation of G_I).

Considering G as the plant for synthesis purposes, the LPV controller structure reduces to a parameter-dependent static output feedback of the form,

$$K(\theta) = D_{c,0} + \sum_{i=1}^{6} \rho_i(\theta) D_{c,i} + \theta_f D_{c,7}, \quad D_{c,n} := \begin{bmatrix} D_{p,n} & D_{i,n} & D_{\dot{q},n} \end{bmatrix},$$

$$n = 0, 1, \ldots, 7.$$

Controller tuning follows a procedure similar to the $\mathcal{H}_\infty$ design. Notice that, for fixed values of the varying parameter θ, and initially neglecting the tower velocity feedback, the controller design becomes a mixed sensitivities optimization problem intended to minimize the norm,

$$\left\| \begin{array}{c} W_{z1} \, G_I \, S \, G_v \\ W_u \, G_{PI} \, S \, G_v \end{array} \right\|_\infty$$

where S is the sensitivity defined as $S := (I + G_p \, G_{PI})^{-1}$, G_v is the transfer function from $\hat{V}$ to $\hat{\Omega}_g$, W_{z1} and W_u are weighting fuctions. The weight W_{z1} applied to the generator speed deviations can be used to shape the closed-loop response of rotational speed in face of wind disturbances, given by $\hat{\Omega}(t) = S \, G_v \hat{V}(t)$. The desired sensitivity in closed-loop is,

$$S_\Omega(s) := \frac{s^2 + 2\xi_\Omega \omega_\Omega s}{s^2 + 2\xi_\Omega \omega_\Omega s + \omega_\Omega^2}.$$

where the natural frequency ω_Ω and damping ratio ξ_Ω are design parameters that select the desired second-order closed-loop behavior. The desired sensitivity S_Ω can be applied as a loop-shaping weight by defining W_{z1} as

$$W_{z1}(s) := \frac{1}{G_I(s) S_\Omega(s)} = \frac{s^2 + 2\xi_\Omega \omega_\Omega s + \omega_\Omega^2}{(s + z_I)(s + 2\xi_\Omega \omega_\Omega)}.$$

W_u is a first order high-pass filter that penalizes high-frequency content on the pitch angle,

$$W_u(s) := k_3 \frac{s + z_3}{s + p_3}.$$

W_{z1} and W_u governs the tradeoff between rotational speed regulation and pitch wear. Due to the resonance characteristics of the transfer function from $\hat{V}$ to $\dot{q}$, the weighting function W_{z2} is chosen as a scalar k_2, that tradeoffs the desired tower damping.

Two LPV controllers are designed, one fault-intolerant and another tolerant to pitch actuator faults. The only difference on their synthesis is the inclusion of the fault dependent terms $P_7 \theta_f$ and $D_{c,7}\theta_f$ of the Lyapunov and controller matrices, respectively. The parameters for the loop-shaping weight W_{z1} are selected as $\omega_\Omega = 0.6283$ rad/s (0.1 Hz) and $\xi_\Omega = 0.7$, with the zero of the integrator filter located at $z_I = 1.0$ rad/s. A special attention must be devoted to the choice of W_u. Due to the fact that the pitch system has slower dynamics in the presence of low oil pressure, the bandwidth of this filter must be

made large enough to allow rotational speed and tower damping control in the occurrence of faults. Defining Ω_{3P} as three times the nominal rotational speed $\Omega_{r,N}$, in the present example, $k_3 = 1$, $p_3 = 1.5\Omega_{3P}$ and $z_3 = 15\Omega_{3P}$.

Remember that the iterative LMI algorithm is a synthesis procedure in discrete time. Therefore, the augmented LPV plant in continuous time is discretized using a bilinear (Tustin) approximation [25] with sampling time $T_s = 0.02$ s, at each point $\Theta_g \times \text{Vert}\mathcal{V}$. The rate of variation of the scheduling variables in continuous-time must as well be converted to discrete-time by the relation $\Delta\theta(k) = T_s\Delta\theta(t)$.

The initial slack matrices $G(\theta, \Delta\theta)^{\{0\}}$, $\forall(\theta, \Delta\theta) \in \Theta_g \times \text{Vert } \mathcal{V}$ required to initialize the LMI-based algorithm are determined from the solution of the following LMI optimization problem,

$$\text{Minimize } \gamma \text{ subject to (4.22), (4.26), (4.27), } \forall(\theta, \Delta\theta) \in \Theta_g \times \text{Vert } \mathcal{V}$$

with a given initial controller $K(\theta)$. The resulting Lyapunov matrix determines $G(\theta, \Delta\theta)^{\{0\}}$ $= \mathcal{P}(\theta, \Delta\theta)^{-1}$. The proportional and integral gains of the given initial controller can be computed by placement of the poles of the transfer function from $\hat{V}$ to $\hat{\Omega}_g$. Neglecting pitch actuator dynamics, and considering a pure integrator, the k_p and k_i gains can be described analytically as [26],

$$k_p(\theta) = \frac{2\xi_\Omega\omega_\Omega\left(J_r + N_g^2 J_g\right) - N_g\frac{\partial Q_g}{\partial \Omega_g} + \rho_1(\theta)}{-N_g\rho_3(\theta)}, \; k_i(\theta) = \frac{\omega_\Omega^2\left(1 + \xi_\Omega^2\right)\left(J_r + N_g^2 J_g\right)}{-N_g\rho_3(\theta)}$$

The tower feedback gain of the initial controller is $k_{\dot{q}}(\theta) = 0$, meaning no active tower damping.

Convergence tolerance of the iterative algorithm is set to $\epsilon = 10^{-3}$. After 89 iterations, convergence is achieved to a performance level $\gamma = 0.586$. The evolution of $\gamma^{\{j\}}$ versus the iteration number is depicted in Fig. 4.11, where the monotonically decreasing property of the sequence is noticeable. The proportional and integral gains depicted on the figures are multiplied by the gearbox ratio N_g for better illustration. The controller gains $K(\theta) = [k_p(\theta), k_i(\theta), k_{\dot{q}}(\theta)]$ computed at $\theta_{op} = 15$ m/s, $\theta_f = 0$, during the course of the iterative LMI algorithm, are also shown. The synthesis procedure converge to controller gains different than the gains of the initial controller. The tower feedback gain $k_{\dot{q}}$, null in the initial controller, has converged to a nonzero value, meaning active tower damping.

The proportional, integral and tower feedback gains as three-dimensional surfaces of the scheduling parameters $\overline{V}$ and θ_f are illustrated in Fig. 4.12a to 4.12c. The controller gains capture the dependence of the LPV system on the the wind speed given by the basis functions. Compare the shape of the surfaces with the aerodynamic gains (Fig. 4.8). Also notice the slight changes in k_p and $k_{\dot{q}}$ and the changes in k_i scheduled by θ_f.

Simulation Results

The performance of the LPV controllers are accessed in a nonlinear wind turbine simulation environment [10]. The effective wind speed is estimated by an unknown input observer that uses measurements of generator speed, generator torque and pitch angle [18]. Figures 4.13a to 4.14d depict time series of the variables of interest resulted from
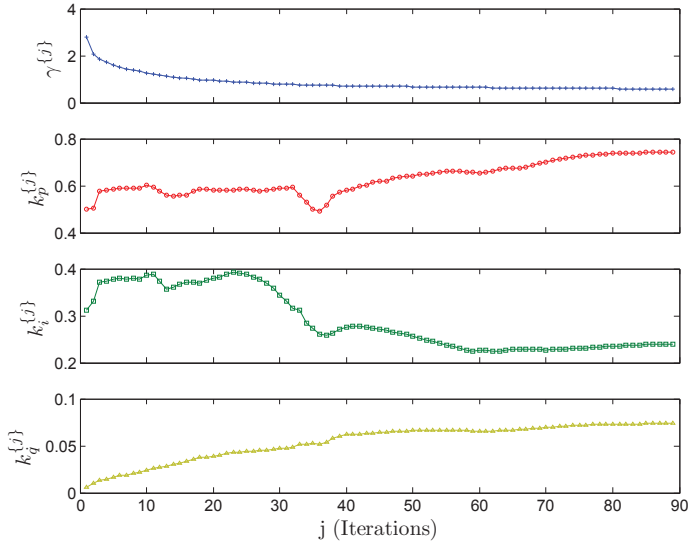
Figure 4.11: Evolution of performance level γ and controller gains k_p, k_i, $k_{\dot{q}}$ during the iterative LMI synthesis. Controller gains computed at $\theta_{op} = 15$ m/s, $\theta_f = 0$.
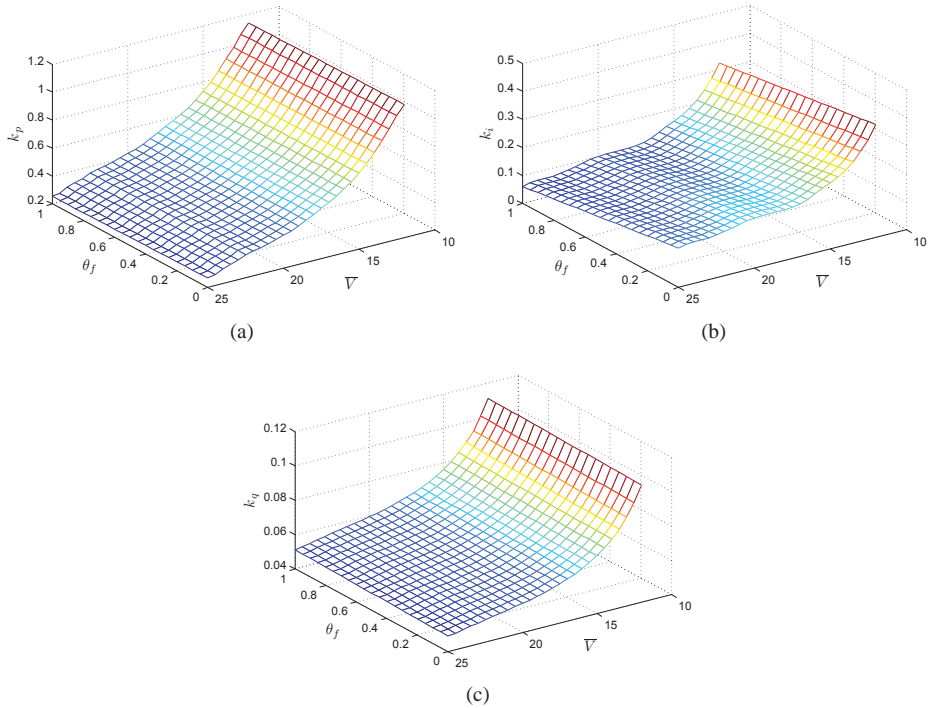

Figure 4.12: Proportional, integral and tower feedback gains as functions of the operating point and fault scheduling variables.

a 600 s simulation. A mean speed of 17 m/s with 12 % turbulence intensity and shear exponent of 0.1 characterizes the wind field (Fig. 4.13a). At time $t = 200$ s, the pitch system experiences a fault with θ_f increasing from 0 to 1 (Fig. 4.13b). At $t = 430$ s, the pitch system comes to normality with θ_f decreasing from 1 to 0. Both variations on the fault scheduling variable are made with maximum rate of variation.

Results of LPV controllers intolerant and tolerant to pitch actuator faults are compared to support a discussion of the consequences of the fault on the closed-loop system as well as fault accommodation. When the wind turbine is controlled by the fault intolerant LPV PI controller, the rotational speed (Fig. 4.13c) experiences poor and oscillatory regulation during the occurrence of faults, more pronouncedly while θ_f is varying. The threshold for a shutdown procedure due to overspeed is usually between 10-15% over the nominal speed [27]; in this particular case, the overspeed would not cause the wind turbine to shutdown. The FT-LPV PI controller successfully accommodates the fault, maintaining rotor speed properly regulated. Oscillatory power overshoots of up to 6% of the nominal power (Fig. 4.13d) degrades power quality; the same does not happen to the FT-LPV controlled system.

More serious than the effects on rotational speed and power are the consequences of faults on the pitch system and tower. Excessive pitch angle excursions during faults (Fig. 4.14a) with the limits on velocity of ± 8 deg/s being reached (Fig. 4.14b) may cause severe wear on pitch bearings. The FT-LPV controller maintain pitch excursions and velocities within normal limits. The tower experiences displacements (Fig. 4.14c) of up to 0.48 m, an increase of approximately 60% when compared to the FT-LPV. The displacements comes along with very high tower velocities of almost 0.4 m/s, 260% higher than the fault accommodated case.

In such a situation, the supervisory controller would shut down the wind turbine due to excessive vibrational levels measured by the nacelle accelerometer. The same would not be necessary if the wind turbine is controlled by the FT-LPV. Therefore, fault tolerance leads to higher energy generation and availability. It also collaborates to a better management of condition-based maintenance; higher priority of maintenance can be given to wind turbines with faults that cannot be accommodated by the control system. These are examples of the benefits that the LPV control design framework presented in this Chapter can bring to wind turbines in closed-loop with industry-standard as well as more elaborate controllers.

5   Conclusions

This chapter initially presents the modeling of a wind turbine model as an LPV system, considering faults on actuators and sensors. Later, an iterative LMI-based algorithm for the design of structured LPV controllers is described. This constitutes a unified LMI-based design framework to address gain-scheduling, fault-tolerance and robustness on the design of wind turbine controllers.

The method is based on parameter-dependent Lyapunov functions, which reduces conservativeness of control for systems with rate bounds, which is the case in this work. The iterative algorithm may be computationally expensive depending on the number of plant states and scheduling variables, but brings desired flexibility in terms of the controller structure: decentralized of any order, dynamic (reduced-order) output feedback,
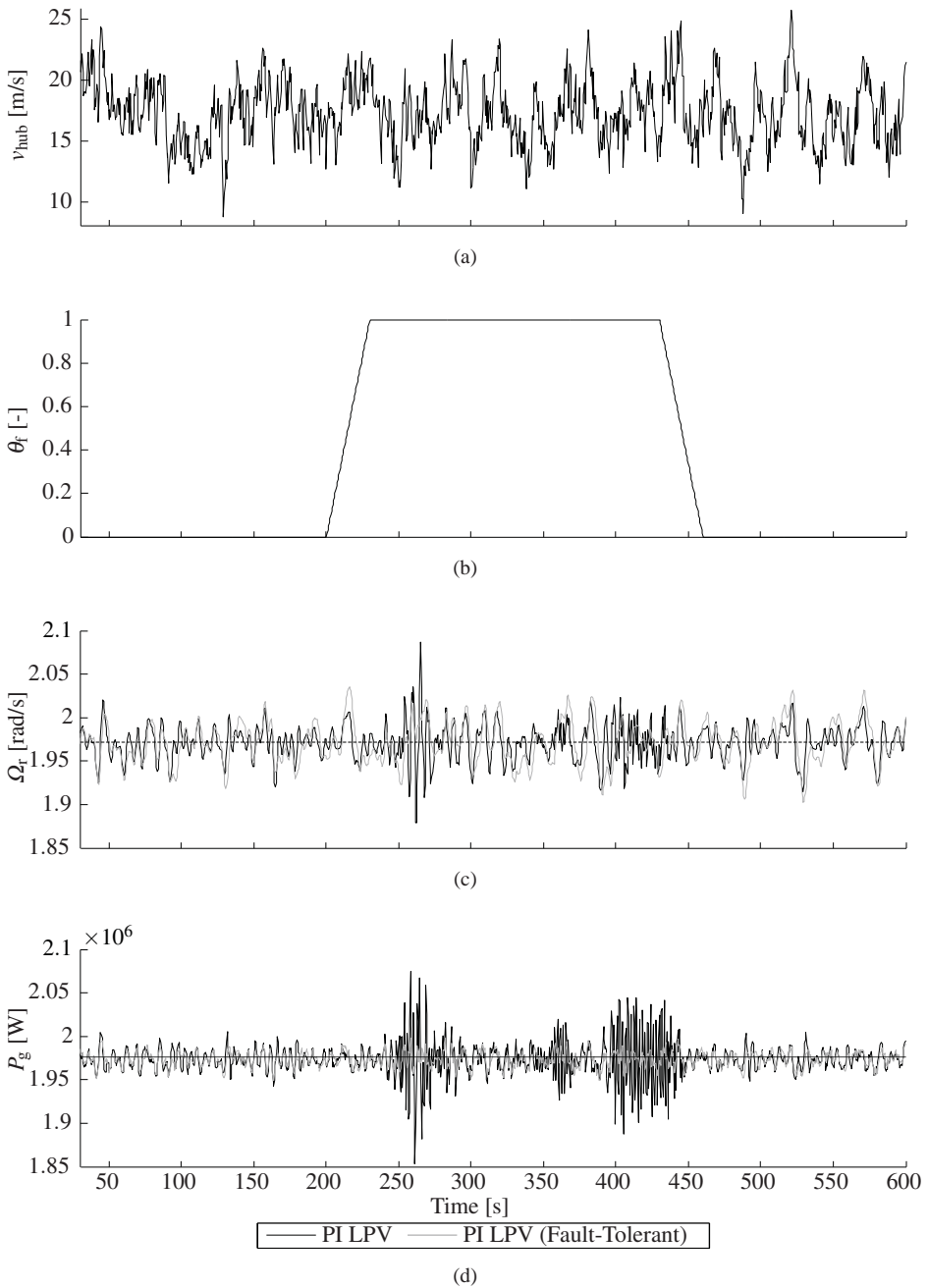
Figure 4.13: Time series of (a) hub height wind speed, (b) fault scheduling variable, (c) rotor speed and (d) electrical power. Simulation results of a 2MW wind turbine controlled by a fault-intolerant and a fault-tolerant LPV PI controller.
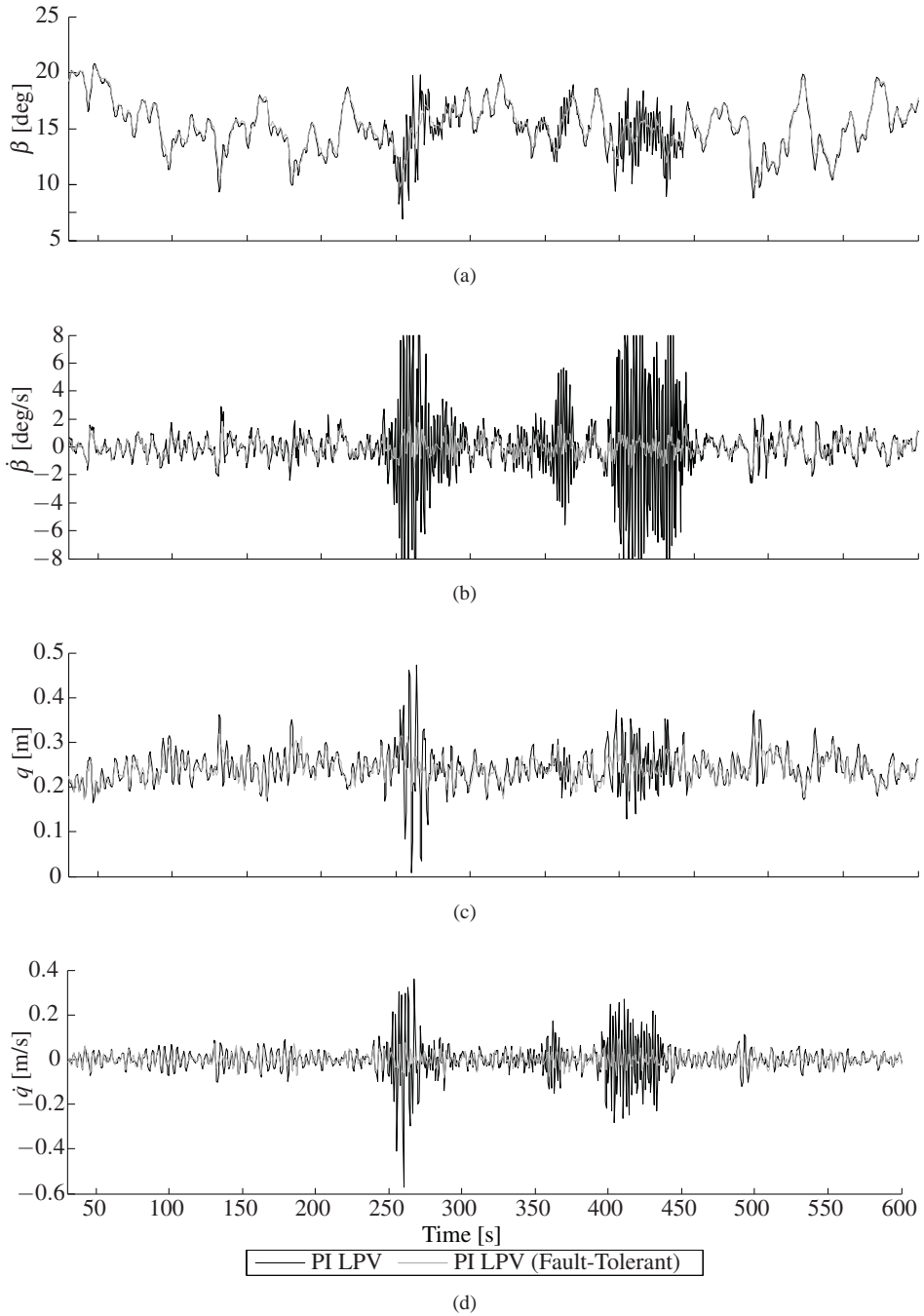
Figure 4.14: Time series of (a) pitch angle, (b) pitch velocity, (c) fore-aft tower position and (d) fore-aft tower velocity. Simulation results of a 2MW wind turbine controlled by a fault-intolerant and a fault-tolerant LPV PI controller.

static output feedback and state feedback are among the possible ones. Moreover, the resulting controller can also be easily implemented in practice due to low data storage and simple math operations. In fact, the required data to be stored on the controller memory is only the controller matrices, and scalar functions of the scheduling variables representing plant nonlinearities. The mathematical operations needed to compute the controller at each sampling time are look-up tables with interpolation, products between a scalar and a matrix, and sums of matrices.

A design example of a fault-tolerant controller for the Region III, with a structure similar to the the state-of-the-art industrial controllers, intends to show that theoretical rigorousness on the design of gain-scheduled controllers may bring advantages in terms of performance and reliability of wind turbines in closed-loop. The presented framework is not limited to the specific example shown. Due to its flexibility, the framework can be applied to other known wind turbine controller structures or even to explore different control philosophies.

Simulations indeed confirm that the fault-tolerant LPV controllers have superior performance in the occurrence of faults. The LPV controller designed for the nominal system start oscillating when the fault is introduced. In a real situation, the supervisory controller would shut down the wind turbine due to excessive vibrational levels measured by the nacelle accelerometer. The same would not be necessary if the wind turbine is controlled by the FT-LPV. Therefore, higher energy generation and availability is achieved. It also ccontributes to a better management of condition-based maintenance; priority on maintenance can be given to wind turbines with faults that cannot be accommodated by the control system.

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Paper B

Structured Control of LPV Systems with Application to Wind Turbines

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Abstract

This paper deals with structured control of linear parameter varying systems (LPV) with application to wind turbines. Instead of attempting to reduce the problem to linear matrix inequalities (LMI), we propose to design the controllers via an LMI-based iterative algorithm. The proposed algorithm can synthesize structured controllers like decentralized, static output and reduced order output feedback for discrete-time LPV systems. Based on a coordinate decent, it relies on a sufficient matrix inequality condition extended with slack variables to an upper bound on the induced $\mathcal{L}_2$-norm of the closed-loop system. Algorithms for the computation of feasible as well as optimal controllers are presented. The general case where no restrictions are imposed on the parameter dependence is treated here due to its suitability for modeling wind turbines. A comprehensive numerical example of a gain-scheduled LPV controller design with prescribed pattern for wind turbines illustrate the utilization of the proposed algorithm.

1 Introduction

Practical considerations often dictate structural constraints on the controller. Control practitioners face the challenge of designing low-order, decentralized, observed-based, PID control structures, among others. These control problems are naturally formulated as Bilinear Matrix Inequalities (BMI), and to which equivalent convex reformulations based on Linear Matrix Inequalities (LMI) are not known to exist. Add to that some systems inherently exhibit time-varying nonlinear dynamics along their nominal operating trajectory, motivating the use of advanced control techniques such as gain-scheduling, to counteract performance degradation or even instability problems by continuously adapting to the dynamics of the plant. A systematic way of designing controllers for systems with linearized dynamics that vary significantly with the operating point is within the framework of linear parameter-varying (LPV) control. Wind turbines are naturally inserted in this context. Firstly, because gain-scheduling is an usual approach to deal with varying dynamics dependent on the operating point [1]. Secondly, the structure of wind turbine industrial controllers often have a prescribed pattern [2].

In this context, our interest lies in the synthesis of LPV controllers with structural constraints, more specifically, the $\mathcal{L}_2$-norm minimization problem,

$$minimize \, \|T_{z \to w}(\theta, K(\theta))\|_2$$
$$K(\theta) \in \mathcal{K}$$

where $T_{z \to w}(\cdot)$ is an input-output system operator, $K(\theta)$ is a linear parameter varying controller dependent on a vector of time-varying parameters θ, and $\mathcal{K}$ represents a structural constraint in the controller matrices.

The diversification of LPV controller structures is not extensively addressed in the literature. The static state feedback and full-order dynamic output feedback are by far the most investigated structures. There are some proposals on the design of static output feedback controllers [3, 4]. A few works can be found on other controller structures like decentralized [5], fixed-order dynamic output for single-input single-output polynomial systems [6]. Synthesis conditions based on Linear Matrix Inequalities (LMI) is a common

feature to all these papers. Recently, static output [7] and full-order dynamic output [8] synthesis procedures relies on extended LMI conditions with slack variables [9].

Instead of an attempt to reduce the problem to linear matrix inequalities (LMI), this paper investigates the design of structured LPV controllers via an LMI-based iterative algorithm. Iterative LMI algorithms with slack matrices were investigated in the context of robust [10] and affine LPV [11] control. Decentralized of any order, fixed-order output, static output and simultaneous plant-control design are among the possible control structures. Based on a coordinate decent, it relies on extended LMI conditions to an upper bound on the induced $\mathcal{L}_2$-norm of the closed-loop system. We propose a relaxation on the LMI condition useful for computing feasible controllers. After a feasible controller is found, the objective is cost minimization until the solution converges to a stationary point. The general case where no restrictions are imposed on the parameter dependence is treated here due to its suitability for modeling wind turbines.

Realizing advanced gain-scheduled controllers can be difficult in practice and may lead to numerical challenges [1, 12]. Usually, several plant and controller matrices must be stored on the controller memory. Moreover, matrix factorizations and inversions are among the operations that must be done online by the controller at each sampling time [13]. The proposed synthesis methodology can be of practical relevance because the resulting controllers have simple implementation.

This paper is organized as follows. Section II describes the system, controller and some possible controller structures. Section III presents known extended matrix inequalities conditions for the induced $\mathcal{L}_2$ norm and the proposed relaxation. Section IV describes the iterative LMI algorithm along with convergence and computational considerations. Section V revisits the design of wind turbine industry-standard controllers under the LPV framework.

2   System and Controller Description

An open-loop, discrete-time augmented LPV system with state-space realization of the form,

$$
\begin{aligned}
x(k+1) &= A(\theta)x(k) + B_w(\theta)w(k) + B_u(\theta)u(k) \\
z(k) &= C_z(\theta)x(k) + D_{zw}(\theta)w(k) + D_{zu}(\theta)u(k) \\
y(k) &= C_y(\theta)x(k) + D_{yw}(\theta)w(k),
\end{aligned}
\tag{5.1}
$$

is considered for the purpose of synthesis, where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ is the vector of disturbance, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $z(k) \in \mathbb{R}^{n_z}$ is the controlled output, and $y(k) \in \mathbb{R}^{n_y}$ is the measured output. $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ are continuous functions of some time-varying parameter vector $\theta = [\theta_1, \ldots, \theta_{n_\theta}]$. Assume θ ranges over a hyperrectangle denoted Θ,

$$
\Theta = \left\{ \theta : \ \underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i, \ i = 1, \ldots, n_\theta \right\}.
$$

The rate of variation $\Delta\theta = \theta(k+1) - \theta(k)$ belongs to a hypercube denoted $\mathcal{V}$,

$$
\mathcal{V} = \left\{ \Delta\theta : \ |\Delta\theta_i| \leq v_i, \ i = 1, \ldots, n_\theta \right\}.
$$

The LPV controller has the form,

$$x_{\mathrm{c}}(k+1) = A_{\mathrm{c}}(\theta)x_c(k) + B_{\mathrm{c}}(\theta)y(k)$$
$$u(k) = C_{\mathrm{c}}(\theta)x_{\mathrm{c}}(k) + D_{\mathrm{c}}(\theta)y(k),$$
(5.2)

where $x_{\mathrm{c}}(k) \in \mathbb{R}^{n_c}$ and the controller matrices are continuous functions of θ. Note that depending on the controller structure, some of the matrices may be zero. The controller matrices can be represented in a compact way,

$$K(\theta) := \begin{bmatrix} D_{\mathrm{c}}(\theta) & C_{\mathrm{c}}(\theta) \\ B_{\mathrm{c}}(\theta) & A_{\mathrm{c}}(\theta) \end{bmatrix}.$$
(5.3)

The interconnection of system (5.1) and controller (5.2) leads to the following closed-loop LPV system denoted S_{cl},

$$S_{\mathrm{cl}}: \quad x(k+1) = \mathcal{A}(\theta, K(\theta))x_{\mathrm{cl}}(k) + \mathcal{B}(\theta, K(\theta))w(k)$$
$$z(k) = \mathcal{C}(\theta, K(\theta))x_{\mathrm{cl}}(k) + \mathcal{D}(\theta, K(\theta))w(k).$$
(5.4)

This general system structure can be particularized to some usual control topologies. In the case $K(\theta)$ is an *unconstrained* matrix, if $n_{\mathrm{c}} = 0$, the problem becomes a static output feedback. The static state feedback is a particular case of static output, when the system output is a full rank linear transformation of the state vector $\forall\theta$. If $n = n_{\mathrm{c}}$, the full-order dynamic output feedback arises. In a structured control context, more elaborate control systems can be designed by constraining $K(\theta)$. A fixed-order dynamic output feedback has $n_{\mathrm{c}} < n$. For decentralized controllers of arbitrary order, the structure of $K(\theta)$ is constrained to be,

$$K(\theta) := \begin{bmatrix} \mathrm{diag}(D_c(\theta)) & \mathrm{diag}(C_c(\theta)) \\ \mathrm{diag}(B_c(\theta)) & \mathrm{diag}(A_c(\theta)) \end{bmatrix}$$

where $\mathrm{diag}(\cdot)$ stands that $(\cdot)$ has a block-diagonal structure.

In the general parameter dependence case, the open-loop system matrices are dependent on arbitrary functions of the varying parameters,

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}(\theta) = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_0 + \sum_i \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_i \rho_i(\theta),$$
$$i = 1, \dots, n_\rho$$
(5.5)

where $\rho_i(\theta)$ are scalar functions known as basis functions that encapsulate possible system's nonlinearities and n_ρ is the number of basis functions. The controller matrices are continuous functions of θ with similar type of dependence,

$$\begin{bmatrix} A_{\mathrm{c}} & B_{\mathrm{c}} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{bmatrix}(\theta) = \begin{bmatrix} A_{\mathrm{c}} & B_{\mathrm{c}} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{bmatrix}_0 + \sum_{i=1}^{n_\theta} \begin{bmatrix} A_{\mathrm{c}} & B_{\mathrm{c}} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{bmatrix}_i \rho_i(\theta)$$
$$i = 1, \dots, n_\theta$$
(5.6)

3 Induced $\mathcal{L}_2$-norm Performance

The design of a closed-loop system usually considers performance specifications that can be characterized in different ways. Define $T_{zw}(\theta)$ as the input-output operator that represents the forced response of (5.4) to an input signal $w(k) \in \mathcal{L}_2$ for zero initial conditions. The induced $\mathcal{L}_2$-norm of a given input-output operator,

$$\|T_{zw}\|_{\mathcal{L}_2} := \sup_{\theta \in \Theta \times \mathcal{V}} \sup_{\|w\|_{\mathcal{L}_2} \neq 0} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}$$

is commonly utilized as a measure of performance of LPV systems and allows formulating the control specification as in $\mathcal{H}_\infty$ control theory. The LPV system (5.4) is said to have performance level γ when it is exponentially stable and $\|T_{zw}\|_{\mathcal{L}_2} < \gamma$ holds. An extension of the bounded real lemma (BRL) for parameter dependent systems is a sufficient condition for checking the $\mathcal{L}_2$ performance level of system S_{cl}.

Lemma 2 (Extended $\mathcal{L}_2$ Performance). *[9, 8] For a given controller $K(\theta)$, if there exist $\mathcal{P}(\theta) = \mathcal{P}(\theta)^T$ and $\mathcal{Q}(\theta)$ satisfying (5.7) with $r = 1$ for all $(\theta, \Delta\theta) \in \Theta \times \mathcal{V}$, then the system S_{cl} is exponentially stabilizable by the controller $K(\theta)$ and $\|T_{zw}(\theta)\|_{\mathcal{L}_2} < \gamma$.*

$$\begin{bmatrix} r^2\mathcal{P}(\theta^+) & \mathcal{A}(\theta, K(\theta))\mathcal{Q}(\theta) & \mathcal{B}(\theta, K(\theta)) & 0 \\ \star & -\mathcal{P}(\theta) + \mathcal{Q}(\theta)^T + \mathcal{Q}(\theta) & 0 & \mathcal{Q}(\theta)^T\mathcal{C}(\theta, K(\theta))^T \\ \star & \star & \gamma I & \mathcal{D}(\theta, K(\theta))^T \\ \star & \star & \star & \gamma I \end{bmatrix} \succ 0 \quad (5.7)$$

The term r^2 multiplying the Lyapunov matrix at the (1,1) entry of (5.7) is, in the present paper, artificially inserted into the formulation. For frozen θ (LTI system) r represents the z-plane circle radius, thus $r = 1$ in the Schur stability criteria. By imposing $r > 1$ the z-plane circle would be enlarged, meaning that even unstable closed-loop systems (eigenvalues lying out of the unit circle) would satisfy Lemma 2. For parameter varying systems, this notion of enlargement still exists when $r > 1$, defined by the following lemma.

Lemma 3 (Enlarged $\mathcal{L}_2$ Performance). *For a given controller $K(\theta)$, if there exist $\mathcal{P}(\theta) = \mathcal{P}(\theta)^T$ and $\mathcal{Q}(\theta)$ satisfying (5.7) with $r = r_e > 1$ for all $(\theta, \Delta\theta) \in \Theta \times \mathcal{V}$, then the system S_{cl} satisfies the enlarged $\mathcal{L}_2$ performance with $\|T_{zw}(\theta)\|_{\mathcal{L}_2, r=r_e} < \gamma$.*

Even systems that are not exponentially stabilizable may satisfy the enlarged $\mathcal{L}_2$-norm condition. This fact will be utilized in the proposed algorithms for finding a feasible controller. The shifted-$\mathcal{H}_\infty$-norm is a similar concept for continuous-time LTI systems [14].

The Lyapunov and slack variables mimic the general parameter dependence of the plant and controller,

$$\mathcal{P}(\theta) = P_0 + \sum_{i=1}^{n_\theta} \rho_i(\theta)P_i \quad (5.8a)$$

$$\mathcal{Q}(\theta) = Q_0 + \sum_{i=1}^{n_\theta} \rho_i(\theta) Q_i \tag{5.8b}$$

The Lyapunov function at $\theta^+ := \theta + \Delta\theta$ can be described as,

$$\mathcal{P}(\theta^+) = P_0 + \rho_i(\theta^+) P_i \tag{5.9}$$

Conveniently, the basis functions at θ^+ are approximated by a linear function of $\rho(\theta)$ and $\Delta\theta$,

$$\rho_i(\theta^+) := \rho_i(\theta) + \frac{\partial \rho_i(\theta)}{\partial \theta} \Delta\theta, \tag{5.10}$$

thereby turning inequality (5.7) affine dependent on the rate of variation $\Delta\theta$. This approximation makes sufficient to verify (5.7) with (5.9)-(5.10) only at Vert $\mathcal{V}$.

4 Optimization Algorithm

The optimization algorithm iterates between LMI problems by fixing the controller variables and the slack variable alternatively. In this way, the parameter dependent Lyapunov matrix remains as a variable during the whole optimization process. In the general parameter dependence case, the controller is designed in a gridded parameter space. A gridding procedure consists of defining a gridded parameter subset denoted $\Theta_g \subset \Theta$, designing a controller that satisfies the matrix inequalities constraints $\forall \theta \in \Theta_g$, and checking the inequalities constraints in a denser grid. If the last step fails, the process is repeated with a finer grid.

In order to save text during the exposure of the algorithms, denote the inequality constrains by

$$\Pi_Q(x) := (5.7), \ \forall (\Theta, \Delta\theta) \in \Theta_g \times \text{Vert } \mathcal{V}$$

The algorithm for computing a feasible structured LPV controller is described next. The aim is to create a sequence of r convergent to 1, that is, for a certain tolerance ϵ, $r^{(j)} \geq 1 - \epsilon$, $r^{(j)} \to 1 \pm \epsilon$, as $j \to \infty$.

Algorithm 3. *(Feasibility) Given initial slack matrix $\mathcal{Q}^{(1)}(\theta) = I$, $\forall \theta \in \Theta_g$, an initial radius $r^{(1)} > 1$, a target radius $r_{tg} \leq 1$ and a convergence tolerance ϵ_1. Set $j = 1$ and start to iterate:*

1. *Find $\mathcal{P}(\theta)$, $K(\theta)$, and γ that solves the LMI problem,*

   Minimize γ subject to $\Pi_Q(x)$ with $r = r^{(j)}$, and frozen $\mathcal{Q}(\theta) = \mathcal{Q}^{(j)}(\theta) \ \forall \theta \in \Theta_g$.

2. *If Step 1 is feasible, $K^{(j)}(\theta) = K(\theta)$. Else, $K^{(j)}(\theta) = K^{(j-1)}(\theta)$.*

3. *Find $\mathcal{P}(\theta)$, $\mathcal{Q}(\theta)$, and γ that solves the LMI problem,*

   Minimize γ subject to $\Pi_Q(x)$ with $r = r^{(j)}$, and frozen $K(\theta) = K^{(j)}(\theta)$, $\forall \theta \in \Theta_g$

4. *If Step 3 is feasible, $\mathcal{Q}(\theta)^{(j+1)} = \mathcal{Q}(\theta)$. Else, $\mathcal{Q}(\theta)^{(j+1)} = \mathcal{Q}^{(j)}(\theta)$*

5. *If Step 1 and step 3 are feasible, $r^{(j+1)} = 0.5\,(r^{(j)} + r_{tg})$ and $\Delta r^{(j+1)} = |r^{(j+1)} - r^{(j)}|$ (Reduced radius).*
Elseif Step 3 is feasible, $r^{(j+1)} = r^{(j)}$ and $\Delta r^{(j+1)} = \Delta r^{(j)}$ (Same radius).
Else, $r^{(j+1)} = r^{(j)} + 0.5\,|r^{(j-1)} - r^{(j)}|$ and $\Delta r^{(j+1)} = |r^{(j+1)} - r^{(j)}|$ (Increased radius).

6. *If $|r^{(j+1)} - r^{(j)}| < \epsilon_1$, stop. Else, $j = j + 1$ and go to step 1.*

The initial radius $r^{(1)}$ should be made large enough to make the first iteration feasible. Our experience shows that $r^{(1)} = 2$ suffices for most situations. The target radius r_{tg} can be made slightly smaller than 1. Once the radius reaches the target radius within a certain tolerance ϵ_1, the objective is only to minimize the performance level γ.

Algorithm 4. *(Performance Level) Given initial controller $K^{(j)}(\theta)$, $\forall \theta \in \Theta_g$, and a convergence tolerance ϵ_2. Set $j = 1$ and start to iterate:*

1. *Find $\mathcal{P}(\theta)$, $K^{(j)}(\theta)$, and γ that solves the LMI problem,*

   Minimize γ subject to $\Pi_Q(x)$ with $r = 1$, and frozen $\mathcal{Q}(\theta) = \mathcal{Q}^{(j)}(\theta)\ \forall \theta \in \Theta_g$.

2. *Find $\mathcal{P}(\theta)$, $\mathcal{Q}(\theta)$, and $\gamma^{(j)}$ that solves the LMI problem,*

   Minimize γ subject to $\Pi_Q(x)$ with $r = 1$, and frozen $K(\theta) = K^{(j)}(\theta)$, $\forall \theta \in \Theta_g$.

3. *If $|\gamma^{(j)} - \gamma^{(j-1)}| < \epsilon_2$, stop. Else, $j = j + 1$ and go to step 1.*

Algorithm 2 generates a convergent sequence of solutions such that the cost is non-increasing, that is, $\gamma^{(1)} \geq \gamma^{(j)} \geq \gamma^{(*)}$. To realize this, notice that taking the slack variable equal to the Lyapunov variable implies sufficiency of Lemma 2 [9]. Therefore, $\mathcal{P}(\theta)$ computed at step 1 is a solution for $\mathcal{Q}(\theta)$ at step 2, implying feasibility of step 2 with at least the same value of γ of step 1. The controller $K(\theta)$ at the iteration j is also a solution for the step 1 at iteration $j + 1$, implying feasibility of step 1 with at least the same performance level as iteration j.

Algorithms 3 and 4 are in fact very similar and can be unified in a single algorithm.

Computational Load

Depending on the system/controller order and number of basis functions, the procedure may be computationally expensive. Slight modifications on the algorithms alleviate computational load at the expense of some conservatism.

- The step at which the slack matrix is computed can be replaced by an update rule of the form
  $$\mathcal{Q}^{(j+1)}(\theta) = \mathcal{P}^{(j)}(\theta), \ \forall \theta \in \Theta_g.$$
  Indeed, taking the slack variable equal to the Lyapunov variable implies sufficiency of Lemma 2 [9].

- The slack matrix can be made parameter independent, e.g. $\mathcal{Q}(\theta) = \mathcal{Q}$.

- Parameter dependent matrix variables may also depend on a fewer number of basis functions/varying parameters than the plant, thus reducing the number of optimization variables. Some basis functions are more representative of system's nonlinearities than others. For example, the LPV controller can be made dependent of some basis functions while being designed robust to the reminiscent ones by including them in the Lyapunov variable.

Controller Implementation

Due to the fact that no linearizing change of variables is involved in the formulation, the resulting controller can be easily implemented in practice. The iterative LMI optimization algorithm provides the controller matrices $A_{c,i}$, $B_{c,i}$, $C_{c,i}$, $D_{c,i}$, for $i = 0, 1, \ldots, n_\rho$. These matrices, the basis functions, and the value of the scheduling variables are the only required information to determine the control signal $u(k)$. At each sample time k, the scheduling variable $\theta(k)$ is measured (or estimated) and a control signal is obtained as follows.

1. Compute the value of the basis functions $\rho_i(\theta(k))$, for $i = 0, 1, \ldots, n_\rho$. The basis functions may be stored in a lookup table that takes $\theta(k)$ as an input and outputs an interpolated value of $\rho(\theta(k))$.

2. With the value of the basis functions in hand, determine the controller matrices $A_c(\theta(k))$, $B_c(\theta(k))$, $C_c(\theta(k))$, $D_c(\theta(k))$ according to (5.6).

3. Once the controller matrices have been found, the control signal $u(k)$ can be obtained by the dynamic equation (5.2) of the LPV controller, only involving multiplications and sums.

5   Wind Turbine LPV Control

At high wind speeds, the power generated by a wind turbine should be maintained at rated value. A common control strategy is to regulate the generator speed (Ω_g) by varying the blade pitch angles (β) while maintaining a constant generator torque (Q_g). The wind energy industry relies on the proportional and integral (PI) controller to accomplish such task. The PI speed control using pitch angle as controlled input strongly couples with the tower dynamics, denoting a multivariable problem, and should be properly designed. The adopted control structure depicted in Fig. 5.1 includes the most common control loops of a industry standard Region III controller [2].

The generator speed is regulated by a PI controller of the form,

$$G_{\mathrm{PI}} := k_{\mathrm{p}}(\theta) + k_{\mathrm{i}}(\theta)G_{\mathrm{I}}(s)$$

where s denotes the Laplace operator. Instead of a pure integrator, the PI controller is composed by an integrator filter,

$$G_{\mathrm{I}}(s) := \frac{s + z_{\mathrm{I}}}{s},$$
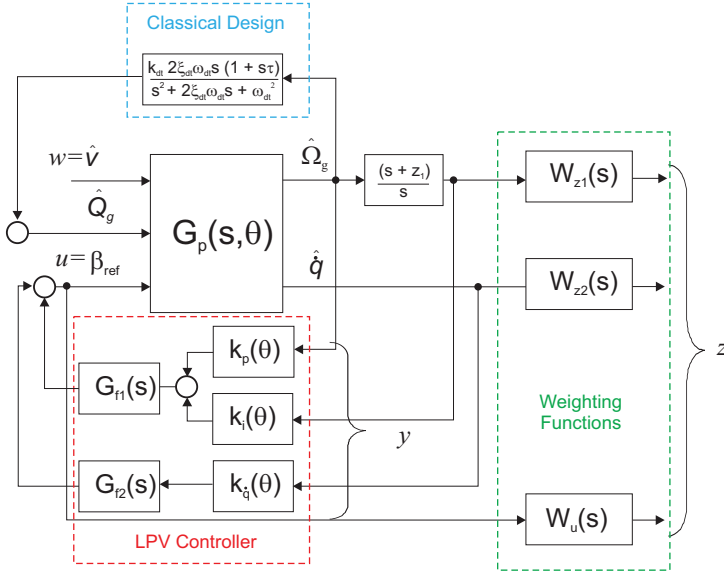
Figure 5.1: Control loops of generator speed and tower damping.

where the filter zero z_I is a design parameter. The PI controller is connected in series with a parameter independent filter $G_{f1}(s)$. It is possible to provide an extra signal by using an accelerometer mounted in the nacelle, allowing the controller to better recognize between the effect of wind speed disturbances and tower motion on the measured power or generator speed. With this extra feedback signal, tower bending moment loads can be reduced without significantly affecting speed or power regulation. Therefore, it is assumed that tower velocity $\dot{q}$ is available for measurement, by integrating tower acceleration $\ddot{q}$, and is multiplied by a parameter-dependent constant $k_{\dot{q}}(\theta)$ for feedback. A parameter independent filter $G_{f2}(s)$ completes the tower feedback loop. The order of the filters $G_{f1}(s)$ and $G_{f2}(s)$ can be arbitrarily chosen. The choice trades-off closed-loop performance and number of controller states. High order filters leads to better performance with the expense of higher controller complexity.

The drive train of a wind turbine presents a poorly damped torsional mode when a constant torque control strategy is adopted. To counteract this, active drive train damping is deployed by adding a signal to the generator torque (Q_g) to compensate for the oscillations in the drive train. For a didactic and clear exposure, the compensation of the drive train damper is considered ideal. Therefore, for synthesis purposes, the drive train torsional mode is neglected and the rotor speed is proportional to the generator speed. The LPV controller can now be designed to trade off the tracking of generator speed and tower oscillations with control effort (wear on pitch actuator). Wind turbine aerodynamics is the main source of nonlinearities. A linearization-based LPV model depends on partial derivatives of aerodynamic torque (Q) and thrust (T) forces with respect to rotor speed (Ω_r), wind speed (V) and pitch angle. These partial derivatives, also known as aerodynamic gains, vary with the operating point. Thus, they are natural candidates for the basis functions [15],

$$\rho_1 := \frac{1}{J_{\mathrm{e}}} \left.\frac{\partial Q}{\partial \Omega}\right|_\theta, \quad \rho_2 := \frac{1}{J_{\mathrm{e}}} \left.\frac{\partial Q}{\partial V}\right|_\theta, \quad \rho_3 := \frac{1}{J_{\mathrm{e}}} \left.\frac{\partial Q}{\partial \beta}\right|_\theta,$$
$$\rho_4 := \frac{1}{M_{\mathrm{t}}} \left.\frac{\partial T}{\partial \Omega}\right|_\theta, \quad \rho_5 := \frac{1}{M_{\mathrm{t}}} \left.\frac{\partial T}{\partial V}\right|_\theta, \quad \rho_6 := \frac{1}{M_{\mathrm{t}}} \left.\frac{\partial T}{\partial \beta}\right|_\theta. \tag{5.11}$$

In the above expressions, J_r and J_g is the rotor and generator inertia, which combined with the gearbox ratio N_g results in the equivalent rotational inertia in the rotor side $J_{\mathrm{e}} := J_{\mathrm{r}} + J_{\mathrm{g}} N_{\mathrm{g}}^2$. M_t is the equivalent modal mass of the first bending moment of the tower. Basis functions with equivalent inertia and tower mass were chosen to improve numerical conditioning. The operating point of a wind turbine varies according to the effective wind speed $\theta(t) = \bar{V}(t)$ driving the rotor. The dynamic model of the variable-speed wind turbine can then be expressed as an LPV model of the form,

$$G : \begin{cases} \dot{x} = A(\theta)\, x + B_w(\theta)\, \hat{V} + B_u(\theta)\, \beta_{\mathrm{ref}} \\ y = C_y\, x \end{cases}$$

where states, controllable input and measurements are,

$$x = \begin{bmatrix} \Omega_{\mathrm{r}} & \dot{q} & q & \dot{\beta} & \beta & x_{\Omega,i} \end{bmatrix}^T, \quad u = \beta_{\mathrm{ref}}, \quad y = \begin{bmatrix} \Omega_{\mathrm{g}} & y_{\Omega,i} & \dot{q} \end{bmatrix}^T,$$

with open-loop system matrices,

$$A(\theta) = \begin{bmatrix} \rho_1(\theta) & -\rho_2(\theta) & 0 & 0 & \rho_3(\theta) & 0 \\ \rho_4(\theta) & -\dfrac{1}{M_{\mathrm{t}}}B_{\mathrm{t}} - \rho_5(\theta) & -\dfrac{K_{\mathrm{t}}}{M_{\mathrm{t}}} & 0 & \rho_6(\theta) & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}2\zeta\omega_{\mathrm{n}} & -\omega_{\mathrm{n}}^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ N_g & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_w(\theta) = \begin{bmatrix} \rho_2(\theta) & \rho_5(\theta) & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$B_u = \begin{bmatrix} 0 & 0 & 0 & \omega_{\mathrm{n}}^2 & 0 & 0 \end{bmatrix}^T, \quad C_y = \begin{bmatrix} N_{\mathrm{g}} & 0 & 0 & 0 & 0 \\ z_{\mathrm{I}} & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Notice the PI controller integrator filter G_{I} conveniently augmented into the state-space of G, represented by the state $x_{\Omega,i}$ and the output $y_{\Omega,i}$. The plant G_{p} is defined as the wind turbine model solely (plant G without the augmentation of G_{I}). A state-space realization of the control structure depicted in Fig. 5.1 is given by,

$$A_{\mathrm{c}} := \begin{bmatrix} A_{\mathrm{f}1} & 0 \\ 0 & A_{\mathrm{f}2} \end{bmatrix}, \quad B_{\mathrm{c}}(\theta) := \begin{bmatrix} B_{k_p}(\theta) & B_{k_i}(\theta) & 0 \\ 0 & 0 & B_{k_{\dot{q}}}(\theta) \end{bmatrix},$$
$$C_{\mathrm{c}} := \begin{bmatrix} C_{\mathrm{f}1} & C_{\mathrm{f}2} \end{bmatrix}, \quad D_{\mathrm{c}} := \begin{bmatrix} D_{\mathrm{f}1} & D_{\mathrm{f}2} \end{bmatrix}.$$

where the size of sub-matrices depends on the chosen orders of the filters. The parameter-dependent controller matrix has the general dependence form,

$$\begin{bmatrix} B_{k_p}(\theta) & B_{k_i}(\theta) & 0 \\ 0 & 0 & B_{k_{\dot{q}}}(\theta) \end{bmatrix} := \begin{bmatrix} B_{k_p} & B_{k_i} & 0 \\ 0 & 0 & B_{k_{\dot{q}}} \end{bmatrix}_0 + \sum_{m=1}^{n_\rho} \begin{bmatrix} B_{k_p} & B_{k_i} & 0 \\ 0 & 0 & B_{k_{\dot{q}}} \end{bmatrix} \rho_{\mathrm{m}}(\theta).$$

The Lyapunov matrix is chosen dependent on all basis functions (5.11), and the slack matrix is chosen parameter independent.

Weight W_{z1} and W_u governs the tradeoff between rotational speed regulation and pitch wear. In this example, W_{z1} is chosen as a scalar k_1, turning the first performance channel similar to an integral square error measure ($z_1 = W_{z1}G_{I}\hat{\Omega}_r$). W_u is taken as a first order high-pass filter that penalizes high-frequency content on the pitch angle. Due to the resonance characteristics of the transfer function from $\hat{V}$ to $\dot{q}$, the weighting function W_{z2} is chosen as a scalar k_2, that tradeoffs the desired tower damping. Considering the plant and weighting functions just mentioned, the augmented plant has 7 states. $G(s)_{\mathrm{fl}}$ and $G(s)_{\mathrm{fl}}$ are chosen as first order and second order filters, respectively, therefore the controller is comprised of 3 states.

Remember that the iterative LMI algorithm is a synthesis procedure in discrete time. Therefore, the augmented LPV plant in continuous time is discretized using a bilinear (Tustin) approximation [16] with sampling time $T_{\mathrm{s}} = 0.02$ s, at each point $\Theta_g \times \mathrm{Vert}\ \mathcal{V}$. The effective wind speed ranges $\theta = \bar{V} \in [12\ \mathrm{m/s},\ 25\ \mathrm{m/s}]$ and its rate of variation ranges $\Delta\theta(t) = \Delta\bar{V}(t) \in [-2\ \mathrm{m/s^2},\ 2\ \mathrm{m/s^2}]$. The grid is comprised of seven equidistant points. The rate of variation of the scheduling variables in continuous-time must as well be converted to discrete-time by the relation $\Delta\theta(k) = T_s\Delta\theta(t)$.

The numerical example is based on data from a typical 2MW utility scale wind turbine. The evolution of radius $r^{(j)}$ and performance level $\gamma^{(j)}$ during the course of the optimization is illustrated on Fig. 5.2. During the feasibility phase, as the radius gradually converges to 1, the performance level value increases. The algorithm switches to optimization phase by maintaining $r = 1$ during the subsequent iterations, being the cost monotonically decreasing to a stationary point.

Wind disturbance step responses under different operating points (frozen θ) are depicted in Fig. 5.3. The rotor speed is well regulated around the origin. A similar response irrespective of the operating point is noticeable, meaning that the controller is gain-scheduling to adapt to the nonlinearities of the plant. This is corroborated by the magnitude plots of transfer functions from wind disturbance to rotor speed and tower velocity, for the open-loop and closed-loop systems. The increased damping of the tower fore-aft motion is noticeable in Fig.5.4d where the magnitude of the open-loop system (dashed line) is plotted for comparison.

ACKNOWLEDGMENTS

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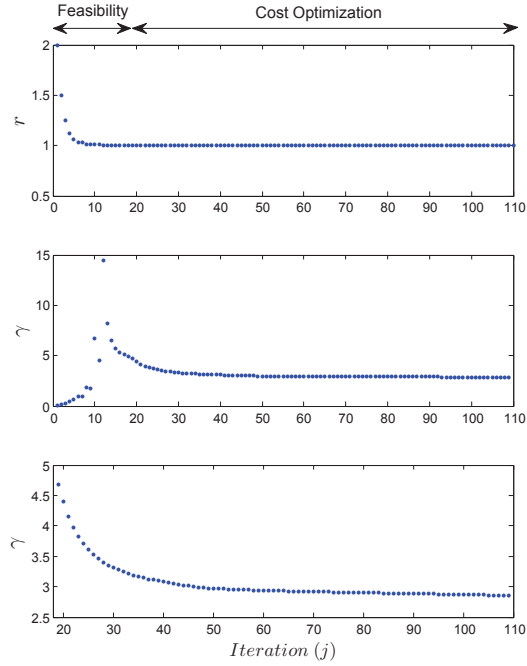
Figure 5.2: Evolution of variables during the iterative LMI optimization.
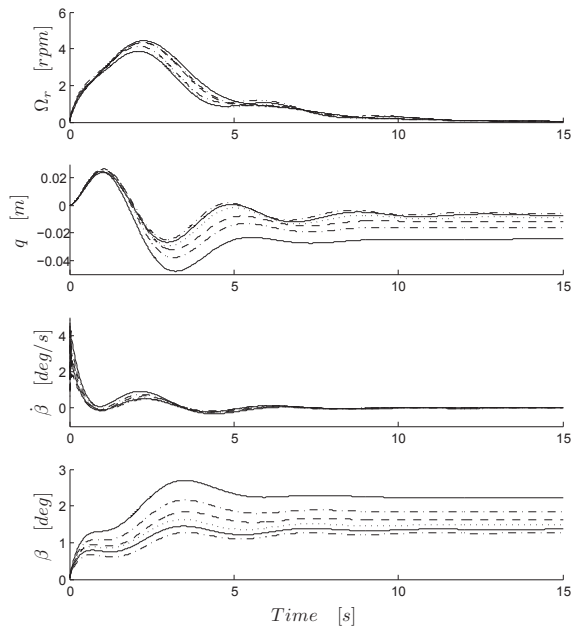
Figure 5.3: Step responses under different operating points.

(a) $T_{\hat{V} \to \Omega_r}$, Open-loop

(b) $T_{\hat{V} \to \Omega_r}$, Closed-loop

(c) $T_{\hat{V} \to \dot{q}}$, Open-loop

(d) $T_{\hat{V} \to \dot{q}}$, Closed-loop
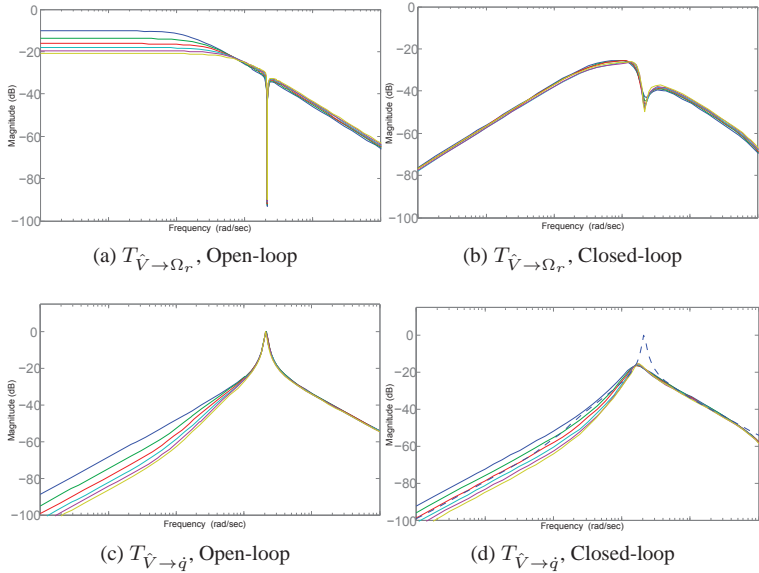
Figure 5.4: Magnitude of transfer functions from wind disturbance to rotor speed and tower velocity.

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Paper C

$\mathcal{H}_\infty/\mathcal{H}_2$ Model Reduction Through Dilated Linear Matrix Inequalities

Fabiano Daher Adegas and Jakob Stoustrup

Abstract

This paper presents sufficient dilated linear matrix inequalities (LMI) conditions to the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ model reduction problem. A special structure of the auxiliary (slack) variables allows the original model of order n to be reduced to an order $r = n/s$ where $n, r, s \in \mathbb{N}$. Arbitrary order of the reduced model can be enforced by including states in the original system with negligible input-to-output system norms. The use of dilated LMI conditions facilitates model reduction of parameter-dependent systems. When a reduced model determined by the sufficient LMI conditions does not satisfactorily approximates the original system, an iterative algorithm based on dilated LMIs is proposed to significantly improve the approximation bound. The effectiveness of the method is accessed by numerical experiments. The method is also applied to the $\mathcal{H}_2$ order reduction of a flexible wind turbine model.

1   Introduction

The model reduction problem consists on the approximation of a given asymptotically stable system by a reduced order model according to a given minimum norm criteria on the approximation error. Several techniques and norm measures were investigated, giving rise to numerically reliable algorithms. A comparison of some of the algorithms for model reduction can be found in [1]. This problem can be formulated as an optimization problem with rank constraints [2] or posed as a set of nonlinear matrix equations [3]. Due to the inherent non-convexity of these problems, they are very difficult to solve.

More recently, model reduction has been investigated under the linear matrix inequalities (LMI) framework, facilitating the use of classical norm criteria for the reduction error like $\mathcal{H}_\infty$ [4] and $\mathcal{H}_2$. This framework is particularly suitable to address multichannel / mixed problems as well as uncertain models [5, 6, 7]. Unfortunately, the difficulties of non-convexity remains when formulating the model reduction problem as an LMI, typically involving an additional rank constraint [8] or resulting in bilinear matrix inequalities [9, 10]. In order to circumvent the non-convexity of the problem, some authors reformulate the non-convex constraint by a linear constraint presenting a matrix variable that is fixed a priori [7, 11]. The choice of the fixed variable influences the degree of suboptimality.

In this paper, we explore the usage of dilated (or extended) LMIs to the model reduction problem. See [12] for a survey on the history and different characterizations proposed in the literature. Dilated LMIs are composed of instrumental (slack) variables which facilitates a linear dependence of the LMI in the Lyapunov variables. This added flexibility is valuable for reducing conservatism in robust and multi-objective control. A sufficient LMI condition with a special structure of the slack variables is here proposed, allowing an original model of order n to be reduced to an order $r = n/s$ where $n, r, s \in \mathbb{N}$. Arbitrary order of the reduced model can be enforced by including states in the original system with negligible input-to-output system norms. This slack variable structure is trivially extended to cope with robust and parameter-dependent model reduction. When a reduced model determined by the sufficient LMI conditions does not satisfactorily approximates the original system, an iterative algorithm based on dilated LMIs is proposed to significantly improve the approximation bound with the expense of higher computational cost. The effectiveness of the method is accessed by numerical experiments. The method is successfully applied to the $\mathcal{H}_2$ order reduction of a flexible wind turbine.

2 Model Reduction Through Dilated LMI

Linear Time-Invariant Systems

We initially consider a stable MIMO LTI dynamical system of order n in state-space form

$$S : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{6.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$, $D \in \mathbb{R}^{n_y \times n_u}$. We seek a model of order $r < n$ denoted S_r

$$S_r : \begin{cases} \dot{x}_r = A_r x_r(t) + B_r u(t) \\ y(t) = C_r x_r(t) + D_r u(t) \end{cases} \tag{6.2}$$

where $A_r \in \mathbb{R}^{r \times r}$, $B_r \in \mathbb{R}^{r \times n_u}$, $C_r \in \mathbb{R}^{n_y \times r}$, $D_r \in \mathbb{R}^{n_y \times n_u}$ such that the input-output difference between the original system S and the reduced system S_r is small in an $\mathcal{H}_\infty$ or $\mathcal{H}_2$-norm sense. That is

$$\|S - S_r\|_{\infty \text{ or } 2} \le \gamma \tag{6.3}$$

where γ represents the upper bound on $\mathcal{H}_\infty$ or $\mathcal{H}_2$, depending on the context. The input-output difference of S and S_r can be represented by the following state-space description denoted ΔS

$$\Delta S : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + \begin{bmatrix} B \\ B_r \end{bmatrix} u(t) \\ y_\Delta(t) = \begin{bmatrix} C & -C_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + \left(D - D_r \right) u(t) \end{cases} \tag{6.4}$$

Hereafter, the system matrices of ΔS are denoted $A_\Delta, B_\Delta, C_\Delta, D_\Delta$. Our results benefit from the dilated LMI conditions for an upper bound on $\mathcal{H}_\infty$ [13] or $\mathcal{H}_2$ [14]. Please consult [12] and references therein for a throughout exposure of dilated LMIs; we state them already in the context of our problem.

Lemma 4. $\|S - S_r\|_\infty \le \gamma$ *holds if, and only if, there exist a general auxiliary matrix Q, symmetric matrix X and a scalar $\mu > 0$ such that*

$$\begin{bmatrix} A_\Delta Q + Q^T A_\Delta^T & \star & \star & \star \\ \mu Q^T A^T - Q + X & -\mu(Q + Q^T) & \star & \star \\ C_\Delta Q & \mu\, C_\Delta Q & -\gamma I & \star \\ B_\Delta^T & 0 & D_\Delta^T & -\gamma I \end{bmatrix} \prec 0, \tag{6.5}$$

is satisfied.

The multiplication between the scalar μ and matrix variables in (6.5) makes a line search in μ necessary.

Lemma 5. $\|S - S_r\|_2 \leq \gamma$ *holds if, and only if, there exist a general auxiliary matrix* Q *and symmetric matrices* X, Z *such that*

$$
\begin{bmatrix}
A_\Delta Q + Q^T A_\Delta^T & \star & \star \\
-Q^T A_\Delta^T + Q - X & -(Q + Q^T) & \star \\
C_\Delta Q & -C_\Delta Q & -\gamma I
\end{bmatrix} \prec 0,
$$

$$
\begin{bmatrix}
Z & \star \\
B_\Delta & X
\end{bmatrix} \succ 0, \quad trace\,(Z) < \gamma
\tag{6.6}
$$

is satisfied.

The previous two lemmas state conditions for analysis of ΔS. In order to derive conditions to synthesize the reduced order matrices $A_r, \ldots, D_r$, let the general auxiliary matrix Q be partitioned as

$$
Q := \begin{bmatrix}
Q_1 & Q_2 & \ldots & Q_{s+1} \\
H & H & \ldots & H
\end{bmatrix}
\tag{6.7}
$$

where $Q_k, \in \mathbb{R}^{n \times r}$, $k = 1, \ldots, s+1$, $H \in \mathbb{R}^{r \times r}$, and $r = n/s$, r, n, $s \in \mathbb{N}$. Also define new matrix variables $\hat{A}_r$ and $\hat{C}_r$ resulting from the nonlinear change of variables

$$
\hat{A}_r := A_r H, \quad \hat{C}_r := C_r H.
\tag{6.8}
$$

With these definitions at hand, the LMI conditions for synthesis can be stated as follows.

Theorem 9. $\|S - S_r\|_\infty \leq \gamma$ *holds if there exist general auxiliary matrices* Q_k, $k = 1, \ldots, s+1$ *and* H, *symmetric matrix* X, *general matrices* $\hat{A}_r$, B_r, $\hat{C}_r$, D_r *and a scalar* $\mu > 0$ *such that*

$$
\begin{bmatrix}
\hat{A}_\Delta + \hat{A}_\Delta^T & \star & \star & \star \\
\mu\,\hat{A}_\Delta^T - Q + X & -\mu\,(Q + Q^T) & \star & \star \\
\hat{C}_\Delta & \mu\,\hat{C}_\Delta & -\gamma I & \star \\
B_\Delta^T & 0 & D_\Delta^T & -\gamma I
\end{bmatrix} \prec 0,
\tag{6.9a}
$$

$$
\hat{A}_\Delta := \begin{bmatrix}
AQ_1 & AQ_2 & \ldots & AQ_{s+1} \\
\hat{A}_r & \hat{A}_r & \ldots & \hat{A}_r
\end{bmatrix},
$$

$$
B_\Delta := \begin{bmatrix} B \\ B_r \end{bmatrix}, \quad D_\Delta := D - D_r
\tag{6.9b}
$$

$$
\hat{C}_\Delta := \begin{bmatrix} CQ_1 - \hat{C}_r & CQ_2 - \hat{C}_r & \ldots & CQ_{s+1} - \hat{C}_r \end{bmatrix},
$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to

$$
A_r = \hat{A}_r H^{-1}, \quad C_r = \hat{C}_r H^{-1}.
\tag{6.10}
$$

Proof. The LMIs (6.9) are obtained by trivial manipulations of (6.5), (6.7) and resorting to the nonlinear change of variables (6.8). To show that $\hat{A}_r$ and $\hat{C}_r$ can always be reconstructed according to (6.10), H should be invertible thus nonsingular. The fact that $-\mu \left(Q + Q^T \right) \prec 0$ with $\mu > 0$ implies nonsingularity of Q. Notice that H is the lower-right block of Q (see (6.7)), thus nonsingularity of H is also guaranteed. $\qquad\square$

The same rationale can be applied to turn Lemma 5 into synthesis conditions.

Theorem 10. $\|S - S_r\|_2 \leq \gamma$ *holds if there exist general auxiliary matrices* Q_k, $k = 1, \ldots, s+1$ *and* H, *symmetric matrices* X, Z *and general matrices* $\hat{A}_r$, B_r, $\hat{C}_r$, D_r *such that*

$$
\begin{bmatrix}
\hat{A}_\Delta + \hat{A}_\Delta^T & \star & \star \\
-\hat{A}_\Delta^T + Q - X & -(Q + Q^T) & \star \\
\hat{C}_\Delta & -\hat{C}_\Delta & -\gamma I
\end{bmatrix} \prec 0,
$$
(6.11)

$$
\begin{bmatrix}
Z & \star \\
B_\Delta & X
\end{bmatrix} \succ 0, \quad trace\,(Z) < \gamma, \quad and\ Eq.\ (9b),
$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to (6.10)*.*

The chosen structure (6.7) of the auxiliary variable Q restrains the dimension of A_r to be $r = n/s$, or in words, the order of the reduced model is an integer fraction of the order of the original model. Therefore, the order of the reduced system can be chosen smaller by redefining the partitioning of Q. For example, in the case of $r = n/3$

$$
Q := \begin{bmatrix}
Q_1 & Q_2 & Q_3 & Q_4 \\
H & H & H & H
\end{bmatrix}
$$
(6.12)

with $\hat{A}_\Delta$ and $\hat{C}_\Delta$ changing accordingly

$$
\hat{A}_\Delta := \begin{bmatrix}
AQ_1 & AQ_2 & AQ_3 & AQ_4 \\
\hat{A}_r & \hat{A}_r & \hat{A}_r & \hat{A}_r
\end{bmatrix},
$$

$$
\hat{C}_\Delta := \begin{bmatrix} CQ_1 - \hat{C}_r & CQ_2 - \hat{C}_r & CQ_3 - \hat{C}_r & CQ_4 - \hat{C}_r \end{bmatrix}.
$$
(6.13)

Being n a multiple of r limits the choice of the order of the reduced model. This fact can be circumvented by adding states on the original system S with negligible input-to-output norms. A convenient way to do so is by augmenting the system with modes appearing in the diagonal of A

$$
A \rightarrow \begin{bmatrix} A & 0 \\ 0 & A_a \end{bmatrix}, \quad A_a = \mathrm{diag}(A_{a,i}), \quad B \rightarrow \begin{bmatrix} B \\ B_a \end{bmatrix},
$$

$$
B_a = \begin{bmatrix} B_{a,1} \\ B_{a,2} \\ \vdots \\ B_{a,i} \end{bmatrix}, \quad C \rightarrow \begin{bmatrix} C & C_a \end{bmatrix},
$$

$$
C_a = \begin{bmatrix} C_{a,1} & C_{a,2} & \cdots & C_{a,i} \end{bmatrix}, \quad i = 1, \ldots, n_a.
$$

where $\|C(sI - A)^{-1}B + D\| >> \|C_a(sI - A_a)^{-1}B_a\|$. An arbitrary order r can be chosen by combining both strategies.

Parameter Dependent Systems

The linear time invariant conditions just presented are trivially extended to cope with model reduction of parameter-dependent (PD) systems. Consider the linear PD system of order n

$$S(\alpha) : \begin{cases} \dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t) \\ y(t) = C(\alpha)x(t) + D(\alpha)u(t). \end{cases} \tag{6.14}$$

System matrices are polytopic with respect to the parameter α

$$\begin{matrix} A(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i A_i, & B(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i B_i, \\ C(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i C_i, & D(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i D_i, \end{matrix} \quad \Lambda := \left\{ \alpha : \sum_{i=1}^{N_\alpha} \alpha_i = 1, \ \alpha_i \geq 0 \right\} \tag{6.15}$$

as well as the symmetric matrices

$$X(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i X_i, \qquad Z(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i Z_i, \tag{6.16}$$

where $\alpha \in \Lambda$. The aim is to find a reduced system $S_r(\alpha)$ with order $r < n$ and structure analogous to (6.14), (6.15) such that $\|S(\alpha) - S_r(\alpha)\| \leq \gamma$ for all $\alpha \in \Lambda$. The auxiliary matrices are considered parameter independent defined according to (6.7). New matrix variables $\hat{A}_{r,i}$ and $\hat{C}_{r,i}$ result from the nonlinear change of variables involving the reduced order matrices $A_{r,i}$ and $C_{r,i}$

$$\hat{A}_r(\alpha) := \sum_{i=1}^{N_\alpha} \alpha_i \hat{A}_{r,i}, \quad \hat{A}_{r,i} = A_{r,i}H,$$

$$\hat{C}_r(\alpha) := \sum_{i=1}^{N_\alpha} \alpha_i \hat{C}_{r,i}, \quad \hat{C}_{r,i} = C_{r,i}H, \tag{6.17}$$

$$i = 1, \dots, N_\alpha.$$

Theorem 11. $\|S(\alpha) - S_r(\alpha)\|_\infty \leq \gamma$ *holds if there exist general auxiliary matrices* Q_k, $k = 1, \dots, s+1$ *and* H, *symmetric matrices* X_i, *general matrices* $\hat{A}_{r,i}$, $B_{r,i}$, $\hat{C}_{r,i}$, $D_{r,i}$ *and a scalar* $\mu > 0$ *such that*

$$\begin{bmatrix} \hat{A}_{\Delta i} + \hat{A}_{\Delta i}^T & \star & \star & \star \\ \mu \, \hat{A}_{\Delta i}^T - Q + X_i & -\mu \, (Q + Q^T) & \star & \star \\ \hat{C}_{\Delta i} & \mu \, \hat{C}_{\Delta,i} & -\gamma I & \star \\ B_{\Delta i}^T & 0 & D_{\Delta i}^T & -\gamma I \end{bmatrix} \prec 0, \tag{6.18a}$$

$$\hat{A}_{\Delta i} := \begin{bmatrix} A_i Q_1 & A_i Q_2 & \dots & A_i Q_{s+1} \\ \hat{A}_{r,i} & \hat{A}_{r,i} & \dots & \hat{A}_{r,i} \end{bmatrix},$$

$$B_{\Delta i} := \begin{bmatrix} B_i \\ B_{r,i} \end{bmatrix}, \quad D_{\Delta i} := D_i - D_{r,i} \tag{6.18b}$$

$$\hat{C}_{\Delta i} := \begin{bmatrix} C_i Q_1 - \hat{C}_{r,i} & C_i Q_2 - \hat{C}_{r,i} & \dots & C_i Q_{s+1} - \hat{C}_{r,i} \end{bmatrix},$$

$$i = 1, \dots, N_\alpha,$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to

$$A_{r,i} = \hat{A}_{r,i} H^{-1}, \quad C_{r,i} = \hat{C}_{r,i} H^{-1}. \tag{6.19}$$

Theorem 12. $\|S - S_r\|_2 \le \gamma$ *holds if there exist general auxiliary matrices* Q_k, $k = 1, \dots, s+1$ *and* H, *symmetric matrices* X_i, Z_i *and general matrices* $\hat{A}_{r,i}$, $B_{r,i}$, $\hat{C}_{r,i}$, D_r *such that*

$$\begin{bmatrix} \hat{A}_{\Delta i} + \hat{A}_{\Delta i}^T & \star & \star \\ -\hat{A}_{\Delta i}^T + Q - X_i & -(Q + Q^T) & \star \\ \hat{C}_{\Delta i} & -\hat{C}_{\Delta i} & -\gamma I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} Z_i & \star \\ B_{\Delta i} & X_i \end{bmatrix} \succ 0, \quad trace\,(Z_i) < \gamma, \quad and\ Eq.\ (18b), \tag{6.20}$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to (6.19).

Iterative Algorithm

If the reduced system does not satisfactorily approximate the dynamics of the original system, one can resort to an iterative LMI (ILMI) algorithm based on dilated LMIs to find a better result. The reduced model resulted from the sufficient conditions just presented can be used to initialize the ILMI algorithm. The auxiliary (slack) variable Q is now considered a general matrix without any specific partitioning. The following matrices are also redefined under the ILMI context.

$$\hat{A}_{\Delta i} := \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} Q + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{r,i} & C_{r,i} \\ B_{r,i} & A_{r,i} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q$$

$$B_{\Delta i} := \begin{bmatrix} B_i \\ B_{r,i} \end{bmatrix}, \quad D_{\Delta i} := D_i - D_{r,i} \tag{6.21}$$

$$\hat{C}_{\Delta i} := \begin{bmatrix} C_i & 0 \end{bmatrix} Q + \begin{bmatrix} -I & 0 \end{bmatrix} \begin{bmatrix} D_{r,i} & C_{r,i} \\ B_{r,i} & A_{r,i} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q$$

For a clear exposure, only the ILMI algorithm for the $\mathcal{H}_\infty$ model reduction is described here. The $\mathcal{H}_2$ case can be treated similarly. The notation $(\cdot)^{\{j\}}$ stands for the iteration index. The algorithm solves LMI problems by successively fixing the reduced order matrices $A_{r,i}, \dots, D_{r,i}$ at one step and the slack variable Q at another step.

Algorithm 5. *Consider* $A_{r,i}^{\{1\}}$, $B_{r,i}^{\{1\}}$, $C_{r,i}^{\{1\}}$, $D_{r,i}^{\{1\}}$ *as the solution of Theorem 11. Set a tolerance* ϵ, $j = 1$ *and start to iterate:*

1. *Find* $Q^{\{j\}}$ *and* $\gamma^{\{j\}}$ *that solves the LMI problem:*

   Minimize $\gamma^{\{j\}}$ *subject to* (6.18a) *and* (6.21) *with fixed* $A_{r,i}^{\{j\}}$, ..., $D_{r,i}^{\{j\}}$,
   $$i = 1, \ldots, N_\alpha.$$

2. *Find* $A_{r,i}^{\{j\}}$, ..., $D_{r,i}^{\{j\}}$, $i = 1, \ldots, N_\alpha$ *and* $\gamma^{\{j\}}$ *that solves the LMI problem:*

   Minimize $\gamma^{\{j\}}$ *subject to* (6.18a) *and* (6.21) *with fixed* $Q^{\{j\}}$, $i = 1, \ldots, N_\alpha$.

3. *If* $|\gamma^{\{j\}} - \gamma^{\{j-1\}}| < \epsilon$, *stop. Else, set* $j = j + 1$ *and go to step 1.*

The Lyapunov matrices X_i act as variables during the whole optimization, a benefit of using dilated LMI conditions in an iterative scheme.

3 Numerical Examples

To solve the LMI problems, we have used the interface YALMIP [15] with semidefinite programming solver SeDuMi. Because the interest lies in finding reduced order models with minimal $\mathcal{H}_\infty$ / $\mathcal{H}_2$ norm bounds, the optimization objective *Minimize* γ are included in the LMI conditions just presented.

Comparison With Other Results

In this subsection, some results obtained by the proposed conditions are compared with [7] and references therein.

Example 1

Consider an LTI system with state-space matrices (6.22) [10] from which a first order model should be approximated in an $\mathcal{H}_2$-norm sense. This example gives us a glimpse of the conservativeness of the proposed condition in face of severe order reduction (1/6 of the original system) that may occur due to partitioning the slack variable (6.7).

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.007 & -0.114 & -0.850 & -2.800 & -4.450 & -3.400 \end{bmatrix} \tag{6.22}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$
$$C = \begin{bmatrix} 0.007 & 0.014 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain $\gamma^2 = 0.0283$ by applying Theorem 10 which is considerably close to [7] ($\gamma^2 = 0.0205$) and better than the results in [10] ($0.0557 \leq \gamma^2 \leq 0.0616$). Note that, in contrast to [7], no matrix involved on the formulation should be chosen a priori.

Example 2

A second-order reduced model of the uncertain system with state-space representation [6]

$$A(\alpha) = \begin{bmatrix} -2 & 3 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & a(\alpha) & 12 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B(\alpha) = \begin{bmatrix} -2.5 & b(\alpha) & -1.2 \\ 1.3 & -1 & 1 \\ 1.6 & 2 & 0 \\ -3.4 & 0.1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -2.5 & 1.3 & 1.6 & -3.4 \\ 0 & -1 & 2 & 0.1 \\ -1.2 & 1 & 0 & 2 \end{bmatrix},$$

$$a(\alpha) = -3.5\alpha_1 - 2.5\alpha_2, \quad b(\alpha) = -0.5\alpha_1 + 0.5\alpha_2$$

should be obtained in an $\mathcal{H}_\infty$ sense. Firstly, a parameter-independent reduced system is found by applying Theorem 11 with $A_{r,i} = A_r, \ldots, D_{r,i} = D_r$, $i = 1, \ldots, N_\alpha$. The minimum upper bound $\gamma = 6.2139$ at $\mu = 0.22$ is more conservative than [7] ($\gamma = 5.54$). The resulting reduced-order system is

$$\begin{aligned} A_r &= \begin{bmatrix} 1.8061 & 0.2829 \\ -2.5823 & -4.7980 \end{bmatrix} \\ B_r &= \begin{bmatrix} 4.1664 & -0.1741 & -4.2018 \\ -2.2297 & 0.3139 & 2.9823 \end{bmatrix} \\ C_r &= \begin{bmatrix} 0.4039 & 1.0428 \\ 5.6700 & 8.7890 \\ -5.7262 & -5.3496 \end{bmatrix} \\ D_r &= \begin{bmatrix} 1.7012 & 1.3074 & 0.9660 \\ -2.0854 & 2.3911 & 0.4805 \\ 0.7488 & 1.1724 & -2.0965 \end{bmatrix} \end{aligned} \tag{6.23}$$

The ILMI Algorithm 1 is initialized with system (6.23) in an attempt to find a reduced system that better approximates the original one. Convergence tolerance is set to 1e-3. Figure 6.1a shows the convergence of $\gamma^{\{j\}}$ for three different values of $\mu = \{0.1, 0.22, 0.3\}$. For $\mu = 0.22$, the algorithm converges after 18 iteration to $\gamma = 3.995$. This upper bound is considerably better than [7]. For $\mu = 0.1$, the proposed algorithm finds a parameter-independent reduced model with approximation error $\gamma = 3.578$, less conservative than [6] where a parameter-dependent reduced system is determined by an alternating projection method.

A reduced model with the same parameter dependence as the original one is now desired. Therefore, $A_r(\alpha)$ and $B_r(\alpha)$ depends on the parameter α, while C_r, D_r are parameter independent. The sufficient LMI condition results in an upper bound $\gamma = 6.1080$ (for $\mu = 0.22$), slightly better than the parameter-independent reduced system case. When resorting to the ILMI algorithm to find a parameter-dependent reduced system, an $\mathcal{H}_\infty$ upper bound of $\gamma = 3.506$ is reached for $\mu = 0.1$, expectedly less conservative than the parameter-independent reduced system. The convergence of the algorithm is depicted in Fig. 6.1b.
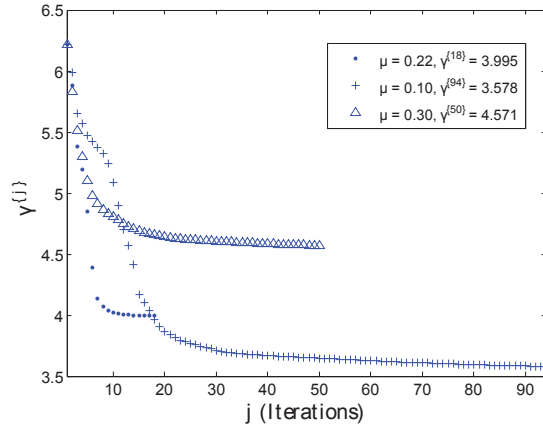
(a) Parameter-Independent
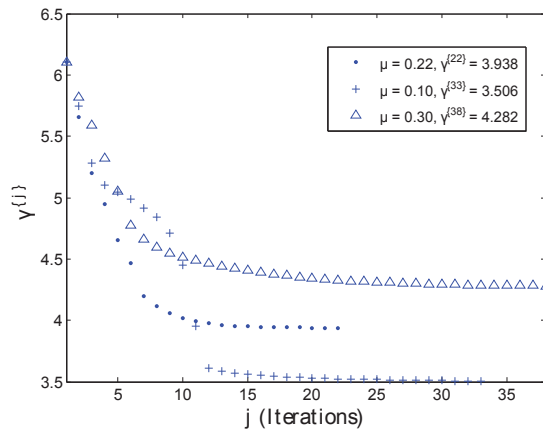
(b) Parameter-Dependent

Figure 6.1: Convergence of ILMI model reduction.

Flexible Wind Turbine

In the wind energy industry, wind turbine models are often derived from high fidelity aeroelastic numerical tools. These large-scale models are not suitable for control analysis and synthesis. The order reduction conditions here presented will be applied to an industrial wind turbine reference model generated by the recently developed aeroelastic code HAWCstab2 [16]. The LTI model contains several hundreds of flexible modes and aerodynamic delays, out of reach for the current capabilities of LMI-based model order reduction techniques. Therefore, the model was initially reduced from 880 to 20 states using a $\mathcal{H}_2$ modal truncation method in which modes with natural frequencies higher than 4 Hz were also discarded. The objective here is to reduce from 20 states to 10 states without compromising the quality of the model in an $\mathcal{H}_2$ sense. Thus, $n = 20, s = 2, r = 10$. Magnitude plots in frequency domain of the original and reduced models are depicted in Fig. 6.2a.

(a) $\mathcal{H}_2$-norm

(b) Balanced Truncation
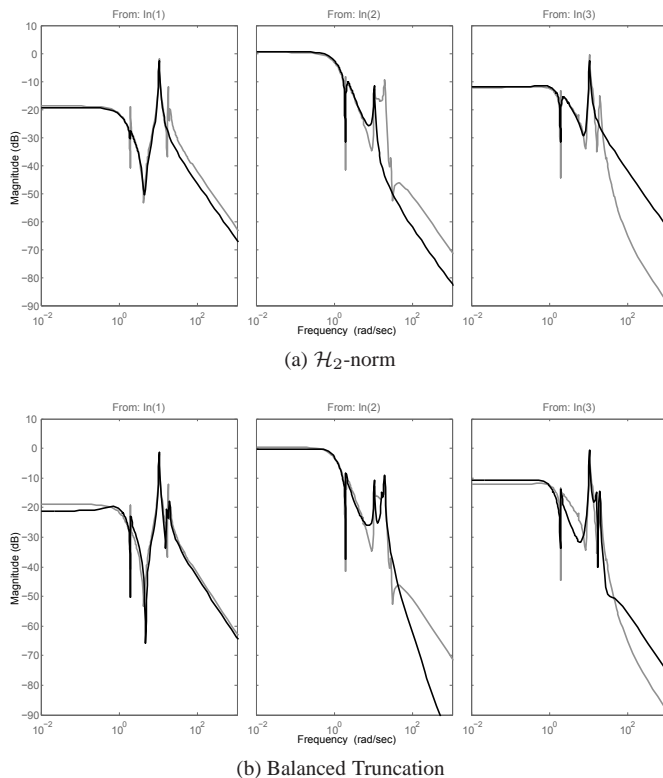
Figure 6.2: Magnitude diagrams of a flexible wind turbine model with 20 states (gray) and reduced 10 states (dark).

Inputs 1 and 2 are controllable signals of generator torque and pitch angle, respectively, while input 3 is the wind speed disturbance. The output channel is the wind turbine rotational speed. Another 10 state model were derived based on the well known balanced truncation model reduction scheme using the MATLAB command *balred*. The comparison with the original model, in this case, is depicted in Fig. (6.2b). When compared to balanced truncation, the $\mathcal{H}_2$ measure seems to be more appropriate in approximating the low frequency range of the model.

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Paper D

Reduced-Order LPV Model of Flexible Wind Turbines from High Fidelity Aeroelastic Codes

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Abstract

Linear aeroelastic models used for stability analysis of wind turbines are commonly of very high order. These high-order models are generally not suitable for control analysis and synthesis. This paper presents a methodology to obtain a reduced-order linear parameter varying (LPV) model from a set of high-order linear time invariant (LTI) models. Firstly, the high-order LTI models are locally approximated using modal and balanced truncation and residualization. Then, an appropriate coordinate transformation is applied to allow interpolation of the model matrices between points on the parameter space. The obtained LPV model is of suitable size for designing modern gain-scheduling controllers based on recently developed LPV control design techniques. Results are thoroughly assessed on a set of industrial wind turbine models generated by the recently developed aeroelastic code HAWCStab2.

1 Introduction

Linear aeroelastic models used for stability analysis of wind turbines are commonly of very high order. Multibody dynamics coupled with unsteady aerodynamics (e.g. dynamic stall) are among the recently developments in wind turbine aeroelasticity [1]. The resulting models contains hundreds or even thousands of flexible modes and aerodynamic delays. In order to synthesize wind turbine controllers, a common practice is to obtain linear time-invariant (LTI) models from a nonlinear model for different operating points. Modern control analysis and synthesis tools are inefficient for such high-order dynamical systems; reducing the model size is crucial to analyze and synthesize model-based controllers.

Model-based control of wind turbines has been extensively researched during the last decade [2]. The linear parameter varying (LPV) framework shown to be suitable to cope, in a systematic manner, with the inherent varying dynamics of a wind turbine over the operating envelope [3, 4, 5]. Wind turbine LPV models are usually simple, first-principles based, often neglecting dynamics related to aerodynamic phenomena and some structural modes. This in turn restricted LPV control of wind turbines to the academic environment only. A procedure to encapsulate high-fidelity dynamics of wind turbines as an LPV system would be beneficial to facilitate industrial use of LPV control.

This paper presents a procedure to obtain a reduced-order LPV wind turbine model from a set of high-order LTI models. Firstly, the high-order LTI models are locally approximated using modal and balanced truncation and residualization. Then, an appropriate manipulation of the coordinate system is applied to allow interpolation of the model matrices between points of the parameter space. The obtained LPV model is of suitable size for synthesizing modern gain-scheduling controllers based on the recent advances on LPV control design. Time propagation of the varying parameter is not explicitly utilized. Therefore, the procedure assumes that the varying parameter do not vary excessively fast in time, in line with common practices in gain-scheduling control [6]. Results are thoroughly assessed on a set of industrial wind turbine models derived by the recently developed aeroelastic code HAWCStab2.

This paper is organized as follows. The modeling principles of the high-order LTI wind turbine models are exposed in Section II. Section III is devoted to present the proposed method. Section IV brings a numerical example along with results. Conclusions and future work are discussed in Section V.

2   Wind Turbine Model

A nonlinear high-fidelity aeroelastic model is the starting point of the modeling procedure. The wind turbine structure is modeled with nonlinear kinematics based on co-rotational Timoshenko elements. Aerodynamics are modeled with Blade Element Momentum (BEM) coupled with unsteady aerodynamics based of shed-vorticity and dynamic stall. Linearization is performed analytically around a steady operational state for a given mean wind speed, rotor speed and collective pitch angle. Hansen [attached] gives a more complete description of the linear aeroelastic model for an isolated blade. Two main equations of motion, one related to structural dynamics and another related to aerodynamics contitutes the LTI model

$$
\begin{aligned}
M\ddot{q}_s(t) + (C + G + C_a)\,\dot{q}_s(t) + (K + K_{sf} + K_a)u_s(t) \\
+ A_f x_a(t) = F_s(t) \\
\dot{x}_a(t) + A_d x_a(t) + C_{sa}\dot{q}_s(t) + K_{sa}q_s(t) = F_a(t)
\end{aligned}
\tag{7.1}
$$

where q_s are the elastic and bearing degrees of freedom, x_a are aerodynamic state variables, M is the structural mass matrix, C the structural damping matrix (Rayleigh), G the gyroscopic matrix, C_a is the aerodynamic damping matrix, K the elastic stiffness matrix, K_{sf} the geometric stiffness matrix, K_a the aerodynamic stiffness matrix, A_f is the coupling of the structure to aerodynamic states, A_d represents aerodynamic time lags, C_{sa} and K_{sa} are coupling matrices to structural states. F_s and F_a represent forces due to actuators and wind disturbance. The equations in first order form are

$$
\begin{aligned}
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\end{aligned}
\tag{7.2a}
$$

$$
x(t) = \begin{bmatrix} x_a(t) & q_s(t) & \dot{q}_s(t) \end{bmatrix}^T \quad u(t) = \begin{bmatrix} Q_g(t) & \beta(t) & V(t) \end{bmatrix}^T
\tag{7.2b}
$$

where the controllable inputs are the generator torque Q_g and collective pitch angle β, and V is the uniform wind speed disturbance input. The model outputs considered here are the generator angular velocity Ω and tower top lateral displacement q. The first output is usually measured and feed to a speed controller that manipulates the pitch angle β. The second output can be utilized for lateral tower load mitigation by generator torque control [7]. The aeroelastic tool offers the possibility to select other inputs and ouputs, but we limit to the ones just mentioned to clearly expose the results.

3   Reduced Order LPV Model

Consider N_s stable multiple-input multiple-output (MIMO) LTI dynamical systems (7.2) of order n corresponding to parameter values $\theta^{(i)}$, $i = 1, 2, \ldots, N_s$,

$$
S_i : \begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i u(t) \\ y(t) = C_i x_i(t) + D_i u(t) \end{cases} \quad, \quad i = 1, \ldots, N_s.
\tag{7.3}
$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times n_u}$, $C_i \in \mathbb{R}^{n_y \times n}$, $D_i \in \mathbb{R}^{n_y \times n_u}$. We seek a reduced-order parameterized model $S(\theta)$ of order $r < n$ which approximates S_i,
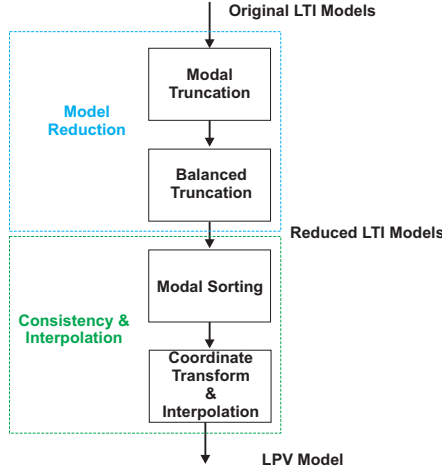
Figure 7.1: Scheme overview.

$$S(\theta) : \begin{cases} \dot{x} = A(\theta)x(t) + B(\theta)u(t) \\ y(t) = C(\theta)x(t) + D(\theta)u(t) \end{cases} \qquad (7.4)$$

where $A(\theta) \in \mathbb{R}^{r \times r}$, $B(\theta) \in \mathbb{R}^{r \times n_u}$, $C(\theta) \in \mathbb{R}^{n_y \times r}$, $D(\theta) \in \mathbb{R}^{n_y \times n_u}$ are continuous functions of a vector of varying parameters $\theta := [\theta_1, \theta_2, \ldots, \theta_{N_\theta}]^T$. The dynamics of the original system S_i and the approximated system $S(\theta)$ are assumed to evolve smoothly with respect to $\theta^{(i)}$ and θ, respectively. The parameter θ may represent the current operating point. It also may describe deviations on aerodynamics and structural properties for the sake of parametric model uncertainties. Plant parameters to be designed under an integrated plant-controller synthesis scheme could also be parameterized.

Variation in aerodynamic forces under structural vibration contributes significantly to changes in natural frequencies and damping of some structural modes. A specific procedure that takes these particularities into account is proposed here. A flowchart containing the required steps is depicted in Fig. 7.1.

Known methods for model reduction constitutes the proposed scheme and are briefly explained in the sequel, in the context of our application. Consult the survey of [8] for a more comprehensive exposure on model reduction.

Model Reduction

A reduced order model is commonly obtained by truncation of appropriate states. Let the state vector x_i be partitioned into $x_i := [x_{r,i} \quad x_{t,i}]^T$ where $x_{r,i}$ is the vector of retained states and $x_{t,i}$ is the vector of truncated states. The original system is partitioned accordingly

$$\begin{bmatrix} \dot{x}_{r,i}(t) \\ \dot{x}_{r,i}(t) \end{bmatrix} = \begin{bmatrix} A_{rr,i} & A_{rt,i} \\ A_{tr,i} & A_{tt,i} \end{bmatrix} \begin{bmatrix} x_{r,i}(t) \\ x_{t,i}(t) \end{bmatrix} + \begin{bmatrix} B_{r,i} \\ B_{t,i} \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} C_{r,i} & C_{t,i} \end{bmatrix} \begin{bmatrix} x_{r,i}(t) \\ x_{t,i}(t) \end{bmatrix} + Du(t) \tag{7.5}$$

and the reduced model is simply given by the state-space equation of the retained states

$$\dot{x}_{r,i} = A_{rr,i} x_i(t) + B_{r,i} u(t),$$
$$y = C_{r,i} x_{r,i} + Du(t) \tag{7.6}$$

If the original model is a stable system so is its truncated counterpart. While truncation tends to produce a good approximation in the frequency domain, the zero frequency gains (DC gains) are not guaranteed to match. This can be of particular importance in a wind turbine model because some aerodynamic states may not influence the transient behaviour but can contribute significantly to low frequency gains. Matching DC gains can be enforced by a model residualization method by setting the derivative of $x_{t,i}$ to zero in (7.5) and solving the resulting equation for $x_{r,i}$. After trivial manipulations, the reduced model is given by

$$\dot{x}_{r,i} = \left[A_{rr} - A_{rt} A_{tt}^{-1} A_{tr} \right]_i x_{r,i} + \left[B_r - A_{rt} A_{tt}^{-1} B_t \right]_i u(t)$$
$$y = \left[C_r - C_t A_{tt}^{-1} A_{tr} \right]_i x_{r,i} + \left[D - C_t A_{tt}^{-1} B_t \right]_i u(t) \tag{7.7}$$

Note that $A_{tt,i}$ is assumed invertible for (7.7) to hold. Residualization is performed in both modal and balanced reduction steps.

Modal Truncation

Due to size and numerical properties associated with large size systems and low damped dynamics, most model reduction algorithms based on Hankel singular values fail to produce a good reduced model. In order to start the reduction process, the original model is truncated to an intermediate size for subsequent reduction in a more accurate way. In modal form the system is put into a modal realization before states are truncated [9]. The modal form realization has the state matrix A is in block diagonal form with either 1×1 or 2×2 blocks when the eigenvalue is real or complex, respectively. Let system S_i be represented in modal form,

$$S_{m,i} : \begin{cases} \dot{x}_i(t) = A_{m,i} x_{m,i}(t) + B_{m,i} u(t) \\ y(t) = C_{m,i} x_{m,i}(t) + D_{m,i} u(t) \end{cases} \tag{7.8a}$$

$$A_{m,i} = \text{diag}(A_{m,k,i}),$$

$$A_{m,k,i} = -e_{k,i} \quad \text{for real eigenvalues},$$

$$A_{m,k,i} = \begin{bmatrix} -\xi_{k,i}\omega_{k,i} & \omega_{k,i}\sqrt{1-\xi^2} \\ \omega_{k,i}\sqrt{1-\xi^2} & -\xi_{k,i}\omega_{k,i} \end{bmatrix} \quad \text{for complex eigen.}$$

$$B_{m,i} = \begin{bmatrix} B_{m,1,i} \\ B_{m,2,i} \\ \vdots \\ B_{m,k,i} \end{bmatrix}, \quad C_{m,i} = \begin{bmatrix} C_{m,1,i} & C_{m,2,i} & \cdots & C_{m,k,i} \end{bmatrix}$$

(7.8b)

$$i = 1, \ldots, N_s, \quad k = 1, \ldots, N_m.$$

where N_m is the number of modes, $\xi_{k,i}$ and $\omega_{k,i}$ are the damping ratio and natural frequency of mode k and model i. The diagonal blocks are usually arranged in ascending order according to their eigenvalue magnitudes. The magnitude of a complex eigenvalue is $\omega_{k,i}$ while for a purely real eigenvalue is $e_{k,i}$. The retained states are then the ones with magnitudes smaller than a chosen treshhold $\bar{\omega}$. The intermediate model must contain all modes within the frequencies of interest for control design. A large number of states (300 to 450) is expected at this stage since many modes are of low frequency.

Balanced Truncation

The order of the intermediate system is further reduced by balanced truncation. In balanced truncation [10] the system is transformed to a balanced realization. A MIMO LTI system of the form (7.3) is said to be balanced if, and only if, its controllability and observability grammians are equal and diagonal, i.e. $P_i = Q_i = \text{diag}(\sigma_1, \ldots, \sigma_n)$, where $\sigma_1, \ldots, \sigma_n$ denotes the Hankel singular values sorted in decreasing order and matrices P_i, Q_i are the controllability and the observability Gramians. The gramians are solutions of the following Lyapunov equations

$$A_i P_i + P_i A_i^T + B_i B_i^T = 0$$
$$A_i^T Q_i + Q_i A_i + C_i^T C_i = 0$$

(7.9)

If this holds, the balanced system is given by

$$S_{b,i} : \begin{cases} \dot{x}_{b,i} = W_i^T A_i V_i x_{b,i}(t) + W_i^T B_i u_i(t) \\ y(t) = C_i V_i x_{b,i}(t) + D_i u_i(t) \end{cases}$$

(7.10)

$$i = 1, \ldots, N_s.$$

where $x_b \in \mathbb{R}^n$, $V = UZ\Sigma^{-1/2}$ and $W = LY\Sigma^{-1/2}$, together with the factorizations $P = UU^T$, $Q = LL^T$ and the singular value decomposition $U^T L = Z\Sigma Y^T$ [11]. This state coordinate equalizes the input-to-state and state-to-output energy transfers, making the Hankel singular values a measure of the contribution of each state to the input/output behavior.

Denote $V_{i(r)}$ and $W_{i(r)}$ the first r columns of V_i and W_i. The reduced-order systems $\hat{S}_i$

$$\hat{S}_i : \begin{cases} \dot{\hat{x}}_i = \hat{A}_i \hat{x}_i(t) + \hat{B}_i u_i(t) \\ \hat{y}(t) = \hat{C}_i \hat{x}_i(t) + \hat{D}_i \hat{u}_i(t) \end{cases} \qquad i = 1, \ldots, N_s. \qquad (7.11)$$

are obtained by truncation when the projectors $V_{i(r)}$ and $W_{i(r)}$ are applied to the intermediate sized model. In words, the balanced truncation removes the states with low Hankel singular values, thus not much information about the system will be lost. When applied to a stable system, balanced truncation preserves stability and guarantes and an upper bound on the approximation error in an H_∞ sense [12]. Expected order of the final reduced system is 7 to 20 states. The choice of the final order depends on the required model complexity and admissible error between the full and reduced model.

State-Space Consistency & Interpolation

Consider the balanced reduced models $\hat{S}_i$ and put them in modal form. The first step towards a consistent state-space representation is to assure that all modes keep their positions in the state matrix throughout the parameter space. The second step to a consistent state-space is to ensure that values of the entries of the system matrices change smoothly between each LTI system. At this point, the system matrices cannot be readily interpolated because the modal and balanced similarity transformations applied to the original system are not unique. One could think of interpolating the system in modal form. Indeed, the state matrix A is unique up to a permutation of the location of the modes and could easily be interpolated, but the similarity transformation that puts the system in modal form is not unique. Therefore, matrices $\hat{B}$ and $\hat{C}$ may have entries with abrupt value changes. The balanced realization is unique up to a sign change and consequently abrubt sign changes in the system matrices may occur from one LTI system to another. As suggested by [13], these issues can be corrected by properly changing the sign of the correspondent eigenvectors. Istead of correcting the eigenvectors before similarity transformations, we propose to transform the reduced order LTI systems into a representation based on the companion canonical form. No unique canonical form for multivariable systems is known to exist [14]. However, there exist algorithms which, for a system under arbitrary similarity transformation, find a unique companion form [15]. One algorithm with such properties is implemented in the function *canon* of MATLAB. The companion form is poorly conditioned for most state-space computations [16]. In order to avoid numerical issues, each mode k of the reduced system in modal coordinates is transformed into a companion realization. The system matrices of this particular realization are

$$A_{c,i} = \text{diag}(A_{c,k,i}), \quad B_{c,i} = \begin{bmatrix} B_{c,1,i} \\ B_{c,2,i} \\ \vdots \\ B_{c,k,i} \end{bmatrix},$$

$$C_{c,i} = \begin{bmatrix} C_{c,1,i} & C_{c,2,i} & \dots & C_{c,k,i} \end{bmatrix},$$

$$\begin{cases} A_{c,k,i} = -a_{k,i} \\ B_{c,k,i} = \begin{bmatrix} 1 & b_{1,k,i} & \dots & b_{n_u-1,k,i} \end{bmatrix} \quad \text{for real eigenvalues,} \\ C_{c,k,i} = \begin{bmatrix} c_{1,k,i} \\ \dots \\ c_{n_y-1,k,i} \\ 0 \end{bmatrix} \\ A_{c,k,i} = \begin{bmatrix} 0 & -a_{k,i,1} \\ 1 & -a_{k,i,2} \end{bmatrix}, \\ B_{c,k,i} = \begin{bmatrix} 0 & b_{11,k,i} & \dots & b_{1n_u-1,k,i} \\ 1 & b_{21,k,i} & \dots & b_{2n_u,k,i} \end{bmatrix} \quad \text{for complex eigen.} \\ C_{c,k,i} = \begin{bmatrix} c_{11,k,i} & \dots & c_{1r,k,i} \\ \vdots & \ddots & \vdots \\ c_{n_y1,k,i} & \dots & c_{n_yr,k,i} \end{bmatrix} \end{cases} \quad (7.12)$$

$$i = 1, \dots, N_s, \quad k = 1, \dots, N_m.$$

The characteristic polynomial of each mode appears in the rightmost column of the matrix $A_{c,k,i}$. The entries of $A_{c,k,i}$, $B_{c,k,i}$ and $C_{c,k,i}$ may be easily checked for possible inconsistencies of a particular mode, by detecting abrupt value changes between LTI systems. The state-space matrices are now at a realization suitable for interpolation. Let $z(\theta)$ be one matrix entry, function of θ. We focus on the polynomial dependence

$$z(\theta) = \sum_{k=1}^{N_p} \eta_k p_k(\theta) \quad (7.13)$$

where p_k is a set of multivariate polynomials on the parameters $\theta_1, \dots, \theta_{N_\theta}$ and η_k are coefficients to be determined. Let z_i be the values of a matrix element for $i = 1, \dots, N_s$. Define the following matrices

$$H = \begin{bmatrix} p_1(\theta^{(1)}) & \dots & p_{N_p}(\theta^{(1)}) \\ \vdots & \ddots & \vdots \\ p_1(\theta^{(N_s)}) & \dots & p_{N_p}(\theta^{(N_s)}) \end{bmatrix} = \begin{bmatrix} P_1 & \dots & P_{n_p} \end{bmatrix} \quad (7.14)$$

$$Z^T = \begin{bmatrix} z_1 & \dots & z_{N_s} \end{bmatrix}, \quad \Gamma^T = \begin{bmatrix} \eta_1 \dots \eta_{N_p} \end{bmatrix}.$$

A linear least squares fit minimizes the quadratic error between $z(\theta^{(i)})$ and z_i, $i = 1, \dots, N_s$

$$\Gamma^* = \arg \min_{\Gamma} (Z - H\Gamma)^T (Z - H\Gamma) \tag{7.15}$$

The optimal Γ^* is determined by initialy computing a singular value decomposition of H

$$\Upsilon \Xi \Psi^T = \mathrm{svd}(H) \tag{7.16}$$

With the decomposition at hand, the solution to the linear least squares problem is given by

$$\Gamma^* = \Psi \Xi^+ \Upsilon^T Z \tag{7.17}$$

where Ξ^+ stands for the Moore-Penrose pseudoinverse of Ξ. Repeating the above procedure for each matrix entry results in the polynomial approximations of the matrices $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ that can be used for subsequent analysis and design of controllers.

4   Numerical Example

In this section, the proposed procedure is applied to the NREL 5MW reference wind turbine model [17]. The aim is to find an LPV model encapsulating the wind turbine dynamics operating at the full load region. Large scale MIMO LTI models with 877 states are computed by the aeroelastic code HAWCStab2 for wind speeds equidistant 1 m/s ($\theta^{(i)} \in \{12, 13, \ldots, 25\}$). The model is parameterized by the mean wind speed $\theta := \bar{V}$. A fifth-order polynomial dependence of the LPV system matrices

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \sum_{d=1}^{5} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \theta^d \tag{7.18}$$

gives a fair trade-off between interpolation accuracy and polynomial order. Due to the different units of inputs and outputs, the LTI systems should be suitably normalized before the order is reduced. Parameter-independent scales are applied to all LTI models such that expected signal excursions are normalized to 1. The generator torque input was scaled to 5% of the rated torque. The pitch angle input and wind speed input remained unscaled. The rotor angular velocity is scaled to the maximum excursion desired in closed-loop, 5 % of its nominal value. The lateral tower top displacement was scaled with the H_∞-norm of the inputs to this output channel.

Bode plots of the original, intermediate and final reduced order models for an operating point $\theta = 15$ m/s are depicted in Fig. 7.2. The intermediate model with 410 states resulted from modal truncation and residualization of the original system. A balanced truncation with residualization further reduced the size to 14 states. The magnitude plots have an excellent agreement in the frequencies of interest. The residualization in both steps assisted to a better fit of the low frequency content of the tower displacement output.

The balanced realization "misses" a low frequency anti-ressonance related to the transfer function from wind speed input to tower displacement output. Notice the differences in magnitude and more pronouncedtly phase around 10^{-2} Hz. However, this anti-ressonance does not contribute significantly to the input-output behaviour of the MIMO
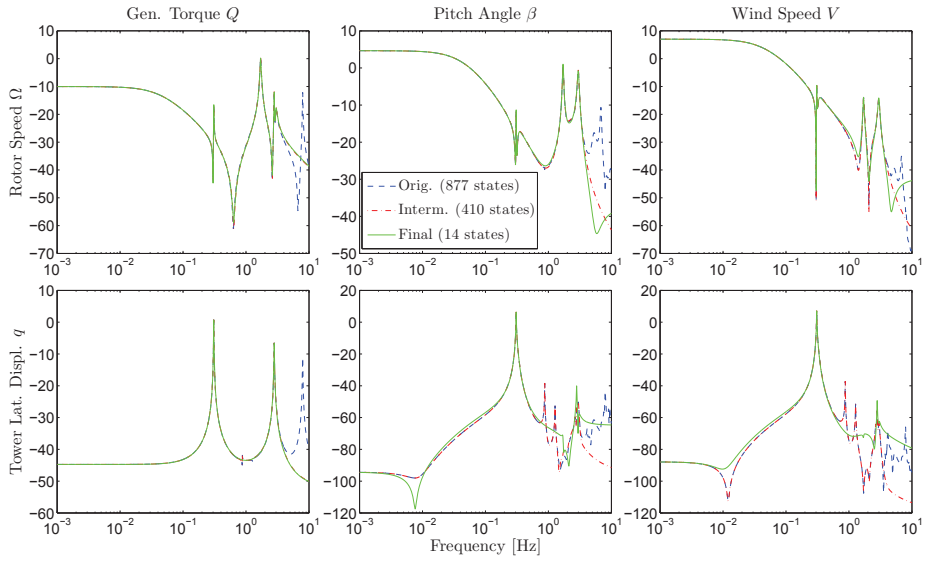
(a) Magnitude
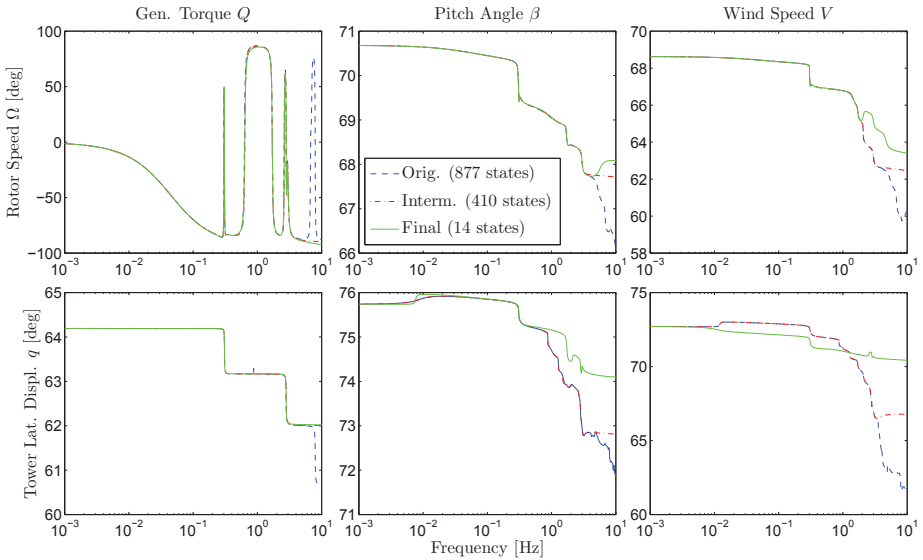
(b) Phase

Figure 7.2: Bode plots of the original, intermediate and final reduced order models for a mean wind speed of 15 m/s.
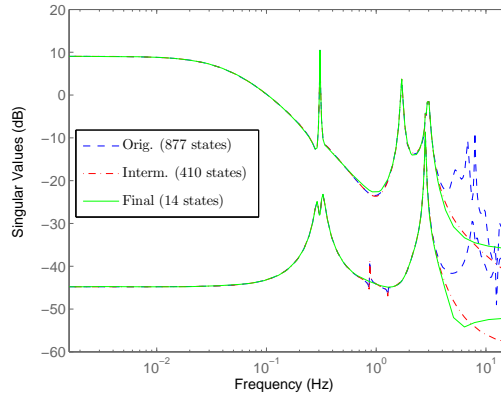
Figure 7.3: Singular values of the original, intermediate and final reduced order models for a mean wind speed of 15 m/s.
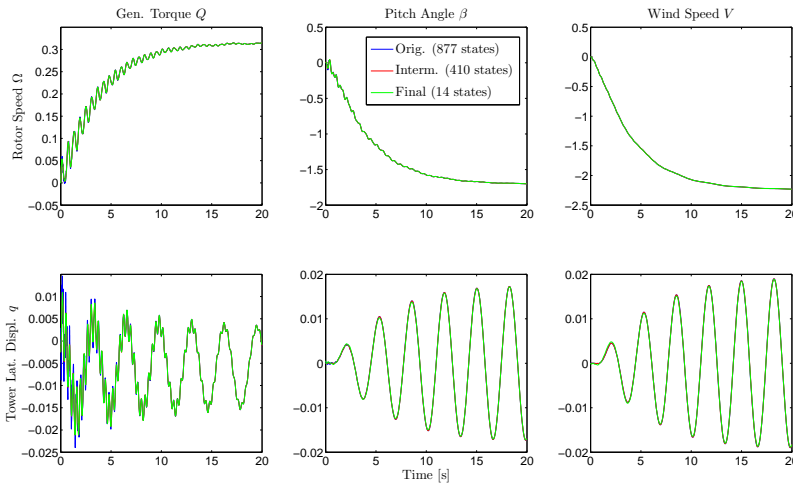
Figure 7.4: Step responses of the original, intermediate and final reduced order models for a mean wind speed of 15 m/s.

system. A comparison of the minimum and maximum singular values is depicted in Fig. 7.3 and shows an excellent agreement.

Step responses of the original, intermediate and final reduced order models for a mean wind speed of 15 m/s are depicted in Fig. 7.4. Except for some high frequency content in the signal from generator torque to tower position, the responses are identical.

The location of the poles of the LPV system for a $2N_s$ grid of equidistant operating points is illustrated in Fig. 7.5. The arrows indicate how the poles move for increasing mean wind speeds. A smooth evolution of the poles along the full load region is notice-
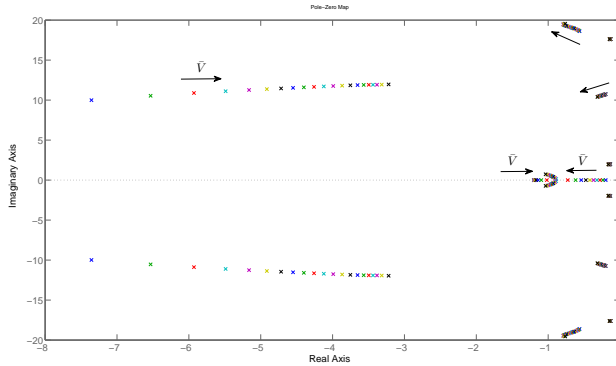
(a) Pole Map
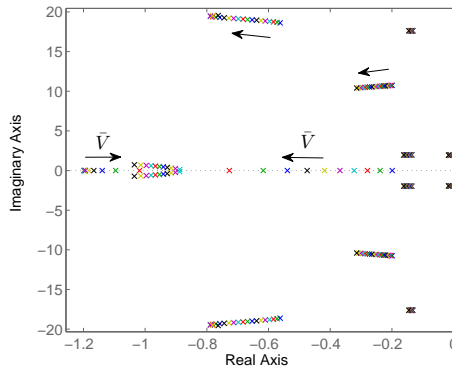
(b) Detail

Figure 7.5: Pole location of the LPV model for frozen values of the varying parameter θ.

able.

The relative difference of the Hankel singular values of the interpolated LPV system and the reduced order system defined as

$$\sigma_{rel,r,i} = \frac{\sigma_{int,r,i} - \sigma_{r,i}}{\sigma_{r,i}} \times 100, \quad i = 1, \ldots, N_s \tag{7.19}$$

serves as a measure of the quality of the interpolation. A good fit can be corroborated by some metrics of $\sigma_{rel,r,i}$ given in Tab. 7.1. The mean difference in the Hankel singular values is only 0.27% and the maximum difference just 2.75%.

5   Conclusion & Future Work

This paper presents a procedure to obtain a reduced-order LPV model of a wind turbine from a set of high-order LTI models. Finding ways to encapsulate high-fidelity LTI aeroe-

Table 7.1: Difference in the Hankel singular values between the LPV and reduced order system for frozen values of θ.

Max	Min	Mean	Std. dev
2.75	0.001	0.27	0.57

lastic models as an LPV system is an important step to increase the utilization of recent advances in LPV control by the wind turbine industry. The proposed procedure starts by model reduction of the high-order LTI systems at different values of the parameter space. Manipulations of the state-space coordinates follows, in order to arrive at low-order consistent LTI systems for subsequent interpolation. The reduced-order LPV system has a suitable size for analysis and synthesis of controllers and presents smoothly varying dynamics along the scheduling parameter range.

A subject for future work is to initially interpolate the set of high-order LTI models and later apply an appropriate reduction method to realize a reduced order LPV model. Preserving structure reduction methods applied directly in the second order vector equations of motion in an interesting topic to be studied. Model complexity versus required polynomial degree and a comparison with models obtained by first-principles is also subject of future work.

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Paper E

New Sufficient LMI Conditions for Static Output Stabilization

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Abstract

This paper presents new linear matrix inequality (LMI) conditions to the static output feedback stabilization (SOFS) problem. Although the conditions are only sufficient, numerical experiments show excellent success rates in finding a stabilizing controller.

1 Introduction

This paper presents new conditions to the static output feedback stabilization (SOFS) problem: given a linear time-invariant (LTI) system,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{8.1}$$

determine if it is stabilizable by a static output feedback,

$$u(t) = Ky(t), \tag{8.2}$$

where $x(t) \in R^n$, $u(t) \in R^{n_u}$ and $y(t) \in R^{n_y}$. The pair (A, B) is stabilizable and (C, A) is detectable. B and C are both full rank matrices. This is one of the most fundamental and challenging control problems which remains open up to date. Previous research efforts resulted in efficient numerical algorithms to solve the SOFS and related problems; see [1, 2] for a survey. Many of these approaches are based on non-convex characterizations of the problem accompanied by a numerical optimization procedure able to compute stationary solutions. A numerical comparison and proposed classification of some of these algorithms is found in [3]. The iterative algorithms proposed in [4, 5] involving bilinear matrix inequalities (BMI) extended with slack variables are related to the present work.

The present work characterizes the SOFS as sufficient linear matrix inequality (LMI) conditions to which efficient convex programming tools are available. The Lyapunov stability theory is the basis for converting a wide variety of problems arising in system and control into LMI characterizations [6]. A drawback of this framework is the natural appearance of products between the Lyapunov matrix variable, certificate of system stability, and the system matrices. When the system matrices are functions of controller variables, the matrix inequality is no longer linear but bilinear (BMI) in the decision variables. For particular controller structures, the BMI can be reformulated into an LMI without loss of generality via a change-of-variables involving the Lyapunov and controller matrices. This notion first appeared in the context of full state feedback [7] and was later extended to the full order dynamic output feedback [8]. A change-of-variables without loss generality for the SOFS remains unknown. The imposition of a block diagonal constraint on both the controller gain and the Lyapunov matrix facilitate change-of-variables that yields sufficient LMI conditions [9]. An important contribution to reduce the conservativeness due to the structural constraint on the Lyapunov variable is the introduction of extended LMI characterizations. Firstly proposed in the context of discrete-time systems [10, 11] and later to continuous-time systems [12, 13, 14], multipliers (also referred to as slack variables) are introduced into the formulation with the objective of decoupling the Lyapunov matrix from other matrix variables. The required constraints to facilitate change-of-variables are enforced on the multiplier rather than the Lyapunov variable, and

thus conservativeness is reduced [11]. Recent contributions [15, 16] decrease the conservativeness of this approach by finding more general LMI results for discrete-time systems, which included the existing ones as particular cases. In all these works, a suitable similarity transformation is applied to the system and the structure of the matrix multiplier is constrained accordingly.

This paper presents new sufficient stability and stabilizability conditions tailored for the SOFS problem. The extended LMI are obtained through an approach distinct from the ones found in the literature. We use the concepts of enlarged spaces combined with Finsler's Lemma [17] to derive stability conditions with multipliers. The enlarged spaces here proposed contains the input vector $u(t)$ or output vector $y(t)$ explicitly, that is, the feedback law (8.2) is not substituted in the differential equation (8.1). Excessive degrees of freedom introduced by the multipliers are removed with the aid of the Elimination Lemma, along the lines of [18]. The LMI conditions with fewer multipliers are more suitable for synthesis of stabilizing controllers. To facilitate change-of-variables involving controller data and multiplier, the input (or output) vector is augmented with redundant inputs (or outputs) such that their dimension equals the dimension of the states. The dimensions of the controller data is also increased accordingly, but due to the redundancy of inputs/outputs, a controller with the original dimensions can be easily computed by summing rows or columns of the augmented controller matrix. The method can cope with continuous and discrete-time systems; only continuous-time is addressed here due to space limitations. Although the conditions are only sufficient, numerical experiments show success rates of over 90% in finding stabilizing controllers.

2   Preliminaries

Let us recall some concepts of Lyapunov stability for autonomous state-space systems. Consider the dynamics of a continuous-time linear time-invariant (LTI) system governed by the differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{8.3}$$

where $x(t) : [0, \infty] \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. Define the quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(x) := x(t)^T P x(t) \tag{8.4}$$

where $P \in \mathbb{S}^n$ and $\mathbb{S}^n$ stands for the field of real symmetric matrices. According to Lyapunov theory, system (8.3) is asymptotically stable if there exists $V(x(t)) > 0, \forall x(t) \neq 0$ such that

$$\dot{V}(x(t)) < 0, \quad \dot{x}(t) = Ax(t), \quad \forall x(t) \neq 0. \tag{8.5}$$

In words, if there exists $P \succ 0$ such that the time derivative of the quadratic Lyapunov function (8.4) is negative along all trajectories of system (8.3). Conversely, if the linear system (8.3) is asymptotically stable then there always exists $P \succ 0$ that satisfies (8.5). These two affirmatives imply the well known fact that Lyapunov theory with quadratic functions is necessary and sufficient to prove stability of LTI systems. A way to obtain an LMI condition equivalent to the existence of the set characterized by (8.5) is to explicitly substitute (8.3) into (8.4) [6]

$$\dot{V}(x(t)) = x(t)^T \left(A^T P + PA \right) x(t) < 0, \quad \forall x(t) \neq 0. \tag{8.6}$$

The condition (8.6) is equivalent to the LMI feasibility problem

$$\exists P \in \mathbb{S}^n \; : \; P \succ 0, \quad A^T P + PA \prec 0. \tag{8.7}$$

The fact that (8.5) is a set characterized by inequalities subject to dynamic equality constraints is explored in [17] to propose a constrained optimization point of view to the stability problem. It is then possible to characterize the set defined by (8.5) without substituting (8.3) explicitly into $\dot{V}(x(t)) < 0$ [17]. The well know Finsler's Lemma [19] is the main mathematical tool to transform the constrained optimization problem into a problem subject to LMI constraints. e

Lemma 6. *(Finsler) Let $x(t) \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$ and $\mathcal{B} \in \mathbb{R}^{m \times n}$ such that $\text{rank}(\mathcal{B}) < n$. The following statements are equivalent.*

   i. $x(t)^T \mathcal{Q} x(t) < 0, \quad \mathcal{B}x(t) = 0, \quad \forall x(t) \neq 0.$

   ii. $\mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^\perp \prec 0.$

   iii. $\exists \mu \in \mathbb{R} \; : \; \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0.$

   iv. $\exists \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^T \mathcal{X}^T \prec 0.$

A similarity between statement i. of the above lemma and (8.5) can be noticed. In contrast to (8.6), the space of statement i. is an *enlarged space* [17] composed of $x(t)$ as well as $\dot{x}(t)$. Statements iii. and iv. can be seen as equivalent unconstrained quadratic forms of i. [17]. The equality constraint $\dot{x}(t) = Ax(t)$ is included in the formulation weighted by the scalar multiplier μ or matrix multiplier $\mathcal{X}$. The Elimination Lemma [6, 20] stated next will serve for the purpose of removing excessive multipliers without adding conservatism into the solution.

Lemma 7. *(Elimination) Let $\mathcal{Q} \in \mathbb{S}^n$, $\mathcal{B} \in \mathbb{R}^{m \times n}$, $\mathcal{C} \in \mathbb{R}^{n \times k}$. The following statements are equivalent.*

   i. $\exists \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{C}^T \mathcal{X} \mathcal{B} + \mathcal{B}^T \mathcal{X}^T \mathcal{C} \prec 0$

   ii. $\qquad \mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^\perp \prec 0 \qquad\qquad \mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^\perp \prec 0$
$$\tag{8.8a} \qquad\qquad\qquad \tag{8.8b}$$

   iii. $\exists \mu \in \mathbb{R} \; : \; \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0, \quad \mathcal{Q} - \mu \mathcal{C}^T \mathcal{C} \prec 0.$

The Elimination Lemma reduces to the Finsler's Lemma when particularized with $\mathcal{C} = I$. In such a case $\mathcal{C}^\perp = \{0\}$ and (8.8b) is removed from the statement.

3 Static Ouput Stability Conditions

The stability conditions presented here rely on enlarged spaces composed by the state $x(t)$ and its time derivative $\dot{x}(t)$, the control input $u(t)$ or the measured output $y(t)$. Different LMI characterizations emerge depending on the considered spaces $(x(t), \dot{x}(t), u(t))$ or

$(x(t), \dot{x}(t), y(t))$. For the former space, let the dynamical equations (8.1)-(8.2) be rewritten in the equivalent form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{8.9a}$$

$$u(t) = KCx(t). \tag{8.9b}$$

Stability of the above dynamical system is characterized by the existence of the set

$$\dot{V}(x(t), \dot{x}(t)) < 0, \quad \forall (x(t), \dot{x}(t), u(t)) \neq 0 \quad \text{satisfying (8.9).} \tag{8.10}$$

where the quadratic form $\dot{V} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\dot{V}(t) := \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \tag{8.11}$$

The LMI feasibility problem stated in the following theorem is equivalent to the existence of a non-empty set (8.10) and serves as a certificate of quadratic stability.

Theorem 13. *(Quadratic Stability)* $A+BKC$ *is Hurwitz if, and only if,* $\exists P \in \mathbb{S}^n, \Lambda_1, \Phi_1 \in \mathbb{R}^{n \times n}, \Gamma_1 \in \mathbb{R}^{n_u \times n}, \Lambda_2, \Phi_2 \in \mathbb{R}^{n \times n_u}, \Gamma_2 \in \mathbb{R}^{n_u \times n_u},$

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad P \succ 0. \tag{8.12a}$$

$$\mathcal{J} := \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{H} := \begin{bmatrix} \Lambda_1 A + \Lambda_2 KC & -\Lambda_1 & \Lambda_1 B - \Lambda_2 \\ \Phi_1 A + \Phi_2 KC & -\Phi_1 & \Phi_1 B - \Phi_2 \\ \Gamma_1 A + \Gamma_2 KC & -\Gamma_1 & \Gamma_1 B - \Gamma_2 \end{bmatrix}. \tag{8.12b}$$

Proof. Assign

$$x(t) \leftarrow \begin{pmatrix} x(t) \\ \dot{x}(t) \\ u(t) \end{pmatrix}, \quad \mathcal{Q} \leftarrow \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B}^T \leftarrow \begin{bmatrix} A^T & C^T K^T \\ -I & 0 \\ B^T & -I \end{bmatrix}, \quad \mathcal{X} \leftarrow \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Phi_1 & \Phi_2 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}$$

and apply Finsler's lemma to the constrained Lyapunov problem (8.10) with $P \succ 0$. $\quad\square$

The matrix inequality (8.12) is a function of the multipliers Λ, Φ, Γ. The Elimination Lemma is here invoked to constraint multipliers without adding conservatism to the solution.

Lemma 8. *The multipliers can be constrained as* $\Phi_1 := \alpha\Lambda_1$, $\Phi_2 := \alpha\Lambda_2$ *without loss of generality, where* $\alpha > 0$ *is an arbitrary positive scalar*

Proof. Assign

$$\mathcal{Q} \leftarrow \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B}^T \leftarrow \begin{bmatrix} A^T & C^T K^T \\ -I & 0 \\ B^T & -I \end{bmatrix}, \quad \mathcal{X} \leftarrow \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}$$

$$\mathcal{C}^\perp \leftarrow \begin{bmatrix} \alpha I \\ -I \\ 0 \end{bmatrix}, \quad \mathcal{C}^T \leftarrow \begin{bmatrix} I & 0 \\ \alpha I & 0 \\ 0 & I \end{bmatrix}$$

and apply the Elimination Lemma with $P \succ 0$. The inequality (8.8b) does not bring conservatism to the solution. To see this, expand $\mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^\perp \prec 0$ and realize that $-2\alpha P \prec 0$ holds $\forall \alpha > 0$. $\quad\square$

In order to obtain a condition suitable for computing stabilizable controllers, let an extended input matrix $\tilde{B} \in \mathbb{R}^{n \times n}$ be the result of augmenting redundant inputs to the system, such that the number of inputs are equal to the number of states. That is

$$\tilde{B} := \begin{bmatrix} B_{u_1} & B_{u_2} & \ldots & B_{u_{n_u}} & B_{u_1} & B_{u_2} & \ldots \end{bmatrix} \tag{8.13}$$

where B_{u_i} stands for the i-th column of matrix B related to the i-th input. Let also define an augmented feedback gain $\tilde{K} \in \mathbb{R}^{n \times n_y}$ partitioned accordingly

$$\tilde{K} := \begin{bmatrix} K_{1,u_1}^T & K_{1,u_2}^T & \ldots & K_{1,u_{n_u}}^T & K_{2,u_1}^T & \ldots & K_{2,u_2}^T & \ldots \end{bmatrix}^T \tag{8.14}$$

where $K_{j,u_i} \in \mathbb{R}^{1 \times n_y}$ are j-th feedback gain from $y(t)$ to the i-th input u_i. The contribution of the different gains K_{j,u_i} to a particular input is just their sum

$$K_{u_i} := \sum_{j=1}^{N} K_{j,u_i} \tag{8.15}$$

Let also define the change-of-variables involving the augmented controller $\tilde{K}$ and the multiplier Λ_2

$$\hat{K} := \Lambda_2 \tilde{K}. \tag{8.16}$$

Due to the augmentation of input matrix and controller, the multiplier is now a square matrix $\Lambda_2 \in \mathbb{R}^{n \times n}$. Whenever Λ_2 is non-singular, the original controller gains can be reconstructed according to $\tilde{K} = \Lambda_2^{-1} \hat{K}$. With these definitions at hand, the stabilizability condition can now be stated.

Theorem 14. *(Stabilizability) There exists a static output feedback that renders $A + BKC$ Hurwitz if $\exists\, P \in \mathbb{S}^n$, $\hat{K} \in \mathbb{R}^{n \times n_y}$, Λ_1, Λ_2, $\Gamma_1 \in \mathbb{R}^{n \times n}$, $\mu \in \mathbb{R}$:*

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad P \succ 0, \quad \mathcal{J} := \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.17a}$$

$$\mathcal{H} := \begin{bmatrix} \Lambda_1 A + \hat{K}C & -\Lambda_1 & \Lambda_1 \tilde{B} - \Lambda_2 \\ \alpha(\Lambda_1 A + \hat{K}C) & -\alpha\Lambda_1 & \alpha(\Lambda_1 \tilde{B} - \Lambda_2) \\ \Gamma_1 A + \mu\hat{K}C & -\Gamma_1 & \Gamma_1 \tilde{B} - \mu\Lambda_2 \end{bmatrix} \tag{8.17b}$$

for an arbitrary scalar $\alpha > 0$ and if the solution yields Λ_2 non-singular.

The above inequality arise from (8.12), the multiplier constraints of Lemma 8, as well as the constraint $\Gamma_2 := \mu\Lambda_2$ enforced to facilitate the change-of-variables (8.16). This constraint brings conservatism to the above theorem. The structure of (8.17) does not guarantee a non-singular multiplier Λ_2. The later multiplier constraint and the possibility of solutions that renders Λ_2 singular thus not invertible are the reasons for sufficiency of this condition. Notice the required line search in μ.

The enlarged space containing the output vector $(x(t),\ \dot{x}(t),\ y(t))$ is now considered. This time, let the dual of the dynamical equations (8.1)-(8.2) be re-written in the equivalent form

$$\dot{x}'(t) = A^T x'(t) + C^T y'(t) \tag{8.18a}$$

$$y'(t) = K^T B^T x'(t). \tag{8.18b}$$

where the subscript $(\cdot)^{'}$ stands for dual state and input vectors with appropriate dimensions. Working with the dual system facilitates the appearance of multipliers in a suitable position for change-of-variables with controller gains, as will become clear later. It is worth to mention that duality preserves the eigenvalues of the state matrix, i.e. $\lambda(A + BKC) = \lambda(A^T + C^T K^T B^T)$. Therefore, a stabilizing controller for the dual system also stabilizes the primal one. Stability of the above dynamical system is characterized by the existence of the set

$$\dot{V}(x^{'}(t), \dot{x}^{'}(t)) < 0, \ \forall (x^{'}(t), \dot{x}^{'}(t), y^{'}(t)) \neq 0 \ \text{satisfying (8.18)} \qquad (8.19)$$

where the quadratic form $\dot{V} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is also defined as (8.11). The existence of set (8.19) is equivalent to the LMI feasibility problem stated in the following theorem.

Theorem 15. *(Quadratic Stability, Dual System)* $A + BKC$ *is Hurwitz if, and only if,* $\exists \, P \in \mathbb{S}^n$, Λ_1, $\Phi_1 \in \mathbb{R}^{n \times n}$, $\Gamma_1 \in \mathbb{R}^{n \times n_y}$, Λ_2, $\Phi_2 \in \mathbb{R}^{n_y \times n}$, $\Gamma_2 \in \mathbb{R}^{n_y \times n_y}$,

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad P \succ 0, \quad \mathcal{J} := \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (8.20a)$$

$$\mathcal{H} := \begin{bmatrix} A\Lambda_1 + BK\Lambda_2 & A\Phi_1 + BK\Phi_2 & A\Gamma_1 + BK\Gamma_2 \\ -\Lambda_1 & -\Phi_1 & -\Gamma_1 \\ C\Lambda_1 - \Lambda_2 & C\Phi_1 - \Phi_2 & C\Gamma_1 - \Gamma_2 \end{bmatrix} \qquad (8.20b)$$

Proof. Assign

$$x(t) \leftarrow \begin{pmatrix} x^{'}(t) \\ \dot{x}^{'}(t) \\ y^{'}(t) \end{pmatrix}, \ \mathcal{Q} \leftarrow \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathcal{B}^T \leftarrow \begin{bmatrix} A & BK \\ -I & 0 \\ C & -I \end{bmatrix}, \ \mathcal{X} \leftarrow \begin{bmatrix} \Lambda_1^T & \Lambda_2^T \\ \Phi_1^T & \Phi_2^T \\ \Gamma_1^T & \Gamma_2^T \end{bmatrix}$$

and apply Finsler's lemma to the constrained Lyapunov problem (8.19) with $P \succ 0$. $\quad\square$

A condition suitable for computing stabilizable controllers can be obtained analogously to what was done for the enlarged space containing $u(t)$. Let an extended coutput matrix $\tilde{C} \in \mathbb{R}^{n \times n}$ be the result of augmenting redundant outputs to the system, such that the number of outputs are equal to the number of states.

$$\tilde{C} := \begin{bmatrix} C_{y_1}^T & C_{y_2}^T & \dots & C_{y_{n_y}}^T & C_{y_1}^T & C_{y_2}^T & \dots \end{bmatrix}^T \qquad (8.21)$$

where C_{y_i} stands for the i-th line of matrix C related to the i-th output. Let an augmented feedback gain $\tilde{K} \in \mathbb{R}^{n_u \times n}$ be partitioned accordingly

$$\tilde{K} := \begin{bmatrix} K_{1,y_1} & K_{1,y_2} & \dots & K_{1,y_{n_y}} & K_{2,y_1} & \dots & K_{2,y_2} & \dots \end{bmatrix} \qquad (8.22)$$

where $K_{j,y_i} \in \mathbb{R}^{n_u \times 1}$ are j-th feedback gain from the i-th output y_i to the input $u(t)$. The contribution of the different gains K_{j,y_i} from a particular output is just their sum

$$K_{y_i} := \sum_{j=1}^{N} K_{j,y_i} \qquad (8.23)$$

Due to the augmentation of output matrix and controller, the multiplier $\Lambda_2 \in \mathbb{R}^{n \times n}$ is now a square matrix facilitating the change-of-variables (8.16). The elimination of multipliers without added conservatism follows analogously to Lemma 8 and will be omitted for brevity. The stabilizability condition for the enlarged space containing $y(t)$ is stated next.

Theorem 16. *(Stabilizability, Dual System) There exists a static output feedback that renders $A + BKC$ Hurwitz if* $\exists P \in \mathbb{S}^n$, $\hat{K} \in \mathbb{R}^{n_u \times n}$, Λ_1, Λ_2, $\Gamma_1 \in \mathbb{R}^{n \times n}$, $\mu \in \mathbb{R}$:

$$
\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad P \succ 0, \quad \mathcal{J} := \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.24a}
$$

$$
\mathcal{H} := \begin{bmatrix} A\Lambda_1 + B\hat{K} & \alpha(A\Lambda_1 + B\hat{K}) & A\Lambda_1 + \mu B\hat{K} \\ -\Lambda_1 & -\alpha\Lambda_1 & -\Gamma_1 \\ \tilde{C}\Lambda_1 - \Lambda_2 & \alpha(\tilde{C}\Lambda_1 - \Lambda_2) & \tilde{C}\Gamma_1 - \mu\Lambda_2 \end{bmatrix} \tag{8.24b}
$$

for an arbitrary scalar $\alpha > 0$ and if the solution yields Λ_2 non-singular.

Sufficiency of this condition is occasioned by the constraint $\Gamma_2 := \mu\Lambda_2$ enforced to facilitate the change-of-variables (8.16) and the possibility of solutions that yields Λ_2 singular, thus not invertible.

Ensuring an Invertible Multiplier

The numerical experiments to be shown later suggest that the assumption of a solution rendering a non-singular multiplier Λ_2 is not strong. However, it is possible to ensure an invertible multiplier involved in the change-of-variables with the drawback of increased LMI complexity and (possibly) higher conservatism. For doing so, an extra equation of the time derivative of the input

$$
\dot{u}(t) = KC\dot{x}(t). \tag{8.25}
$$

is included on the description of the dynamical system (8.9). Similarly, the time derivative of the dual output

$$
\dot{y}'(t) = K^T B^T \dot{x}'(t). \tag{8.26}
$$

is included on the description of the dual system (8.18). The enlarged space should also be extended with $\dot{u}(t)$ or $\dot{y}'(t)$. The following stability condition originates from applying Finsler's Lemma to the set (8.10) with the extended enlarged space and equality constraint (8.25), and subsequent use of the Elimination Lemma to remove excessive multipliers.

Theorem 17. $A + BKC$ *is Hurwitz if, and only if,* $\exists P \in \mathbb{S}^n$, $\Lambda_1 \in \mathbb{R}^{n \times n}$, $\Gamma_1 \in \mathbb{R}^{n_u \times n}$, $\Lambda_2, \Lambda_3 \in \mathbb{R}^{n \times n_u}$, $\Gamma_2, \Gamma_3 \in \mathbb{R}^{n_u \times n_u}$,

$$
\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad P \succ 0, \quad \mathcal{J} := \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{8.27a}
$$

$$
\mathcal{H} := \begin{bmatrix} \Lambda_1 A + \Lambda_2 KC & -\Lambda_1 + \Lambda_3 KC & \Lambda_1 B - \Lambda_2 & -\Lambda_3 \\ \alpha(\Lambda_1 A + \Lambda_2 KC) & \alpha(-\Lambda_1 + \Lambda_3 KC) & \alpha(\Lambda_1 B - \Lambda_2) & -\alpha\Lambda_3 \\ \Phi_1 A + \Phi_2 KC & -\Phi_1 + \Phi_3 KC & \Phi_1 B - \Phi_2 & -\Phi_3 \\ \Gamma_1 A + \Gamma_2 KC & -\Gamma_1 + \Gamma_3 KC & \Gamma_1 B - \Gamma_2 & -\Gamma_3 \end{bmatrix} \tag{8.27b}
$$

for an arbitrary scalar $\alpha > 0$.

The proof follows similar arguments of Theorem 13 and Lemma 8 and is omitted for brevity. When considering the augmented input matrix in (8.27), i.e. $B \leftarrow \tilde{B}$, the following constraints on the square multipliers

$$\Lambda_2 := \mu_1 \Gamma_3, \quad \Lambda_3 := \mu_2 \Gamma_3, \quad \Phi_2 := \mu_3 \Gamma_3,$$
$$\Phi_3 := \mu_4 \Gamma_3, \quad \Gamma_2 = \mu_5 \Gamma_3$$

facilitate the change-of-variables $\hat{K} = \Gamma_3 \tilde{K}$. Note that $\Gamma_3 + \Gamma_3^T \prec 0$ on the lower right block of (8.27) implies a non-singular Γ_3 and ensure, theoretically, the controller reconstruction. Line searches on the scalars $\mu_1, \mu_2, \ldots, \mu_5$ are now required making the characterization less attractive.

Some Applications

The LMI characterizations composed of multipliers facilitate the solution of some hard control problems [11]. The Lyapunov variable P can be made multiple or parameter-dependent due to its appearance free from products with system matrices. This can be exploited in the well known simultaneous and robust stabilization problems.

Simultaneous stabilization

The simultaneous stabilization problem can be posed as follows: given a family of open-loop plants,

$$G_i := \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array} \right], \qquad i = 1, 2, \ldots, N_p$$

find a controller K that simultaneously stabilizes the plants, that is,

$$\text{Re} \left(\lambda \left(A_i + B_i K C_i \right) \right) < 0, \qquad i = 1, 2, \ldots, N_p.$$

Considering the enlarged space $(x(t), \dot{x}(t), u(t))$, this problem can be cast as the sufficient LMI feasibility condition: $\exists P_i \in \mathbb{S}^n, \hat{K} \in \mathbb{R}^{n \times n_y}, \Lambda_{1,i}, \Lambda_2, \Gamma_{1,i} \in \mathbb{R}^{n \times n}, \mu \in \mathbb{R}$:

$$\mathcal{J}_i + \mathcal{H}_i + \mathcal{H}_i^T \prec 0, \quad P_i \succ 0, \quad \mathcal{J}_i := \begin{bmatrix} 0 & P_i & 0 \\ P_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.28a}$$

$$\mathcal{H} := \begin{bmatrix} \Lambda_{1,i} A_i + \hat{K} C_i & -\Lambda_{1,i} & \Lambda_{1,i} \tilde{B}_i - \Lambda_2 \\ \alpha(\Lambda_{1,i} A_i + \hat{K} C_i) & -\alpha \Lambda_{1,i} & \alpha(\Lambda_{1,i} \tilde{B}_i - \Lambda_2) \\ \Gamma_{1,i} A_i + \mu \hat{K} C_i & -\Gamma_{1,i} & \Gamma_{1,i} \tilde{B}_i - \mu \Lambda_2 \end{bmatrix} \tag{8.28b}$$

for $i = 1, \ldots, N_p$ where $\alpha > 0$ is an arbitrary scalar, and if the solution yields Λ_2 non-singular. A single Λ_2 is adopted for the family of open-loop plants in order to facilitate change-of-variables involving the controller gain and multiplier, while multiple $\Lambda_{1,i}$ and $\Gamma_{1,i}$ help to reduce conservativeness.

Robust stabilization

Consider the linear time-invariant uncertain polytopic system,

$$\dot{x} = \sum_{i=1}^{N_\theta} \theta_i A_i x + \sum_{i=1}^{N_\theta} \theta_i B_i u, \quad y = \sum_{i=1}^{N_\theta} \theta_i C_i x,$$

$$\theta_i \geq 0, \quad \sum_{i=1}^{N_\theta} \theta_i = 1.$$

The polytopic symmetric matrix variable $P(\theta)$ is also defined

$$P(\theta) := \sum_{i=1}^{N_\theta} \theta_i P_i. \tag{8.29}$$

When synthesis of robust SOF is of concern, the LMI condition for the dual system might find a controller: $\exists\, P_i \in \mathbb{S}^n, \hat{K} \in \mathbb{R}^{n_u \times n}, \Lambda_1, \Lambda_2, \Gamma_1 \in \mathbb{R}^{n \times n}, \mu, \alpha \in \mathbb{R}$:

$$\mathcal{J}_i + \mathcal{H}_i + \mathcal{H}_i^T \prec 0, \quad P_i \succ 0, \quad \mathcal{J}_i := \begin{bmatrix} 0 & P_i & 0 \\ P_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.30a}$$

$$\mathcal{H} := \begin{bmatrix} A_i \Lambda_1 + B_i \hat{K} & \alpha(A_i \Lambda_1 + B_i \hat{K}) & A\Lambda_1 + \mu B_i \hat{K} \\ -\Lambda_1 & -\alpha \Lambda_1 & -\Gamma_1 \\ \tilde{C}_i \Lambda_1 - \Lambda_2 & \alpha(\tilde{C}_i \Lambda_1 - \Lambda_2) & \tilde{C}_i \Gamma_1 - \mu \Lambda_2 \end{bmatrix} \tag{8.30b}$$

holds for $i = 1, \ldots, N_\theta$ and if the solution yields Λ_2 non-singular. Notice that the choice of α does influence on the conservativeness and, in this case, should be considered a variable.

4 Numerical Experiments

The success of the proposed conditions in finding stabilizable controllers is verified empirically via numerical experiments. Sets of 1.000 random triplices (A, B, C) satisfying the generic stabilizability results of Kimura [21] were generated with the aid of *rss* MATLAB function. The Kimura condition guarantees the existence of a stabilizing static output feedback if the dimensions of (A, B, C) satisfies $n > n_u + n_y$. The state dimension takes the values $n \in \{4, 6\}$. The set of triplices are characterized by different combinations of input and output dimensions - $(n_u, n_y) \in \{(2, 3), (3, 2)\}$ for $n = 4$ and $(n_u, n_y) \in \{(3, 4), (4, 3)\}$ for $n = 6$ - with the aim of investigating whether the characterization of stabilizability of a particular enlarged space ($u(t)$ or $y(t)$) is better suited for inputs/outputs with larger dimensions. The random systems were made unstable by updating the state matrix according to $A \leftarrow A - 1.1\lambda^+ I$, where $\lambda^+ = \max(\text{Re}\,(\lambda(A)))$. Each state matrix had a minimum of one up to four unstable eigenvalues. The redundant inputs of the augmented matrix $\tilde{B}$ are the first up to $n - n_u$ inputs of B, i.e. $\tilde{B} := \begin{bmatrix} B & B(:, 1 : n - n_u) \end{bmatrix}$ in MATLAB notation. The augmented matrix $\tilde{C}$ is similarly constructed.

The stabilizability conditions of the enlarged spaces containing $u(t)$ and $y(t)$ (Theorems 14 and 16, respectively) were solved for each random unstable system. The line search in μ was performed in the discrete set

$$\mathcal{M} := \{\mu : \mu \in \{10^0, 10^2, 10^1, 10^{-1}, 10^{-2}, -10^{-2}, -10^{-1}, -10^0, -10^1, -10^2\}\}$$

until either a valid solution is found or infeasibility resulted for all $\mu \in \mathcal{M}$. We consider a valid solution as a feasible solution to which the solver did not accuse any problems during the optimization like numerical problems, exceeded maximum number of iterations, or lack of progress. The occurrences of valid solutions that yielded a multiplier Λ_2 close to singular were registered according to the rule $\min(|\lambda(\Lambda_2)|) < 10^{-3}$ with the purpose of accessing if non-singularity of this multiplier is a strong assumption. To enforce strict LMI solutions the right hand side of the inequalities $\prec 0$ ($\succ 0$) were replaced by $\prec -\epsilon I$ ($\succ \epsilon I$) where $\epsilon = 10^{-6}$. The Lyapunov matrix were constrained to be $P \succeq I$ with the purpose of improving numerical conditioning. This can be done without loss of generality due to homogeneity arguments [6]. The interface YALMIP [22] and semidefinite programming SeDuMi [23] had been used with standard configuration.

Table 8.1 summarizes the success of finding valid solutions for different dimensions (n, n_u, n_y). The number of successes related to the primal dynamical system (Theorem 14), dual dynamical system (Theorem 16), and either one of these two are depicted. The rates of at least one characterization succeed are over 90%, remarkably high. A trend on the results suggest that the condition for the enlarged space containing $u(t)$ is more suited to systems in which $n_u > n_y$. Conversely, the success rates of the condition for the enlarged space containing $y(t)$ are higher when $n_y > n_u$. Out of all valid solutions, the smallest absolute eigenvalue of the multiplier Λ_2 was smaller than 10^{-3} only at two occasions (0.03%). The controller reconstruction was disturbed by the close-to-singular multiplier and yielded an unstable closed-loop system only once.

(n_u, n_y)	Thm. 14	Thm. 16	Thm. 14 or 16
	n=4		
(3,2)	924 (92.4%)	859 (85.9%)	983 (98.3%)
(2,3)	847 (84.7%)	943 (93.4%)	972 (97.2%)
	n=6		
(3,4)	796 (79.6%)	856 (85.6%)	943 (94.3%)
(4,3)	881 (88.1%)	819 (81.9%)	954 (95.4%)

Table 8.1: Success rate on the SOFS problem.

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Paper F

Linear Matrix Inequalities for Analysis and Control of Linear Vector Second-Order Systems

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This paper was submitted

Abstract

Many dynamical systems are modelled as vector second order differential equations. This paper presents analysis and synthesis conditions in terms of Linear Matrix Inequalities (LMI) with explicit dependence in the coefficient matrices of vector second-order systems. These conditions benefit from the separation between the Lyapunov matrix and the system matrices by introducing matrix multipliers, which potentially reduce conservativeness in hard control problems. Multipliers facilitate the usage of parameter-dependent Lyapunov functions as certificates of stability of uncertain and time-varying vector second-order systems. The conditions introduced in this work have the potential to increase the practice of analyzing and controlling systems directly in vector second-order form.

1   Introduction

Many physical systems have dynamics governed by linear time-invariant ordinary differential equations (ODEs) formulated in the vector second order form

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = Ff(t) \tag{9.1}$$

where $q(t) \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times n_f}$ and $f(t) \in \mathbb{R}^{n_f}$ is the force input vector. Depending on the type of loads (i.e. conservative, non-conservative), matrices M, C, K have a particular structure. Conservative systems (i.e. pure structural systems) possess symmetric system matrices. Non-conservative systems yielding from the fields of aeroelasticity, rotating machinery, and interdisciplinary system dynamics usually possess nonsymmetric system matrices. For control purposes, system (9.1) is often re-written as first-order differential equations

$$\dot{x}(t) = Ax(t) + Bf(t) \tag{9.2a}$$

commonly referred to as state-space form. The relationship between the physical coordinate description (9.1) and the state-space description (9.2) is simply

$$x(t) := \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}, \quad A := \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} \tag{9.3}$$

where a nonsingular matrix M is assumed. Working with the model in physical coordinates has some advantages over the model in state-space form [1], [2], [3]:

- Physical interpretation of the coefficient matrices and insight of the original problem are preserved;

- Natural properties of the coefficient matrices like bandedness, definiteness, symmetry and sparsity are preserved;

- Unlike first-order systems in which the acceleration is composed as a linear combination of position and velocity states by an additional circuitry, the acceleration feedback can be utilized in its original form;

- Physical coordinates favour computational efficiency, because the dimension of the vector $x(t)$ is twice that of the vector $q(t)$;

- Complicating nonlinearities in the parameters introduced by inversion of a parameter-dependent mass matrix are avoided.

The stability of vector second-order systems received considerable interest during the last four decades. In [4] several sufficient conditions for stability and instability using Lyapunov theory are derived. Necessary and sufficient conditions of Lyapunov stability, semistability and asymptotic stability are proposed in [5]. This work also brings a substantial literature survey up to 1995. In [2] the necessary and sufficient conditions of stability are based on the Generalized Hurwitz criteria. A desirable property of these works is the explicit dependence of the conditions on the system coefficient matrices. An undesirable fact is that conditions are particular to systems under different dynamic loadings.

Most of the research on feedback-control design of vector-second order systems has focused on stabilization, pole assignment, eigenstructure assignment and observer design. Identification errors in mechanical systems might be quite large. Therefore, robust stability of the closed-loop system is of utmost importance. The fact that stability of some classes of vector-second order systems can be ensured by qualitative condition on the coefficient matrices has facilitated the design of robust stabilizing controllers. In [6] conditions for robust stabilization via static feedback of velocity and displacement were motivated by the stability condition $M^T = M \succ 0, C^T + C \succ 0, K^T = K \succ 0$ in the coefficient matrices An extension to dynamic displacement feedback control law is presented in [7]. Dissipative system theory is exploited in [8, 9] for the synthesis of stabilizing controllers. All these approaches result in closed-loop systems inherently insensitive to plant uncertainties. Based on the eigenvalue analysis of real symmetric interval matrices, in [10] the authors propose sufficient conditions for robust stabilizability considering structured uncertainty in the system matrices. A transformation on the system matrices suitable for modal control is proposed in [3]. Partial pole assignment techniques via state feedback control are proposed in [11, 12]. Robustness in the partial pole assignment problem is considered in [13]. An effective method for partial eigenstructure assignment for systems with symmetric mass, damping and stiffness coefficients is presented in [14]. Robust eigenstructure assignment is treated in [15]. Vector second-order observer and their design are addressed in [16, 17, 18].

Despite these efforts, the first-order state-space remains the preferred representation due to the abundance of control techniques and numerical algorithms tailored for such. As far as modern, optimization-based control theories are concerned, the literature lacks on results to handle the systems directly in second-order form. An interesting contribution towards this goal is the stability results of [19] for systems in standard phase-variable canonical form, given in terms of linear matrix inequalities (LMI) extended with multipliers. The numerical tools of modern convex optimization can solve these problems efficiently [20]. The authors also mentioned the possibility of generalizing these results to systems described by higher order vector differential equations.

The present manuscript extend the results in [19] by presenting conditions for analysis and synthesis of vector second-order systems given in terms of LMI. We believe that the conditions here introduced have the potential to increase the practice in analyzing and controlling mechanical systems explicitly in physical coordinates. Necessary and sufficient LMI criteria for checking stability of vector second-order systems is presented in Section 2. Some of these benefit from the separation between the Lyapunov matrix and

the system matrices by introducing Lagrange multipliers, which potentially reduce conservativeness in robust and other hard control problems [21, 19, 22, 23]. The multipliers facilitate the usage of parameter-dependent Lyapunov functions as certificates of stability of uncertain and time-varying systems. They also allow structural constraints on the controller to be addressed less conservatively. Elimination of multipliers is investigated to determine in which circumstances multipliers can be removed without loss of generality. The stabilization problem by a full vector second-order feedback as well as the problem of clustering the closed-loop system poles in a convex region of the complex plane, namely $\mathcal{D}$-Stability, completes the results related to stability. A gradual extension for systems with inputs and outputs during Section 3 leads to the criteria of synthesis subject to Integral Quadratic Constrains (ICQ). Conditions for the design of static state and output feedback controllers in vector second-order form are addressed, with focus on the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain performance measure due to its importance in robust control. Section 4 concludes the paper and suggest topics for future work.

2 Asymptotic Stability

Let us recall some concepts of Lyapunov stability for first-order state-space systems before working with vector second-order representation. Consider the dynamics of a continuous-time linear time-invariant (LTI) system governed by the differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{9.4}$$

where $x(t) : [0, \infty] \to \mathbb{R}^{2n}$ and $A \in \mathbb{R}^{2n \times 2n}$. Define the quadratic Lyapunov function $V : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$V(x) := x(t)^T P x(t) \tag{9.5}$$

where $P \in \mathbb{S}^{2n}$. According to Lyapunov theory, system (9.4) is asymptotically stable if there exists $V(x(t)) > 0, \forall x(t) \neq 0$ such that

$$\dot{V}(x(t)) < 0, \quad \dot{x}(t) = Ax(t), \quad \forall x(t) \neq 0. \tag{9.6}$$

In words, if there exists $P \succ 0$ such that the time derivative of the quadratic Lyapunov function (9.5) is negative along all trajectories of system (9.4). Conversely, if the linear system (9.4) is asymptotically stable then there always exists $P \succ 0$ that satisfies (9.6). These two affirmatives imply the well known fact that Lyapunov theory with quadratic functions is necessary and sufficient to prove stability of LTI systems. The usual way to obtain a linear matrix inequalities (LMI) condition equivalent to (9.6) is to explicitly substitute (9.4) into (9.5) [20], that is

$$\dot{V}(x(t)) = x(t)^T \left(A^T P + PA \right) x(t) < 0, \quad \forall x(t) \neq 0. \tag{9.7}$$

The condition (9.7) is equivalent to the LMI feasibility problem

$$\exists P \in \mathbb{S}^{2n} : P \succ 0, \quad A^T P + PA \prec 0. \tag{9.8}$$

The fact that (9.6) is a set characterized by inequalities subject to dynamic equality constraints is explored in [19] to propose a constrained optimization solution to the stability problem. It is then possible to characterize the set defined by (9.6) without substituting

(9.4) explicitly into $\dot{V}(x(t)) < 0$ of (9.6) [19]. The well know Finsler's Lemma [24] is the main mathematical tool to transform the constrained optimization problem into a problem subject to LMI constraints.

Lemma 9 (Finsler). *Let $x(t) \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$ and $\mathcal{B} \in \mathbb{R}^{m \times n}$ such that $\mathrm{rank}(\mathcal{B}) < n$. The following statements are equivalent.*

  i. $x(t)^T \mathcal{Q} x(t) < 0, \quad \forall \mathcal{B} x(t) = 0, \quad x(t) \neq 0.$

  ii. $\mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^{\perp} \prec 0.$

  iii. $\exists \mu \in \mathbb{R} \,:\, \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0.$

  iv. $\exists \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{X} \mathcal{B} + \mathcal{B}^T \mathcal{X}^T \prec 0.$

A similarity between statement i. of the above lemma and (9.6) can be noticed. In contrast to (9.7), the space of statement i. is composed of $x(t)$ and $\dot{x}(t)$ that can be seen as an *enlarged space* [19]. Statements iii. and iv. can be seen as an equivalent unconstrained quadratic forms of i. [19]. The equality constraint $\forall \dot{x}(t) = Ax(t)$ is included in the formulation weigthed by the Lagrangian scalar multiplier μ or matrix multiplier $\mathcal{X}$.

In order to obtain a stability condition for an unforced system of the form (9.1) ($f(t) = 0$), define the quadratic Lyapunov function $V : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$V(q(t), \dot{q}(t)) := \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}^T P \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} := \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} \tag{9.9}$$

where $P \in \mathbb{S}^{2n}$ is conveniently partitioned into $P_1, P_3 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^n$. Resorting to Lyapunov theory once again, system (9.1) is asymptotically stable if, and only if, there exists $V(q(t), \dot{q}(t)) > 0, \forall q(t), \dot{q}(t) \neq 0$ such that

$$\dot{V}(q(t), \dot{q}(t)) < 0, \quad \forall M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0, \quad \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} \neq 0 \tag{9.10}$$

with the time derivative of the quadratic function as

$$\dot{V}(q(t), \dot{q}(t)) = \begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} + \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} < 0. \tag{9.11}$$

Let an enlarged state space vector be defined as $x(t) := \left(q(t)^T, \, \dot{q}(t)^T, \, \ddot{q}(t)^T \right)^T$. For this enlarged space, the constrained Lyapunov stability problem becomes

$$\begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix}^T \begin{bmatrix} 0 & P_1 & P_2 \\ P_1 & P_2 + P_2^T & P_3 \\ P_2^T & P_3 & 0 \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} < 0,$$

$$\forall M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0, \quad \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} \neq 0. \tag{9.12}$$

An LMI stability condition for vector second-order systems results from the direct application of Finsler's Lemma 9 to the problem above.

Theorem 18. *System* (9.1) *is asymptotically stable if, and only if,*

i. $\exists P_2 \in \mathbb{R}^{n \times n}$, P_1, $P_3 \in \mathbb{S}^n$:

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ E_{31} & E_{32} \end{bmatrix}^T \begin{bmatrix} 0 & P_1 & P_2 \\ P_1 & P_2 + P_2^T & P_3 \\ P_2^T & P_3 & 0 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ E_{31} & E_{32} \end{bmatrix} \prec 0, \quad (9.13a)$$

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ E_{31} & E_{32} \end{bmatrix} := \begin{bmatrix} K & C & M \end{bmatrix}^\perp,$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \quad (9.13b)$$

ii. $\exists P_2 \in \mathbb{R}^{n \times n}$, P_1, $P_3 \in \mathbb{S}^n$, $\lambda \in \mathbb{R}$:

$$\begin{bmatrix} -\lambda K^T K & P_1 - \lambda K^T C & P_2 - \lambda K^T M \\ \star & P_2 + P_2^T - \lambda C^T C & P_3 - \lambda C^T M \\ \star & \star & -\lambda M^T M \end{bmatrix} \prec 0, \quad (9.14a)$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \quad (9.14b)$$

iii. $\exists \Phi$, Γ, Λ, $P_2 \in \mathbb{R}^{n \times n}$ *and* P_1, $P_3 \in \mathbb{S}^n$:

$$\begin{bmatrix} K^T \Phi^T + \Phi K & P_1 + K^T \Gamma^T + \Phi C & P_2 + K^T \Lambda^T + \Phi M \\ \star & P_2 + P_2^T + C^T \Gamma^T + \Gamma C & P_3 + C^T \Lambda^T + \Gamma M \\ \star & \star & M^T \Lambda^T + \Lambda M \end{bmatrix} \prec 0,$$
$$(9.15a)$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \quad (9.15b)$$

Proof. Assign

$$x(t) \leftarrow \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix}, \quad \mathcal{Q} \leftarrow \begin{bmatrix} 0 & P_1 & P_2 \\ P_1 & P_2 + P_2^T & P_3 \\ P_2^T & P_3 & 0 \end{bmatrix}, \quad \mathcal{B}^T \leftarrow \begin{bmatrix} K^T \\ C^T \\ M^T \end{bmatrix}, \quad \mathcal{X} \leftarrow \begin{bmatrix} \Phi \\ \Gamma \\ \Lambda \end{bmatrix}$$

and apply Lemma 9 to the constrained Lyapunov problem (9.12) with $P \succ 0$. □

Notice the diagonal entries of the first inequality of statement ii., i.e. $\lambda K^T K \succ 0$ and $\lambda M^T M \succ 0$, which implies that $\lambda > 0$ and K, M nonsingular. The condition reflects that *asymptotic* stability of mechanical systems requires that no eigenvalues of matrix K should lie on the imaginary axis, i.e. no rigid body modes. At last, notice that the condition does not enforce any specific requirement on the structure of the damping matrix C (except $C \neq 0$). Thus, applicable to system under different loadings.

When applied to systems in first order form, the LMI conditions extended with multipliers are obtained by applying the Finsler's lemma with [19]

$$x(t) \leftarrow \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \quad \mathcal{Q} \leftarrow \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}, \quad \mathcal{B}^T \leftarrow \begin{bmatrix} A^T \\ -I \end{bmatrix}, \quad \mathcal{B}^\perp \leftarrow \begin{bmatrix} I \\ A \end{bmatrix}.$$

In this case, the standard Lyapunov LMI condition (9.8) can be easily recovered from Lemma 9 item ii.. Notice that the basis for the null space $\mathcal{B}^\perp$ can be computed in a straighforward manner from $\mathcal{B}$ (the identity matrix I is convenient). However, for vector second order systems, it is not trivial to find a basis $\mathcal{B}^\perp$ from $\mathcal{B} := \begin{bmatrix} K & C & M \end{bmatrix}$ without resorting to matrix operations on the system matrices. This fact reflects the inherent difficulties in finding standard Lyapunov stability conditions for general vector second order systems. The statement i. of Theorem 18 depends solely on the Lyapunov matrices and thus can be interpreted as a standard Lyapunov condition, with the drawback that matrix inversion is required to determine $\mathcal{B}^\perp$. Taking advantage of the non-singularity of the mass matrix, a trivial choice for $\mathcal{B}^\perp$ is

$$\mathcal{B}^\perp := \begin{bmatrix} -I & 0 \\ 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix}$$

yielding the LMI stability condition $\exists P_1, P_3 \in \mathbb{S}^n, P_2 \in \mathbb{R}^n$:

$$\begin{bmatrix} -P_2 M^{-1}K - K^T M^{-T} P_2^T & P_1 - P_2 M^{-1}C - K^T M^{-T} P_3 \\ P_1 - P_3 M^{-1}K - C^T M^{-T} P_2^T & P_2 + P_2^T - P_3 M^{-1}C - C^T M^{-T} P_3 \end{bmatrix} \prec 0,$$
(9.16a)

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0.$$
(9.16b)

Less trivial conditions involve inverses of both mass and stiffness matrices. Consider the factorization

$$\mathcal{B} := \mathcal{B}_1 \mathcal{B}_2 := \begin{bmatrix} I & C & I \end{bmatrix} \begin{bmatrix} K & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M \end{bmatrix} = \begin{bmatrix} K & C & M \end{bmatrix}.$$

The nonsingularity assumption on M and K implies that $\mathcal{B}_2$ is also nonsingular and thus invertible. Notice that

$$\mathcal{B}_1 \mathcal{B}_2 x = 0, \quad x \neq 0 \quad \Leftrightarrow \quad \mathcal{B}_1 y = 0, \quad y = \mathcal{B}_2 x \neq 0.$$

Therefore

$$y = \mathcal{B}_1^\perp z \quad \Rightarrow \quad x = \mathcal{B}_2^{-1} y = \mathcal{B}_2^{-1} \mathcal{B}_1^\perp z \quad \Rightarrow \quad \mathcal{B}^\perp = \mathcal{B}_2^{-1} \mathcal{B}_1^\perp.$$

The basis for the null space $\mathcal{B}^\perp$ can now be determined from $\mathcal{B}_1$. As an example

$$\mathcal{B}_1^\perp := \begin{bmatrix} C & 0 \\ -I & -I \\ 0 & C \end{bmatrix} \quad \Rightarrow \quad \mathcal{B}^\perp := \mathcal{B}_2^{-1} \mathcal{B}_1^\perp = \begin{bmatrix} K^{-1}C & 0 \\ -I & -I \\ 0 & M^{-1}C \end{bmatrix}$$

This particular choice of $\mathcal{B}_1^\perp$ yields the LMI stability condition: $\exists P_1, P_3 \in \mathbb{S}^n, P_2 \in \mathbb{R}^n$:

$$\begin{bmatrix} -C^T K^{-T} P_1 - P_1 K^{-1}C + P_2 + P_2^T & F(x) \\ \star & -P_3 M^{-1}C - C^T M^{-T} P_3 + P_2 + P_2^T \end{bmatrix} \prec 0$$

$$F(x) := -C^T K^{-T} P_1 + C^T K^{-T} P_2 M^{-1}C - P_3 M^{-1}C + P_2 + P_2^T$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0.$$

Some of the Lyapunov stability conditions available in the literature [5] could, in principle, be replicated by choosing $\mathcal{B}^\perp$ appropriately. The presented conditions of asymptotic stability with only Lyapunov matrices as decision variables are quite intricate because of the presence of matrix inverses M^{-1} and K^{-1}. When M and K are given real, the inverse can be computed numerically. Complicating nonlinearities arise as soon as system matrices are functions of uncertain parameters or controller variables.

Linearizing Lyapunov Function

The undesirable dependence of (9.16a) in the inverse of the mass matrix can be circumvented by a suitable modification of the quadratic Lyapunov function (9.9). The next theorem presents a standard Lyapunov stability condition with linear dependence on the system matrices.

Theorem 19. *System (9.1) is asymptotically stable if, and only if, $\exists P_1 \in \mathbb{S}^n$, W_2, $W_3 \in \mathbb{R}^{n \times n}$:*

$$\begin{bmatrix} -(W_2 K + K^T W_2^T) & P_1 - W_2 C - K^T W_3^T \\ P_1 - W_3 K - C^T W_2^T & P_2 + P_2^T - (W_3 C + C^T W_3^T) \end{bmatrix} \prec 0, \tag{9.17a}$$

$$\begin{bmatrix} P_1 & W_2 M \\ M^T W_2^T & W_3 M \end{bmatrix} + \begin{bmatrix} P_1 & W_2 M \\ M^T W_2^T & W_3 M \end{bmatrix}^T \succ 0, \tag{9.17b}$$

$$W_3 M = M^T W_3^T \tag{9.17c}$$

Proof. Inequality (9.17a) results from (9.16a) with the change-of-variables $P_2 := W_2 M$ and $P_3 := W_3 M$. This change-of-variables substitutes the symmetric matrix P_3 with a general matrix $W_3 M$ yielding a non-symmetric Lyapunov matrix (9.16b). Constraints (9.17b), (9.17c) are equivalent versions of the Lyapunov matrix and restores the symmetry of the problem. Equality constraint (9.17c) enforces symmetry to $W_3 M$, while inequality (9.17b) arises from the fact that $P \succ 0 \Leftrightarrow (P + P^T) \succ 0$. $\qquad\square$

Although the above theorem offers a quadratic stability certificate with linear dependence on the system matrices, the conditions extended with multipliers are appealing due to the extra degrees of freedom introduced by these variables.

Elimination of Multipliers

The matrix inequality (9.15) is a function of the multipliers Φ, Γ, Λ. It is worth questioning if all degrees of freedom introduced by the multipliers are really necessary. Would it be possible to constrain or eliminate multipliers without loss of generality? The Elimination Lemma [20, 25] will serve for the purpose of removing multipliers without adding conservatism to the solution.

Lemma 10 (Elimination Lemma). *Let $\mathcal{Q} \in \mathbb{S}^n$, $\mathcal{B} \in \mathbb{R}^{m \times n}$, $\mathcal{C} \in \mathbb{R}^{n \times k}$. The following statements are equivalent.*

i. $\exists \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{C}^T \mathcal{X} \mathcal{B} + \mathcal{B}^T \mathcal{X}^T \mathcal{C} \prec 0$

ii. $\mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^{\perp} \prec 0$ (9.18a) $\mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^{\perp} \prec 0$ (9.18b)

iii. $\exists \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0, \quad \mathcal{Q} - \mu \mathcal{C}^T \mathcal{C} \prec 0.$

Notice that Elimination Lemma reduces to the Finsler's Lemma when particularized with $\mathcal{C} = I$. In such a case $\mathcal{C}^{\perp} = \{0\}$ and (9.18b) is removed from the statement. A discussion on the relation between these two lemmas can be found in [20, 25]. The elimination of multipliers on LMI conditions for systems in first-order form was studied in [3]. In general terms, the idea is to select a suitable $\mathcal{C}$ such that (9.18b) does not introduce conservatism to the original problem while reducing the size of the multiplier $\mathcal{X}$. The next theorems result from a similar rationale.

Theorem 20. *System* (9.1) *is asymptotically stable if, and only if,* $\exists \Phi, \Lambda, P_2 \in \mathbb{R}^{n \times n}$ *and* $P_1, P_3 \in \mathbb{S}^n$:

$$\begin{bmatrix} \Phi K + \star & P_1 + K^T(\alpha\Phi + \Lambda)^T + \Phi C & P_2 + \alpha K^T \Lambda^T + \Phi M \\ \star & P_2 + (\alpha\Phi + \Lambda)C + \star & P_3 + \alpha C^T \Lambda^T + (\alpha\Phi + \Lambda)M \\ \star & \star & \alpha\Lambda M + \star \end{bmatrix} \prec 0,$$
$$(9.19a)$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \tag{9.19b}$$

for an arbitrary scalar $\alpha > 0$.

Proof. Assign

$$\mathcal{Q} \leftarrow \begin{bmatrix} 0 & P_1 & P_2 \\ P_1 & P_2 + P_2^T & P_3 \\ P_2^T & P_3 & 0 \end{bmatrix}, \quad \mathcal{B}^T \leftarrow \begin{bmatrix} K^T \\ C^T \\ K^T \end{bmatrix}, \quad \mathcal{C}^{\perp} \leftarrow \begin{bmatrix} \alpha^2 I \\ -\alpha I \\ I \end{bmatrix},$$

$$\mathcal{C}^T \leftarrow \begin{bmatrix} I & 0 \\ \alpha I & I \\ 0 & \alpha I \end{bmatrix}^T, \quad \mathcal{X} \leftarrow \begin{bmatrix} \Phi \\ \Lambda \end{bmatrix}.$$

and apply the Elimination Lemma with $P \succ 0$. The chosen $\mathcal{C}^{\perp}$ does not introduce conservativeness to the condition. To see this expand (9.18b)

$$\mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^{\perp} = -\alpha^3 P_1 - \alpha P_3 + \alpha^2 P_2 + \alpha^2 P_2^T \prec 0 \tag{9.20a}$$

$$\Updownarrow$$

$$\alpha^3 P_1 + \alpha P_3 - \alpha^2 P_2 - \alpha^2 P_2^T \succ 0. \tag{9.20b}$$

Notice that the following support inequality

$$-NW^{-1}N^T \preceq W - N - N^T \tag{9.21}$$

holds whenever $W \succ 0$. Resorting to the support inequality with $N := \alpha^2 P_2$, $W := \alpha P_3 \succ 0$, (9.20b) is satisfied whenever

$$\alpha^3 (P_1 - P_2 P_3^{-1} P_2^T) \succ 0. \tag{9.22}$$

$P_1 - P_2 P_3^{-1} P_2^T \succ 0$ is equivalent to (9.19b) by a Schur complement argument and thus positive definite. Therefore (9.22) and consequently (9.20b) holds for an arbitrary real scalar $\alpha > 0$. □

A similar, equivalent characterization of the Theorem above can be derived by assigning

$$\mathcal{C}^{\perp} \leftarrow \begin{bmatrix} I \\ -\alpha I \\ \alpha^2 I \end{bmatrix}, \quad \mathcal{C}^T \leftarrow \begin{bmatrix} \alpha I & 0 \\ I & \alpha I \\ 0 & I \end{bmatrix}^T$$

and following the same steps presented on the proof.

The number of multipliers can be further reduced by constraining $\Phi := \mu \Lambda$ in (9.19a) where $\mu > 0$ is a real scalar. Unfortunately, this constraint introduce conservativeness leading to a sufficient condition.

Theorem 21. *System (9.1) is asymptotically stable if $\exists P_2, \Lambda \in \mathbb{R}^{n \times n}$, $P_1, P_3 \in \mathbb{S}^n$, $\mu, \in \mathbb{R}$:*

$$\begin{bmatrix} \mu \Lambda K + \star & P_1 + (1 + \alpha \mu) K^T \Lambda^T + \mu \Lambda C & P_2 + \alpha K^T \Lambda^T + \mu \Lambda M \\ \star & P_2 + (1 + \alpha \mu) \Lambda C + \star & P_3 + \alpha C^T \Lambda^T + (1 + \alpha \mu) \Lambda M \\ \star & \star & \alpha \Lambda M + \star \end{bmatrix} \prec 0 \tag{9.23a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \alpha > 0, \quad \mu > 0. \tag{9.23b}$$

for an arbitrary scalar $\alpha > 0$.

A source of conservatism is the appearance of a single multiplier on the block diagonal entries of (9.23a). For $M \succ 0$, usual property of the mass matrix, the (1,1) block $\alpha(M^T \Lambda^T + \Lambda M) \prec 0$ with $\alpha > 0$ is satisfied only if $\Lambda \prec 0$. As a consequence, stability cannot be certified when $M \succ 0$ and K is indefinite because $\mu(K^T \Lambda^T + \Lambda K) \prec 0$ never holds when $\mu > 0$ and $\Lambda \prec 0$. Numerical experiments suggest that a similar situation is encountered when the matrix C is indefinite and M or K are positive definite. The condition was unable to attest stability of randomly generated stable systems (M, C, K) in which C had at least one negative eigenvalue and $M, K \succ 0$. $P_2 + P_2^T \succ 0$ holds whenever the condition was able to find a certificate of stability, another contributing fact to why the (2-2) block cannot be verified as negative definite when C is indefinite.

Stabilization by Static State Feedback

The dependence of the stability condition to a single multiplier Λ is particularly interesting in the context of feedback stabilization. The vector second order system is augmented with a controllable input $u(t) \in \mathbb{R}^{n_u}$

$$M\ddot{q}(t) + C\dot{q}(t) + K\dot{q}(t) = F_u u(t), \qquad q(0), \ \dot{q}(0) = 0 \tag{9.24}$$

where $F_u \in \mathbb{R}^{n \times n_u}$. Consider a static state feedback controller of the form

$$u(t) = -G_a \ddot{q}(t) - G_v \dot{q}(t) - G_p q(t) \tag{9.25}$$

where G_a, G_v, $G_p \in \mathbb{R}^{n \times n}$ are static feedback gains from acceleration, velocity and position, respectively. The plant (9.24) in closed-loop with the controller (9.25) yields the equations of motion

$$\mathbf{M}\ddot{q}(t) + \mathbf{C}\dot{q}(t) + \mathbf{K}q(t) = 0 \tag{9.26a}$$

$$\mathbf{M} := (M + F_u G_a), \quad \mathbf{C} := (C + F_u G_v), \quad \mathbf{K} := (K + F_u G_p) \tag{9.26b}$$

Conditions for controller synthesis often involve products between controller gains and Lyapunov matrices or multipliers, resulting in nonlinear matrix inequalities. The nonlinear terms can be linearized by resorting to the change-of-variables, firstly introduced in [26] in which only the Lyapunov variable is involved, and later in the context of conditions extended with multipliers [21]. Define the following nonlinear change-of-variables

$$\hat{G}_a := G_a \Lambda, \quad \hat{G}_v := G_v \Lambda, \quad \hat{G}_p := G_p \Lambda. \tag{9.27}$$

Notice from (9.23) that the matrix Λ multiplies the system matrices in a position not suitable for linearization of the nonlinear terms, i.e. $\Lambda(K + F_u G_p)$. A dual transformation of the closed-loop system

$$\mathbf{M} \leftarrow \mathbf{M}^T, \quad \mathbf{C} \leftarrow \mathbf{C}^T, \quad \mathbf{K} \leftarrow \mathbf{K}^T \tag{9.28}$$

makes the linearizing change of variables possible. A discussion on algebraic duality of vector second-order system with inputs and outputs is given in the appendix. It is worth mentioning that the above dual transformation preserves the eigenvalues of the system cast in first-order form, that is

$$\lambda \left(\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \right) = \lambda \left(\begin{bmatrix} 0 & I \\ -M^{-T}K^T & -M^{-T}C^T \end{bmatrix} \right).$$

With these definitions at hand, the stabilizability conditions by static feedback can now be stated.

Theorem 22. *System* (9.24) *is stabilizable by a static feedback law of the form* (9.25) *if* $\exists \Lambda, P_2 \in \mathbb{R}^{n \times n}, P_1, P_3 \in \mathbb{S}^n, \hat{G}_a, \hat{G}_v, \hat{G}_p \in \mathbb{R}^{n_u \times n}, \alpha, \mu \in \mathbb{R}:$

$$\begin{bmatrix} \mu(K\Lambda + F_u\hat{G}_p) + \star & P_1 + (1 + \alpha\mu)(K\Lambda + F_u\hat{G}_p) + \mu(C\Lambda + F_u\hat{G}_v)^T \\ \star & P_2 + (1 + \alpha\mu)(C\Lambda + F_u\hat{G}_v) + \star \\ \star & \star \end{bmatrix}$$

$$\begin{matrix} P_2 + \alpha(K\Lambda + F_u\hat{G}_p) + \mu(M\Lambda + F_u\hat{G}_a)^T \\ P_3 + \alpha(C\Lambda + F_u\hat{G}_v) + (1 + \alpha\mu)(M\Lambda + F_u\hat{G}_a)^T \\ \alpha(M\Lambda + F_u\hat{G}_a) + \star \end{matrix} \end{bmatrix} \prec 0, \tag{9.29a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \mu > 0, \tag{9.29b}$$

and Λ is nonsingular.

Proof. The LMI (9.29) results from a direct application of Proposition 20 to the dual of closed-loop system (9.26), together with a dual transformation $\Lambda \leftarrow \Lambda^T$ of the multiplier and the change-of-variables (9.27). The change-of-variables are without loss of generality when Λ is nonsingular thus invertible. The original controller gains can then be recovered by the inverse map

$$G_a = \hat{G}_a \Lambda^{-1}, \quad G_v = \hat{G}_v \Lambda^{-1}, \quad G_p = \hat{G}_p \Lambda^{-1}. \tag{9.30}$$

which characterizes the stabilizable control law. □

A nonsingular multiplier Λ is not implied by inequality (9.29). This fact contrasts with stability criteria for systems in first-order form where nonsingularity of multipliers is a direct consequence of the structure of the LMI [22, 27]. With some restrictions imposed on the problem formulation, it is possible to ensure a nonsingular Λ. For instance, the multiplier can be confined to the positive cone of symmetric matrices, i.e. $\Lambda \in \mathbb{S}^n, \Lambda \succ 0$, or to the negative cone of symmetric matrices $\Lambda \in \mathbb{S}^n, \Lambda \prec 0$. In these cases, extra conservativeness is brought into the condition.

An assumption that facilitates a nonsingular Λ without adding conservativeness is to exclude the acceleration feedback, i.e. $G_a = 0$. In this case, the lower right block $M\Lambda + \Lambda^T M^T \prec 0$ of the LMI (9.29) with M nonsingular implies a nonsingular multiplier Λ. Therefore, a stabilizing controller can be computed according to (9.30) whenever (9.29) is feasible. Note that the control law (9.25) with $G_a = 0$ is a full state feedback in the first-order state-space sense.

As metioned in the introduction section, acceleration feedback is often desirable due to practical reasons. When position feedback is excluded from the control law ($G_p = 0$) the multiplier is assured to be nonsingular. The entry $\mu(K\Lambda + \Lambda^T K^T) \prec 0$ located in the upper left of (9.29) with K nonsingular and $\mu > 0$ implies Λ nonsingular. Therefore, once a solution for the LMI problem above is found, the controller gains can be reconstructed according to (9.30).

$\mathcal{D}$-Stability

Performance specifications like time response and damping in closed-loop can often be achieved by clustering the closed-loop poles into a suitable subregion of the complex plane. The subclass of *convex regions* of the complex plane can be characterized in terms of LMI constraints [28]. A class of convex subregions representable as LMI conditions extended with multipliers was proposed in [29]. Let $R_{11}, R_{22} \in \mathbb{S}^d, R_{12} \in \mathbb{R}^d, R_{22} \succeq 0$. The $\mathcal{D}_R$ region of the complex plane is defined as the set [29]

$$\mathcal{D}_R(s) := \left\{ s \in \mathbb{C} \ : \ R_{11} + R_{12}s + R_{12}^T s^H + R_{22} s^H s \prec 0 \right\} \tag{9.31}$$

where s is the Laplace operator. An LMI characterization for $\mathcal{D}_R$-stability of vector second-order systems can be derived from $\mathcal{D}_R$-stability condition of a system in first-order form. The autonomous system (9.4) is $\mathcal{D}_R$-stable if and only if $\exists P \in \mathbb{S}^{2n}$: [29]

$$R_{11} \otimes P + R_{12} \otimes (PA) + R_{12}^T \otimes (A^T P) + R_{22} \otimes (A^T PA) \prec 0 \tag{9.32}$$

A relation between regions of the complex plane and a particular Lyapunov constrained problem can be deduced from the above LMI. First define the d-stacked system as

$$x_d(t) := 1_d \otimes x(t), \quad A_d := I_d \otimes A \quad \Rightarrow \quad \dot{x}_d(t) = I_d \otimes A x_d(t) \tag{9.33}$$

where 1_d represents a column vector composed of 1's and I_d is the identity matrix both with dimension d. For example, the d-stacked system for $d = 2$ yields

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}(t) \end{pmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{pmatrix} x(t) \\ x(t) \end{pmatrix}.$$

The time derivative of the Lyapunov function tailored for $\mathcal{D}_R$-stability analysis is defined as

$$\begin{aligned} \dot{V}(x_d(t), \dot{x}_d(t)) := &\; x_d(t)^T R_{11} \otimes P x_d(t) + \dot{x}_d(t)^T R_{12} \otimes P x_d(t) \\ &+ x_d(t)^T R_{12}^T \otimes P \dot{x}_d(t) + \dot{x}_d(t)^T R_{22} \otimes P \dot{x}_d(t) < 0 \end{aligned} \tag{9.34}$$

Substitute (9.33) into (9.34) and expand to arrive at (9.32). The usual time derivative of a quadratic Lyapunov function, i.e. $\dot{V}(x(t)) = \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t)$ is recovered from the above by choosing $R_{11} = R_{22} = 0$, $R_{12} = 1$. The set of solutions of the $\mathcal{D}$-stability problem in time-domain is defined as

$$\mathcal{D}_R(x(t)) := \left\{ x(t) \in \mathbb{R}^n \; : \; \dot{V}(x(t), \dot{x}(t)) < 0, \; \dot{V}(x(t), \dot{x}(t)) \text{ as (9.34)}, \; P \succ 0 \right\}. \tag{9.35}$$

For the sake of $\mathcal{D}_R$-stability of vector second-order systems the d-stacked system is defined

$$q_d(t) := 1_d \otimes q(t), \quad \dot{q}_d(t) := 1_d \otimes \dot{q}(t), \quad \ddot{q}_d(t) := 1_d \otimes \ddot{q}(t), \tag{9.36a}$$

$$M_d := I_d \otimes M, \quad C_d := I_d \otimes C, \quad K_d := I_d \otimes K \tag{9.36b}$$

$$M_d \ddot{q}_d(t) + C_d \dot{q}_d(t) + K_d q_d(t) = 0. \tag{9.36c}$$

Let the constrained Lyapunov problem in the enlarged space be formalized

$$\begin{pmatrix} q_d(t) \\ \dot{q}_d(t) \\ \ddot{q}_d(t) \end{pmatrix}^T \begin{bmatrix} R_{11} \otimes P_1 & R_{11} \otimes P_2 + R_{12} \otimes P_1 & R_{12} \otimes P_2 \\ \star & R_{11} \otimes P_3 + R_{12}^T \otimes P_2 + R_{12} \otimes P_2^T + R_{22} \otimes P_1 & R_{12} \otimes P_3 + R_{22} \otimes P_2 \\ \star & \star & R_{22} \otimes P_3 \end{bmatrix} \begin{pmatrix} q_d(t) \\ \dot{q}_d(t) \\ \ddot{q}_d(t) \end{pmatrix} < 0, \tag{9.37a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \forall M_d \ddot{q}_d(t) + C_d \dot{q}_d(t) + K_d q_d(t) = 0, \quad \begin{pmatrix} q_d(t) \\ \dot{q}_d(t) \\ \ddot{q}_d(t) \end{pmatrix} \neq 0. \tag{9.37b}$$

The $\mathcal{D}$-stability condition for vector second-order systems is stated in the next theorem.

Theorem 23. *System* (9.1) *is* $\mathcal{D}_R$*-stable if, and only if,* $\exists P_1, P_3 \in \mathbb{S}^n, P_2 \in \mathbb{R}^n, \Phi, \Gamma, \Lambda \in \mathbb{R}^{dn \times dn}$:

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad \mathcal{H} := \begin{bmatrix} \Phi(I_d \otimes K) & \Phi(I_d \otimes C) & \Phi(I_d \otimes M) \\ \Gamma(I_d \otimes K) & \Gamma(I_d \otimes C) & \Gamma(I_d \otimes M) \\ \Lambda(I_d \otimes K) & \Lambda(I_d \otimes C) & \Lambda(I_d \otimes M) \end{bmatrix}, \quad (9.38a)$$

$$\mathcal{J} := \begin{bmatrix} R_{11} \otimes P_1 & R_{11} \otimes P_2 + R_{12} \otimes P_1 \\ \star & R_{11} \otimes P_3 + R_{12}^T \otimes P_2 + R_{12} \otimes P_2^T + R_{22} \otimes P_1 \\ \star & \star \end{bmatrix}$$

$$\begin{matrix} R_{12} \otimes P_2 \\ R_{12} \otimes P_3 + R_{22} \otimes P_2 \\ R_{22} \otimes P_3 \end{matrix} \Bigg], \quad (9.38b)$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \quad (9.38c)$$

The proof follows similarly to Theorem 18 and is omitted for brevity. Multipliers need to be eliminated to make the above condition suitable for computing stabilizing controllers. The same choice of $\mathcal{C}^{\perp}$ and $\mathcal{C}$ of Theorems 20 and 21 serve this purpose. However, conservativeness when eliminating multipliers depend on the particular $\mathcal{D}_R$ region. Taking $(\mathcal{C}^{\perp}, \mathcal{C})$ similarly to Theorem 20, $\mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^{\perp} \prec 0$ after expansion and some algebraic manipulations yields

$$R_{11} \otimes \left(\alpha^4 P_1 - \alpha^3 P_2 - \alpha^3 P_2^T + \alpha^2 P_3 \right) + R_{12} \otimes \left(-\alpha^3 P_1 + \alpha^2 P_2 + \alpha^2 P_2^T - \alpha P_3 \right)$$
$$+ R_{12}^T \otimes \left(-\alpha^3 P_1 + \alpha^2 P_2 + \alpha^2 P_2^T - \alpha P_3 \right) + R_{22} \otimes \left(\alpha^2 P_1 - \alpha P_2 - \alpha P_2^T + P_3 \right) \prec 0 \quad (9.39)$$

This inequality depends on the matrices R_{11}, R_{12}, R_{22} that defines the stability region. Although it is not trivial to state non-conservativeness independently of the chosen region, one can attest if the elimination of a multiplier brings any conservativeness for a particular $\mathcal{D}_R$-region. To do so, first note that all of the addends of the above inequality are similar in structure. A correspondence with the Lyapunov matrix P can be established via a congruence transformation involving α, and multiplications with a matrix and its transpose, e.g.

$$Y^T H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} HY \succ 0, \quad H := \mathrm{diag}(\alpha I, I), \quad Y := \begin{bmatrix} I & -I \end{bmatrix}^T.$$

Whenever $H^T P H \succ 0$ holds, which is always the case because of (9.38c), $\alpha^2 P_1 - \alpha P_2 - \alpha P_2^T + P_3 \succ 0$ also holds. Let us take some typical regions as examples. The continuous-time stability region is determined by $R_{11} = R_{22} = 0$, $R_{12} = 1$, rendering non-conservativeness as shown in Theorem 20. For a region with minimum decay rate $\beta > 0$ set with $R_{11} = 2\beta$, $R_{22} = 0$, $R_{12} = 1$, if

$$2\alpha H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} H \succ 2\alpha^2 \beta H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} H \succ 0 \quad (9.40)$$

holds, then (9.39) also holds. Indeed, multiply the inequality above with $Y := \begin{bmatrix} I & -I \end{bmatrix}^T$ from the right and Y^T from the left to obtain (9.39). A set of values of α which does not

introduce conservativeness to the condition can be inferred from (9.40), that is, $\{\alpha : \alpha - \alpha^2\beta > 0, \ \alpha > 0, \ \beta > 0\}$. For the discrete-time stability region, represented as a circle centred at the origin of the complex plane with $R_{11} = -1$, $R_{22} = 1$, $R_{12} = 0$, if

$$\alpha^2 H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} H \succ H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} H \succ 0$$

is satisfied than inequality (9.39) is satisfied. Therefore, any $\alpha > 1$ does not bring conservativeness.

3 Quadratic Performance

The following linear time-invariant vector second-order system with inputs and outputs

$$M\ddot{q}(t) + D\dot{q}(t) + K\dot{q}(t) = F_w w(t), \qquad q(0), \ \dot{q}(0) = 0 \tag{9.41a}$$
$$z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw}w(t) \tag{9.41b}$$

is considered in this section, where $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$ are the disturbance input and performance output vectors, respectively, $U, \ V, \ X \in \mathbb{R}^{n_z \times n}$. The presence of input signals $w(t)$ requires a definition of stability.

$\mathcal{L}_2$ to $\mathcal{L}_2$ Stability

The notion of stability of a system with inputs it related to the characteristics of the input signal $w(t)$. Assume $w(t) : [0, \infty) \to \mathbb{R}^{n_w}$ a piecewise continuous function in the Lebesgue function space $\mathcal{L}_2$

$$\|w(t)\|_{\mathcal{L}_2} := \left(\int_0^\infty w(\tau)^T w(\tau) d\tau \right)^{1/2} < \infty.$$

In the control literature, the quantity $\|w(t)\|_{\mathcal{L}_2}$ is often referred to as the energy of signal $w(t)$. The system (9.41) is said to be $\mathcal{L}_2$ stable if the output signal $z(t) \in \mathcal{L}_2$ for all $w(t) \in \mathcal{L}_2$. Define the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain as the quantity

$$\gamma_\infty := \sup_{w(t) \in \mathcal{L}_2} \frac{\|z\|_2}{\|w\|_2}. \tag{9.42}$$

This quantity can serve as a certificate of $\mathcal{L}_2$ stability. If the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain of a system is finite, i.e. $0 < \gamma_\infty < \infty$, then one can conclude that the system is $\mathcal{L}_2$ stable. Because γ_∞ is bounded by below, it sulffices to find an upper bound γ such that $0 < \gamma_\infty < \gamma < \infty$. Consider the modified Lyapunov stability condition

$$\dot{V}(q(t), \dot{q}(t), \ddot{q}(t)) < 0, \quad z(t)^T z(t) \geq \gamma^2 w(t)^T w(t) \tag{9.43a}$$
$$\forall (q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \text{ satisfying (9.41)}, \tag{9.43b}$$
$$(q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \neq 0. \tag{9.43c}$$

where $\gamma > 0$ is a given scalar. Invoking the S-procedure [20] produces a necessary and sufficient equivalent condition [19]

$$\dot{V}\left(q(t), \dot{q}(t), \ddot{q}(t)\right) < \gamma^2 w(t)^T w(t) - z(t)^T z(t), \tag{9.44a}$$

$$\forall (q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \text{ satisfying (9.41)}, \tag{9.44b}$$

$$(q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \neq 0. \tag{9.44c}$$

To realize that (9.44) implies an $\mathcal{L}_2$ to $\mathcal{L}_2$ gain less than γ, integrate both sides of (9.44a) over time $t > 0$ to get

$$\int_0^t \dot{V}(q(\tau), \dot{q}(\tau), \ddot{q}(\tau)) \, d\tau < \int_0^t \gamma^2 w(\tau)^T w(\tau) - z(\tau)^T z(\tau) \, d\tau \tag{9.45}$$

For $t \to \infty$, the resulting Lyapunov function

$$\int_0^t \dot{V}(q(\tau), \dot{q}(\tau), \ddot{q}(\tau)) \, d\tau = V(q(\tau), \dot{q}(\tau), \ddot{q}(\tau)) > 0 \tag{9.46}$$

is positive by definition. From the above and (9.45) it can be inferred that

$$\|z(t)\|_{\mathcal{L}_2}^2 < \gamma^2 \|w(t)\|_{\mathcal{L}_2}^2. \tag{9.47}$$

which compared to (9.42) implies $\gamma > \gamma_\infty$.

Synthesis of controllers are usually attached to some performance indicator or measure of a system. The $\mathcal{L}_2$ gain also serve as a system performance measure.

$\mathcal{L}_2$ Performance

The $\mathcal{L}_2$ to $\mathcal{L}_2$ gain may also be interpreted as a performance measure. The system is supposed to have good performance when $z(t)$ is small regardless of the disturbance $w(t)$. The quantity γ provides a measure (in an $\mathcal{L}_2$ sense) of the size of the output signal $z(t)$ in response to the worst-case disturbance $w(t)$ with zero initial conditions. This is the time domain interpretation. A frequency domain interpretation can also be attributed to the $\mathcal{L}_2$ gain. Obtain the frequency domain couterpart of system (9.41)

$$s^2 M Q(s) + s C Q(s) + K Q(s) = F_w W(s), \tag{9.48a}$$

$$Z(s) = s^2 U Q(s) + s V Q(s) + X Q(s) + D_{zw} W(s) \tag{9.48b}$$

by applying the Laplace transform, where s in the Laplace operator. Define the input-output transfer matrix $H_{zw}(s)$ as

$$H_{zw}(s) := \left(s^2 U + s V + X\right) \left(s^2 M + s C + K\right)^{-1} F_w + D_{zw}. \tag{9.49}$$

It is well know that

$$\sup_\omega |H_{zw}(j\omega)| = \|H_{zw}\|_{\mathcal{H}_\infty} = \sup_{w(t) \in \mathcal{L}_2} \frac{\|z\|_2}{\|w\|_2}.$$

due to Parseval's theorem [30]. Hence, $\|H_{zw}\|_{\mathcal{H}_\infty} < \gamma$ is equivalent to $\mathcal{L}_2$ to $\mathcal{L}_2$ stability. Moreover, the quantity γ gives an upper bound on the maximum singular value of the transfer matrix $H_{zw}(j\omega)$.

Integral Quadratic Constraints

The notion of system performance can be further generalized by enforcing an integral quadratic constraint on the input and output signals [31, 32]

$$\int_0^t \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0 \tag{9.50}$$

where $Q \in \mathbb{S}^{n_z}$, $R \in \mathbb{S}^{n_w}$, $S \in \mathbb{R}^{n_z \times n_w}$. Similarly to (9.45), pose the inequality

$$\int_0^t \dot{V}(q(\tau), \dot{q}(\tau), \ddot{q}(\tau))\, d\tau < -\int_0^t \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \tag{9.51}$$

The right hand side of the above inequality can be seen as a quadratic contraint on the Lyapunov quadratic function $V(q(t), \dot{q}(t), \ddot{q}(t))$. The modified Lyapunov problem then becomes

$$\dot{V}(q(t), \dot{q}(t), \ddot{q}(t)) < -\begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}, \tag{9.52a}$$

$$\forall(q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \text{ satisfying (9.41)}, \tag{9.52b}$$

$$(q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \neq 0, \tag{9.52c}$$

ready to be transformed in an LMI condition by Finsler's Lemma.

Theorem 24 (Integral Quadratic Constraints). *The following statements are equivalent.*

i. The set of solutions of the Lyapunov problem (9.52) *with* $\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0$ *is not empty.*

ii. $\exists P_1,\ P_3 \in \mathbb{S}^n,\ \Phi_1,\ \Gamma_1,\ \Lambda_1,\ P_2 \in \mathbb{R}^n,\ \Pi_1 \in \mathbb{R}^{n_z \times n},\ \Xi_1 \in \mathbb{R}^{n_w \times n},\ \Phi_2,\ \Gamma_2,\ \Lambda_2, \in \mathbb{R}^{n \times n_z},\ \Pi_2 \in \mathbb{R}^{n_z \times n_z},\ \Xi_2 \in \mathbb{R}^{n_w \times n_z}\ :$

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \text{where} \tag{9.53a}$$

$$\mathcal{J} := \begin{bmatrix} 0 & P_1 & P_2 & 0 & 0 \\ P_1 & P_2 + P_2^T & P_3 & 0 & 0 \\ P_2^T & P_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q & S \\ 0 & 0 & 0 & S^T & R \end{bmatrix}, \tag{9.53b}$$

$$\mathcal{H} := \begin{bmatrix} \Phi_1 K - \Phi_2 X & \Phi_1 C - \Phi_2 V & \Phi_1 M - \Phi_2 U & \Phi_2 & -\Phi_1 F_w - \Phi_2 D_{zw} \\ \Gamma_1 K - \Gamma_2 X & \Gamma_1 C - \Gamma_2 V & \Gamma_1 M - \Gamma_2 U & \Gamma_2 & -\Gamma_1 F_w - \Gamma_2 D_{zw} \\ \Lambda_1 K - \Lambda_2 X & \Lambda_1 C - \Lambda_2 V & \Lambda_1 M - \Lambda_2 U & \Lambda_2 & -\Lambda_1 F_w - \Lambda_2 D_{zw} \\ \Pi_1 K - \Pi_2 X & \Pi_1 C - \Pi_2 V & \Pi_1 M - \Pi_2 U & \Pi_2 & -\Pi_1 F_w - \Pi_2 D_{zw} \\ \Xi_1 K - \Xi_2 X & \Xi_1 C - \Xi_2 V & \Xi_1 M - \Xi_2 U & \Xi_2 & -\Xi_1 F_w - \Xi_2 D_{zw} \end{bmatrix}. \tag{9.53c}$$

Proof. Assign

$$
x(t) \leftarrow \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ z(t) \\ w(t) \end{pmatrix} , \quad \mathcal{Q} \leftarrow (9.53b), \quad \mathcal{B}^T \leftarrow \begin{bmatrix} K^T & -X^T \\ C^T & -V^T \\ M^T & -U^T \\ 0 & I \\ -F_w^T & -D_{zw}^T \end{bmatrix} , \quad \mathcal{X} \leftarrow \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Gamma_1 & \Gamma_2 \\ \Lambda_1 & \Lambda_2 \\ \Pi_1 & \Pi_2 \\ \Xi_1 & \Xi_2 \end{bmatrix} \quad (9.54)
$$

and apply Finsler's lemma to the constrained Lyapunov problem (9.52) with $P \succ 0$. $\qquad\square$

The above condition yields specialized quadratic peformance criterias depending on the choice of Q, S, R. Assign

$$
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad \leftarrow \quad \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} .
$$

to verify the $\mathcal{L}_2$ performance criteria

$$
\int_0^t \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0 \Leftrightarrow \int_0^t z(t)^T z(t) \, dt < \gamma^2 \int_0^t w(t)^T w(t) \, dt
$$
$$
\Leftrightarrow \|z(t)\|_{\mathcal{L}_2}^2 < \gamma^2 \|w(t)\|_{\mathcal{L}_2}^2
$$
$$
\Leftrightarrow \|H_{zw}(j\omega)\|_{\mathcal{H}_\infty}^2 < \gamma^2
$$

also known as *bounded real lemma*. To check *passivity* of a vector second order system, select

$$
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad \leftarrow \quad \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} .
$$

reducing the integral quadratic constraint to

$$
\int_0^t \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0 \Leftrightarrow -2 \int_0^t z(t)^T w(t) \, dt < 0
$$
$$
\Updownarrow
$$
$$
\int_0^t z(t)^T w(t) \, dt > 0 \Leftrightarrow H_{zw}(j\omega) + H_{zw}(j\omega)^* \succ 0 \Leftrightarrow H_{zw}(j\omega) \text{ is passive,}
$$

condition also known as *positive real lemma*. Sector bounds on the signals $z(t)$ and $w(t)$ can be enforced by choosing

$$
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad \leftarrow \quad \begin{bmatrix} I & -\frac{1}{2}(\alpha + \beta)I \\ -\frac{1}{2}(\alpha + \beta)I & -\alpha\beta I \end{bmatrix} .
$$

The integral quadratic constraint yields

$$
\int_0^t \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0 \Leftrightarrow \int_0^t (z(t) - \alpha w(t))^T (z(t) - \beta w(t)) \, dt > 0
$$
$$
\Leftrightarrow (z(t), w(t)) \in \text{sector}(\alpha, \beta).
$$

Similarly to the stability case, in (9.53) the product of the multipliers with the system matrices occurs in a position that does not facilitate possible change-of-variables. One would be tempted to invoke algebraic duality of the vector second-order system once again. However, as discussed in the appendix, the presence of outputs bring complicating issues making such an approach not trivial. Add to this, the multipliers involved in the ICQ condition have different dimensions. An ICQ condition dependent on a single, square, invertible and well located multiplier is desirable for synthesis purposes.

A modification on the constrained Lyapunov problem is the first step towards a condition with such properties. The integral quadratic constraint may depend explicitly on positions, velocities and accelerations by substituting $z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw}w(t)$ into (9.50) yielding

$$\int_0^t \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Z^T Q Z & Z^T(S + QD_{zw}) \\ \star & R + D_{zw}^T Q D_{zw} + D_{zw}^T S + S^T D_{zw} \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ w(t) \end{pmatrix} \geq 0, \quad (9.55a)$$

$$Z := \begin{bmatrix} X & V & U \end{bmatrix} \quad (9.55b)$$

The new constrained Lyapunov problem

$$\dot{V}(q(t), \dot{q}(t), \ddot{q}(t)) < - \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Z^T Q Z & Z^T(S + QD_{zw}) \\ (S + QD_{zw})^T Z & \bar{R} \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ w(t) \end{pmatrix},$$

$$\bar{R} := R + D_{zw}^T Q D_{zw} + D_{zw}^T S + S^T D_{zw} \quad (9.56a)$$

$$\forall (q(t), \dot{q}(t), \ddot{q}(t), w(t)) \text{ satisfying } M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F_w w(t), \quad (9.56b)$$

$$(q(t), \dot{q}(t), \ddot{q}(t), w(t)) \neq 0, \quad (9.56c)$$

is not dependent explicitly on the output vector $z(t)$. Sufficient conditions with reduced number of multipliers can be derived from the above Lyapunov problem by applying the Elimination Lemma. They become also necessary if the acceleration vector (or position vector) is absent in $z(t)$ i.e. $U = 0$ (or $X = 0$).

Theorem 25. *The set of solutions of the Lyapunov problem* (9.56) *with* $P \succ 0$ *is not empty if* $\exists\, P_1,\ P_3 \in \mathbb{S}^n,\ P_2,\ \Phi,\ \Lambda \in \mathbb{R}^{n \times n},\ \alpha \in \mathbb{R}$:

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad \text{where} \quad (9.57a)$$

$$\mathcal{J} := \begin{bmatrix} 0 & P_1 & P_2 & 0 \\ P_1 & P_2 + P_2^T & P_3 & 0 \\ P_2^T & P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} X^T Q X & X^T Q V & X^T Q U & X^T(S + QD_{zw}) \\ \star & V^T Q V & V^T Q U & V^T(S + QD_{zw}) \\ \star & \star & U^T Q U & U^T(S + QD_{zw}) \\ \star & \star & \star & \bar{R} \end{bmatrix},$$

$$(9.57b)$$

$$\mathcal{H} := \begin{bmatrix} \Phi K & \Phi C & \Phi M & -\Phi F_w \\ (\alpha\Phi + \Lambda)K & (\alpha\Phi + \Lambda)C & (\alpha\Phi + \Lambda)M & -(\alpha\Phi + \Lambda)F_w \\ \alpha\Lambda K & \alpha\Lambda C & \alpha\Lambda M & -\alpha\Lambda F_w \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha > 0,$$

$$(9.57c)$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \tag{9.57d}$$

This is necessary and sufficient whenever $U = 0$ in (9.56).

Proof. Assign

$$\mathcal{Q} \leftarrow \begin{bmatrix} 0 & P_1 & P_2 & 0 \\ P_1 & P_2 + P_2^T & P_3 & 0 \\ P_2^T & P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} X^T Q X & X^T Q V & X^T Q U & X^T(S + Q D_{zw}) \\ \star & V^T Q V & V^T Q U & V^T(S + Q D_{zw}) \\ \star & \star & U^T Q U & U^T(S + Q D_{zw}) \\ \star & \star & \star & \bar{R} \end{bmatrix}$$

$$\mathcal{B}^T \leftarrow \begin{bmatrix} K^T \\ C^T \\ M^T \\ -F_w^T \end{bmatrix}, \quad \mathcal{C}^\perp \leftarrow \begin{bmatrix} \alpha^2 I & 0 \\ -\alpha I & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{C}^T \leftarrow \begin{bmatrix} I & 0 \\ \alpha I & I \\ 0 & \alpha I \\ 0 & 0 \end{bmatrix}, \quad \mathcal{X} \leftarrow \begin{bmatrix} \Phi \\ \Lambda \end{bmatrix}$$

and apply the Elimination Lemma with $P \succ 0$. This lemma renders a condition without extra conservatism whenever $\mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^\perp \prec 0$, that is

$$\begin{bmatrix} F_{11}(x) & F_{12}(x) \\ F_{12}(x)^T & F_{22}(x) \end{bmatrix} \succ 0, \quad F_{22}(x) := -\bar{R}, \quad \bar{R} \prec 0,$$

$$F_{11}(x) := 2(\alpha^3 P_1 - \alpha^2(P_2 + P_2^T) + \alpha P_3) - (\alpha^4 X^T Q X - \alpha^3 X^T Q V + \alpha^2 X^T Q U$$
$$+ \alpha^2 V^T Q V - \alpha V^T Q U + U^T Q U) + \star,$$

$$F_{12}(x) := -\alpha^2 X^T(S + Q D_{zw}) + \alpha V^T(S + Q D_{zw}) - U^T(S + Q D_{zw}).$$
$$\tag{9.58}$$

Use the support inequality (9.21) with $N := \alpha^2 P_2^T$, $W := \alpha^3 P_1 \succ 0$ and a Schur complement with respect to $\bar{R}$ to show that (9.58) is equivalent to

$$P_3 - P_2^T P_1^{-1} P_2 \succ \frac{1}{2} \left(\alpha^3 X^T Q X - \alpha^2 X^T Q V + \alpha X^T Q U + \alpha V^T Q V - V^T Q U \right.$$

$$\left. + \alpha^{-1} U^T Q U \right) - \frac{1}{2}(\alpha^3 X^T \bar{R}^{-1} X - \alpha^2 X^T \bar{R}^{-1} V + \alpha(X^T \bar{R}^{-1} U - V^T \bar{R}^{-1} V)$$

$$+ \alpha^{-1}(U^T \bar{R}^{-1} U - U^T \bar{R}^{-1} V)) + \star \succeq 0.$$
$$\tag{9.59}$$

Note that $P_3 - P_2^T P_1^{-1} P_2 \succ 0$ implies $P \succ 0$ due to a Schur complement argument. When $U = 0$ the right hand side of (9.59) is polynomial in α with no constant term. Thus, there exists a sufficiently small α such that (9.59) holds which implies no added conservatism as long as $\alpha > 0$ is considered a variable in the formulation. □

For a constant α the above constraint is an LMI. However, the condition requires a line search in α. When $X = 0$ the same rationale with slightly modified $\mathcal{C}^\perp$ and $\mathcal{C}$

$$\mathcal{C}^\perp \leftarrow \begin{bmatrix} I & 0 \\ -\alpha I & 0 \\ \alpha^2 I & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{C}^T \leftarrow \begin{bmatrix} 0 & \alpha I \\ \alpha I & I \\ I & 0 \\ 0 & 0 \end{bmatrix}$$

also yields a necessary and sufficient condition.

The second step towards an ICQ condition for synthesis is to define a nonlinear change-of-variables between the Lyapunov matrices and a multiplier. Let a congruence transformation be $Y := \text{diag}(\Gamma, \Gamma)$, $\Gamma := \Lambda^{-T}$ where Λ is assumed invertible. Apply it to the partitioned Lyapunov variable, leading to the change-of-variables

$$Y^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} Y := \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} \succ 0 \tag{9.60a}$$

$$\hat{P}_1 := \Gamma^T P_1 \Gamma, \quad \hat{P}_2 := \Gamma^T P_2 \Gamma, \quad \hat{P}_2^T := \Gamma^T P_2^T \Gamma \quad \hat{P}_3 := \Gamma^T P_3 \Gamma. \tag{9.60b}$$

The original Lyapunov matrices can be reconstructed by the inverse congruence transformation

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} = Y^{-T} \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} Y^{-1} \succ 0 \tag{9.61}$$

With the results of Theorem 25 and the previously defined nonlinear change-of-variables at hand, an ICQ criteria suitable for synthesis can be stated.

Theorem 26. *The set of solutions of the Lyapunov problem* (9.56) *with* $P \succ 0$ *is not empty if* $\exists \, \hat{P}_1, \, \hat{P}_3 \in \mathbb{S}^n$, $\hat{P}_2$, $\Gamma \in \mathbb{R}^{n \times n}$, α, $\mu \in \mathbb{R}$:

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} \succ 0, \quad where \tag{9.62a}$$

$$\mathcal{J} := \begin{bmatrix} 0 & \hat{P}_1 & \hat{P}_2 & 0 \\ \hat{P}_1 & \hat{P}_2 + \hat{P}_2^T & \hat{P}_3 & 0 \\ \hat{P}_2^T & \hat{P}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} \Gamma^T X^T Q X \Gamma & \Gamma^T X^T Q V \Gamma & \Gamma^T X^T Q U \Gamma & \Gamma^T X^T (S + Q D_{zw}) \\ \star & \Gamma^T V^T Q V \Gamma & \Gamma^T V^T Q U \Gamma & \Gamma^T V^T (S + Q D_{zw}) \\ \star & \star & \Gamma^T U^T Q U \Gamma & \Gamma^T U^T (S + Q D_{zw}) \\ \star & \star & \star & \bar{R} \end{bmatrix}, \tag{9.62b}$$

$$\bar{R} := R + D_{zw}^T Q D_{zw} + D_{zw}^T S + S^T D_{zw}$$

$$\mathcal{H} := \begin{bmatrix} \mu K \Gamma & \mu C \Gamma & \mu M \Gamma & -\mu F_w \\ (1 + \alpha\mu) K \Gamma & (1 + \alpha\mu) C \Gamma & (1 + \alpha\mu) M \Gamma & -(1 + \alpha\mu) F_w \\ \alpha K \Gamma & \alpha C \Gamma & \alpha M \Gamma & -\alpha F_w \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha > 0, \; \mu > 0. \tag{9.62c}$$

Proof. To derive (9.62) from (9.57), first introduce the constraint $\Phi = \mu\Lambda$ where $\mu > 0$. Apply the congruence transformation $Y_a := \text{diag}(\Gamma, \Gamma, \Gamma, I)$, $\Gamma := \Lambda^{-T}$ to (9.57a), congruence transformation $Y_d := \text{diag}(\Gamma, \Gamma)$ to (9.57d), and the change-of-variables (9.60). Notice that the upper left entry of $\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0$ in (9.62), i.e. $K\Gamma + \Gamma^T K^T + \Gamma^T X^T Q X \Gamma \prec 0$ with K nonsingular implies Γ nonsingular. This fact corroborates the assumption of an invertible Γ in the change-of-variables (9.60). $\qquad \square$

The condition from Theorem 26 benefits from some convenient properties. It depends on a single multiplier Γ in products with M, C, K matrices as well as U, V, X matrices. Moreover the product occurs at the "right side" of the matrices. Both properties facilitate change-of-variables involving the controller data, as will become clear later in this manuscript.

Synthesis of controllers is the subject of the reminder of this paper. It will be given focus to the design of controllers with guaranteed $\mathcal{L}_2$-gain performance for clarity and its practical relevance. Synthesis conditions considering other ICQ criterias can be derived similarly by particularizing Q, R, S and appropriate Schur complements involving these matrices.

Static Full Vector Feedback

The proposed ICQ condition offers the possibility of synthesizing controllers. Consider the vector second-order system with disturbance and controllable inputs

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F_w w(t) + F_u u(t) \tag{9.63a}$$

$$z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw}w(t) + D_{zu}u(t) \tag{9.63b}$$

in loop with a static full vector feedback yielding the closed-loop system denoted $\mathbf{H}_{zw}$:

$$\mathbf{M}\ddot{q}(t) + \mathbf{C}\dot{q}(t) + \mathbf{K}q(t) = F_w w(t) \tag{9.64a}$$

$$z(t) = \mathbf{U}\ddot{q}(t) + \mathbf{V}\dot{q}(t) + \mathbf{X}q(t) + D_{zw}w(t) \tag{9.64b}$$

$$\mathbf{M} := (M + F_u G_a), \quad \mathbf{C} := (C + F_u G_v), \quad \mathbf{K} := (K + F_u G_p) \tag{9.64c}$$

$$\mathbf{U} := (U - D_{zu}G_a), \quad \mathbf{V} := (V - D_{zu}G_v), \quad \mathbf{X} := (X - D_{zu}G_p) \tag{9.64d}$$

The same issues regarding the nonsingularity of the multiplier in the stabilizability case have also to be consider here. Therefore, the next theorem states the existence of a static controller in which the acceleration feedback is absent ($G_a = 0$). This controller structure corresponds to a full state feedback in the first-order state-space sense.

Theorem 27. *There exists a controller of the form* (9.25) *with* $G_a = 0$ *such that* $\|\mathbf{H}_{zw}\|_{\mathcal{L}_2} < \gamma^2$ *if* $\exists\, \hat{P}_1, \hat{P}_3 \in \mathbb{S}^n$, $\hat{P}_2$, $\Gamma \in \mathbb{R}^n$, $\hat{G}_v$, $\hat{G}_p \in \mathbb{R}^{n_u \times n}$, $\alpha, \mu \in \mathbb{R}$:

$$
\left[
\begin{array}{ccc}
\mu(K\Gamma + F_u\hat{G}_p) + \star & \hat{P}_1 + \mu(C\Gamma + F_u\hat{G}_v) + (1+\alpha\mu)(K\Gamma + F_u\hat{G}_p)^T & \\
\star & \hat{P}_2 + (1+\alpha\mu)(C\Gamma + F_u\hat{G}_v) + \star & \\
\star & & \star \\
\star & & \star \\
\star & & \star \\[4pt]
\hat{P}_2 + \mu M\Gamma + \alpha(K\Gamma + F_u\hat{G}_p)^T & -\mu F_w & \Gamma^T X^T - \hat{K}_p^T D_{zu}^T \\
\hat{P}_3 + (1+\alpha\mu)M\Gamma + \alpha(C\Gamma + F_u\hat{G}_v)^T & -(1+\alpha\mu)F_w & \Gamma^T V^T - \hat{K}_v^T D_{zu}^T \\
\alpha(M\Gamma + \Gamma^T M^T) & -\alpha F_w & \Gamma^T U^T \\
\star & -\gamma^2 I & D_{zw}^T \\
\star & \star & -I
\end{array}
\right] \prec 0.
$$
$$\tag{9.65a}$$

$$\alpha > 0, \quad \mu > 0, \quad \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} \succ 0 \tag{9.65b}$$

Proof. In order to obtain the above inequalities from (9.62), first particularize it with $Q = I$, $R = -\gamma^2 I$ and apply a Schur complement with respect to Q. A direct application of the resulting inequalities to the closed-loop system (9.26) together with a change-of-variables of the form (9.27) involving the multiplier Γ and the controller data G_v, G_p yields (9.65). Nonsingularity of Γ is implied by the entry $M\Lambda + \Lambda^T M^T \prec 0$ with M nonsingular. Once a solution to the problem above is found, invertibility of Γ assures the reconstruction of the controller gains from the auxiliary ones according to $G_v = \hat{G}_v \Gamma^{-1}$ and $G_p = \hat{G}_p \Gamma^{-1}$. □

The acceleration feedback was removed from the feedback law for theoretical reasons: ensure a nonsingular Γ. As discussed in the stabilizability section, a nonsingular Γ could also be enforced by neglecting the position feedback G_p. If all feedback gains are desired, in practice the LMI above could be augmented with the acceleration gain and solved. The multiplier Γ could be invertible. In case this happens, the acceleration, velocity and position gains can all be recovered from the auxiliary controller gains.

Working with the closed-loop system in vector form facilitates the feedback of only the position or the velocity vector without introducing extra conservatism to the presented formulation. These controller structures would correspond to partial state feedback in the first-order state-space sense, to which convex reformulations without loss of generality are not known to exist.

Static Output Feedback

The acceleration, velocity or position vectors are often partially available for feedback. In such a case, the vector second order system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F_w w(t) + F_u u(t) \tag{9.66a}$$

$$z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw}w(t) + D_{zu}u(t) \tag{9.66b}$$

$$y(t) = R\ddot{q}(t) + S\dot{q}(t) + Tq(t) + D_{yw}w(t) \tag{9.66c}$$

is augmented with a measurement vector $y(t) \in \mathbb{R}^{n_y}$. The interest lies on the synthesis of a static output feedback controller of the form

$$u(t) = -G_y y(t) \tag{9.67}$$

where $G_y \in \mathbb{R}^{n_u \times n_y}$. To facilitate the derivations that follows, the measurement vector is not corrupted by noise ($D_{yw} = 0$). Assume, without loss of generality, that the output matrices R, S and T are of full row rank. Then, there exist nonsingular transformation matrices W_a, W_v, $W_p \in \mathbb{R}^{n \times n}$ such that

$$RW_a = \begin{bmatrix} I & 0 \end{bmatrix}, \quad SW_v = \begin{bmatrix} I & 0 \end{bmatrix}, \quad TW_p = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{9.68}$$

For any given triplice (R, S, T), the corresponding (W_a, W_v, W_p) are not unique in general. A particular (W_a, W_v, W_p) can be obtained by

$$W_a := \begin{bmatrix} R^T(RR^T)^{-1} & R^\perp \end{bmatrix}, \ W_v := \begin{bmatrix} S^T(SS^T)^{-1} & S^\perp \end{bmatrix}, \ W_p := \begin{bmatrix} T^T(TT^T)^{-1} & T^\perp \end{bmatrix}.$$

The feedback of a single quantity, that is either accelerations, velocities or positions are addressed here. Let the measurement vector be composed of position feedback only,

i.e. $y(t) = Tq(t)$. From the coordinate transformation defined as $\ddot{q} := W_p\ddot{\tilde{q}}$, $\dot{q} := W_p\dot{\tilde{q}}$ and $q := W_p\tilde{q}$, the system matrices of (9.66) are substituted according to

$$
\begin{aligned}
M &\leftarrow MW_p, & C &\leftarrow CW_p, & K &\leftarrow KW_p \\
U &\leftarrow UW_p, & V &\leftarrow VW_p, & X &\leftarrow XW_p \\
R &\leftarrow RW_p, & S &\leftarrow SW_p, & T &\leftarrow \begin{bmatrix} I & 0 \end{bmatrix}.
\end{aligned}
$$

The closed-loop matrices of the transformed system related to positions are then

$$
\mathbf{K} := \left(K + F_u \begin{bmatrix} G_y & 0 \end{bmatrix} \right), \quad \mathbf{X} := \left(X - D_{zu} \begin{bmatrix} G_y & 0 \end{bmatrix} \right)
$$

while the other closed-loop matrices are the same as the open-loop ones. A static output-feedback gain can be obtained by imposing on the auxiliary controller gain $\hat{G}$ and the multiplier Γ the structure

$$
\hat{G} := \begin{bmatrix} \hat{G}_y & 0 \end{bmatrix}, \quad \Gamma := \begin{bmatrix} \Gamma_1 & 0 \\ \Gamma_3 & \Gamma_4 \end{bmatrix}. \tag{9.69}
$$

This kind of controller/multiplier constraint was firstly prpoposed in [22] in the context of first-order state-space systems. This structure is merged in Theorem 27 by imposing the structural constraints $\hat{G}_a := \hat{G}_v := \hat{G}_p := \hat{G}$ and Γ as (9.69). Supposing Γ nonsingular, and consequently the upper left block Γ_1 invertible, the original controller data can be recovered by the inverse change-of-variables

$$
G = \begin{bmatrix} G_y & 0 \end{bmatrix} = \begin{bmatrix} \hat{G}_y\Gamma_1^{-1} & 0 \end{bmatrix}. \tag{9.70}
$$

Thus, the structure imposed to the state feedback gain matrix G facilitates the output feedback law $u(t) = G_y y(t)$. The same procedure can be made when the mesurement vector is $y(t) = R\ddot{q}(t)$ or $y(t) = S\dot{q}(t)$.

Robust Control

The inherent decoupling of the Lyapunov and system matrices occasioned by the introduction of multipliers facilitates the usage of parameter-dependent Lyapunov functions [33]. This decoupling property was firstly exploited under the context of robust stability of first-order state-space systems in [21] and latter extended to performance specifications [22]. Assume that the matrices of system (9.63) are uncertain but belong to a convex and bounded set. This set is such that the matrix

$$
\mathcal{S} := \begin{bmatrix} M & C & K & F_u & F_w \\ U & V & X & D_{zu} & D_{zw} \end{bmatrix}
$$

takes values in a domain defined as a polytopic combination of N given matrices $\mathcal{Q}_1, \ldots, \mathcal{Q}_N$, that is,

$$
\mathsf{S} := \left\{ \mathcal{S}(\alpha) : \mathcal{S}(\alpha) := \sum_{i=1}^N \mathcal{S}_i\alpha_i, \quad \sum_{i=1}^{N_\alpha} \alpha_i = 1, \quad \alpha_i \geq 0. \right\}
$$

The operator $\mathrm{Vert}(\mathsf{S}) := \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ reduces the infinite dimensional set S to the vertices $\mathcal{S}_i$, $i = 1, \ldots, N$. The LMI conditions for vector second-order systems presented

here can turn into sufficient conditions for robust analysis and synthesis by defining a parameter-dependent Lyapunov matrix

$$P(\alpha) := \sum_{i=1}^{N_\alpha} P_i \alpha_i \tag{9.71}$$

and maintaining the multipliers as parameter-independent. In this case, the LMIs are infinite-dimensional functions of the uncertain vector α. A finite-dimensional problem arises with Vert $(F(x, \alpha) \prec 0)$. Consider the robust stability problem as an example. System (9.24) is robustly stabilizable by a static feedback law of the form (9.25) with $G_a = 0$, for all $\mathcal{S} \in \mathsf{S}$, if $\exists \Lambda, P_{2,i} \in \mathbb{R}^{n \times n}, P_{1,i}, P_{3,i} \in \mathbb{S}^n, \hat{G}_v, \hat{G}_p \in \mathbb{R}^{n_u \times n}, \alpha, \mu \in \mathbb{R}:$

$$\mathcal{J}_i + \mathcal{H}_i + \mathcal{H}_i^T \prec 0, \quad \mathcal{J}_i := \begin{bmatrix} 0 & P_{1,i} & P_{2,i} \\ P_{1,i} & P_{2,i} + P_{2,i}^T & P_{3,i} \\ P_{2,i}^T & P_{3,i} & 0 \end{bmatrix}, \tag{9.72a}$$

$$\mathcal{H}_i := \begin{bmatrix} \mu(K_i\Lambda + F_{u,i}\hat{G}_p) & \mu(C_i\Lambda + F_{u,i}\hat{G}_v) & \mu(M_i\Lambda) \\ (1 + \alpha\mu)(K_i\Lambda + F_{u,i}\hat{G}_p) & (1 + \alpha\mu)(C_i\Lambda + F_{u,i}\hat{G}_v) & (1 + \alpha\mu)(M_i\Lambda) \\ \alpha(K_i\Lambda + F_{u,i}\hat{G}_p) & \alpha(C_i\Lambda + F_{u,i}\hat{G}_v) & \alpha(M_i\Lambda) \end{bmatrix} \tag{9.72b}$$

$$\begin{bmatrix} P_{1,i} & P_{2,i} \\ P_{2,i}^T & P_{3,i} \end{bmatrix} \succ 0, \quad \alpha > 0, \mu > 0, \tag{9.72c}$$

for $i = 1, \ldots, N$.

4   Numerical Examples

Three-Mass System

The simplicity of a three-mass system depicted in Fig. 9.1 allows an easy analysis and straightfoward interpretation of the results. In this figure, m_1, m_2 and m_3 are system masses, k_1, k_2, k_3 and k_4 are stiffness coefficients, while d_1, d_2, d_3 and d_4 are damping coefficients.
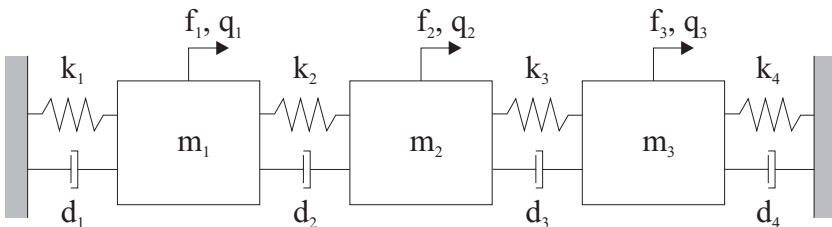

Figure 9.1: Three-mass mechanical system.

The control input $u(t)$ acts at mass 2 and mass 3 in opposite directions. The first disturbance $w_1(t)$ acts at mass 2 and mass 3 in opposite directions, with an amplification factor of 3, and the second disturbance $w_2(t)$ acts at mass 2. The controlled outputs

$(z_1(t), z_2(t), z_3(t))$ are the displacement of mass 2 with an amplification factor of 3, the velocity of mass 3, and the input $u(t)$, respectively. The motion of this mechanical system is described by the differential equations

$$
\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \ddot{q}(t) + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix} \dot{q}(t)
$$
$$
+ \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} q(t) = \begin{bmatrix} 0 & 0 \\ 3 & 1 \\ -3 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u(t),
$$

(9.73a)

$$
z(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \dot{q}(t) + \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t).
$$

(9.73b)

For this system, $m_1 = 3$, $m_2 = 1$, $m_3 = 2$, $k_1 = 30$, $k_2 = 15$, $k_3 = 15$, $k_4 = 30$, and $C = 0.004K + 0.001M$. Magnitude plots of the open-loop transfer functions from disturbances (w_1, w_2) to outputs (z_1, z_2) are depicted in Fig. 9.3a. The lightly damped characteristics of the system modes are noticeable.

$\mathcal{H}_\infty$ control will be used to reject oscillatory response of these modes in face of disturbances. Full vector feedback gains of positions and velocities are synthesized using Theorem 27 for different values of the scalars α, μ. The upper bound γ of the $\mathcal{H}_\infty$-norm for various (α, μ) is illustrated in Fig. 9.2. The minimum achieved upper bound $\gamma^* = 7.679$ occurs at $(\alpha, \mu) = (0.0060, 0.0820)$ with corresponding position and velocity feedback gains

$$
G_p = \begin{bmatrix} 0.2501 & 0.0774 & -0.0786 \end{bmatrix}, \quad G_v = \begin{bmatrix} 5.2757 & 1.9574 & -1.6351 \end{bmatrix}.
$$

Improved vibration performance is corroborated by magnitude plots and impulse responses of the closed-loop system (Fig. 9.3a and 9.3b).

Model Matching Control of Wind Turbines

A different perspective to modern control of wind turbines is given here by considering the design model in its natural form. For clarity, the turbine model contains only the two structural degrees of freedom with lowest frequency contents: rigid body rotation of the rotor and fore-aft tower bending described by the axial nacelle displacement. The simplified dynamics of a wind turbine can be described by the nonlinear differential equations

$$
J\ddot{\psi} = Q_a(v - \dot{q}_t, \dot{\psi}, \beta)(t) - Q_g(t)
$$

(9.74)

$$
M_1\ddot{q}_1 + K_1 q_1 = T_a(v - \dot{q}_1, \dot{\psi}, \beta)
$$

(9.75)

where the aerodynamic torque $Q_a(t)$ and thrust $T_a(t)$ are nonlinear functions of the relative wind speed $v(t) - q_1(t)$ with $v(t)$ being the mean wind speed over the rotor disk, the rotor speed $\dot{\psi}(t)$, and the collective pitch angle $\beta(t)$. Linearization of (9.74) around
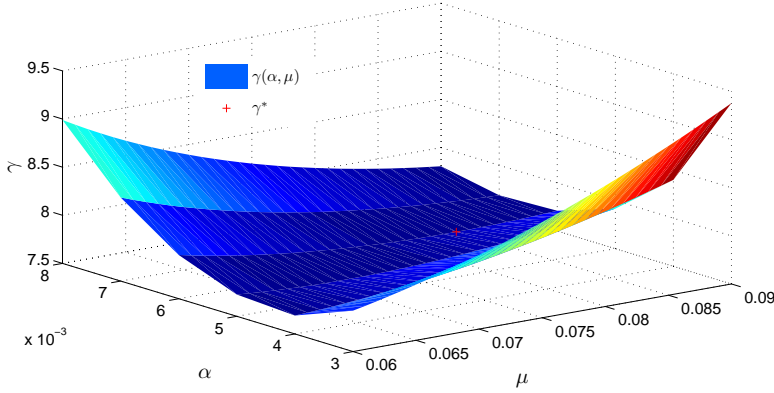
Figure 9.2: Upper bound on the $\mathcal{H}_\infty$-norm of the closed-loop three-mass system with full position and velocity feedback, obtained by Theorem 27.

an equilibrium point θ yields

$$(J_r + N_g^2 J_g)\ddot{\psi}(t) = \left.\frac{\partial Q_a}{\partial \dot{\psi}}\right|_\theta \dot{\psi}(t) + \left.\frac{\partial Q_a}{\partial V}\right|_\theta (v(t) - \dot{q}_1(t)) + \left.\frac{\partial Q_a}{\partial \beta}\right|_\theta \beta(t) - \eta^{-1} N_g Q_g(t)$$

(9.76)

$$M_1 \ddot{q}_1(t) + K_1 q_1(t) = \left.\frac{\partial T_a}{\partial \dot{\psi}}\right|_\theta \dot{\psi}(t) + \left.\frac{\partial T_a}{\partial V}\right|_\theta (v(t) - \dot{q}_1(t)) + \left.\frac{\partial T_a}{\partial \beta}\right|_\theta \beta(t)$$

(9.77)

where J_r and J_g are the rotational inertia of the rotor (low speed shaft part) and the generator (high speed shaft part), K_1 is the stiffness for axial nacelle motion $q_1(t)$ due to fore-aft tower bending, M_1 is the modal mass of the first fore-aft tower bending mode, η is the total electrical and mechanical efficiency, and N_g is the gearbox ratio. The primary control objective of pitch controlled wind turbines operating at rated power is to regulate power generation despite wind speed disturbances. To accomplish this, rotor speed is controlled using the collective blade pitch angle, and generator torque is maintained constant ($Q(t) = 0$ in (9.76)). Tower fore-aft oscillations are induced by the wind turbulence hitting the turbine as well as changes in the thrust force due to pitch angle variations. The collective blade pitch angle can be controlled to supress these oscillations without degrading rotor speed regulation. The vector second-order system

$$\begin{bmatrix} J_r + N_g^2 J_g & 0 \\ 0 & M_1 \end{bmatrix} \begin{pmatrix} \ddot{\psi}(t) \\ \ddot{q}_1(t) \end{pmatrix} + \begin{bmatrix} \left.\frac{\partial Q_a}{\partial \dot{\psi}}\right|_\theta & -\left.\frac{\partial Q_a}{\partial V}\right|_\theta \\ \left.\frac{\partial T_a}{\partial \dot{\psi}}\right|_\theta & -\left.\frac{\partial T_a}{\partial V}\right|_\theta \end{bmatrix} \begin{pmatrix} \dot{\psi}(t) \\ \dot{q}_1(t) \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_1 \end{bmatrix} \begin{pmatrix} \psi(t) \\ q_1(t) \end{pmatrix}$$

$$= \begin{bmatrix} \left.\frac{\partial Q_a}{\partial v}\right|_\theta \\ \left.\frac{\partial T_a}{\partial v}\right|_\theta \end{bmatrix} v(t) + \begin{bmatrix} \left.\frac{\partial Q_a}{\partial \beta}\right|_\theta \\ \left.\frac{\partial T_a}{\partial \beta}\right|_\theta \end{bmatrix} \beta(t)$$

(9.78)
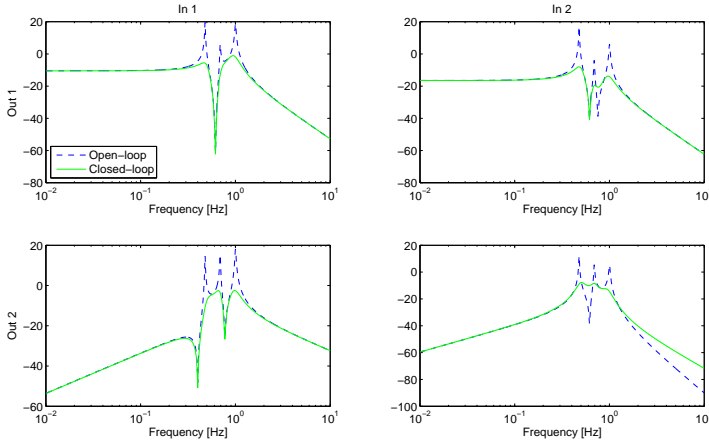
(a) Magnitude Plot
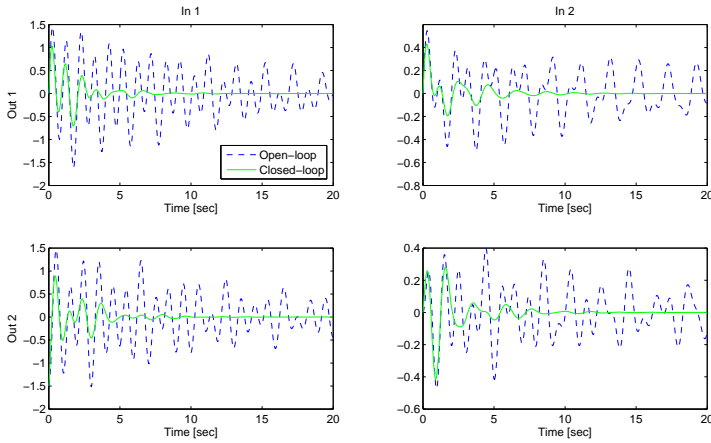
(b) Impulse Response

Figure 9.3: Comparison of open-loop three-mass system and closed-loop with full position and velocity feedback (Theorem 27).

arise from re-arranging expression (9.76). In the above, the disturbance vector is $w(t) := v(t)$ and control input is $u(t) := \beta(t)$. The open-loop system (9.78) has a singular stiffness matrix due to the rigid-body mode of the rotor, which at first may seem inadequate for a direct application of the conditions presented in this work. However, the closed-loop stiffness matrix is non-singular because the position of the rotor is part of the feedback law. Feedback of rotor position is analogous to the inclusion of integral action on rotor speed regulation, usual scheme in wind turbine control.

Controller design follows an $\mathcal{H}_\infty$ model matching criteria, which has an elegant structure when considered in vector second-order form. The performance of the system in

closed-loop should approximate a given a reference model

$$M_r\ddot{q}_r(t) + C_r\dot{q}_r(t) + K_r\ddot{q}_r(t) = F_{wr}w(t) \tag{9.79a}$$

$$z_r(t) = U_r\ddot{q}_r(t) + V_r\dot{q}_r(t) + X_r\ddot{q}_r(t) \tag{9.79b}$$

in an $\mathcal{H}_\infty$-norm sense. The matrices of the reference model are chosen to enforce a desired second-order closed-loop sensitivity function from wind speed disturbance $v(t)$ to rotor speed $\dot{\psi}(t)$. The augmented system for synthesis is

$$\begin{bmatrix} M & 0 \\ -M_{(1,:)} & M_r \end{bmatrix} \begin{pmatrix} \ddot{\psi} \\ \ddot{q}_1 \\ \ddot{\psi}_r \end{pmatrix} \begin{bmatrix} C & 0 \\ -C_{(1,:)} & C_r \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{q}_1 \\ \dot{\psi}_r \end{pmatrix} \begin{bmatrix} K & 0 \\ -K_{(1,:)} & K_r \end{bmatrix} \begin{pmatrix} \psi \\ q_1 \\ \psi_r \end{pmatrix}$$

$$= \begin{bmatrix} F_w \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} F_u \\ F_{u\,(1,:)} \end{bmatrix} u(t) \tag{9.80a}$$

$$z(t) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{q}_1 \\ \dot{\psi}_r \end{pmatrix} + \begin{bmatrix} 0 \\ D_{zu} \end{bmatrix} u(t) \tag{9.80b}$$

where $\dot{\psi}_r(t)$ is the reference model velocity and $(\cdot)_{(1,:)}$ stands for the first line of matrix $(\cdot)$. The reference filter in (9.80a) is forced indirectly by the the open-loop system (9.78), which is convenient for implementation purposes. In this example, $M_r = 6.0776 \cdot 10^6$, $C_r = 6.1080 \cdot 10^6$, and $K_r = 3.9346 \cdot 10^6$ characterizes a reference system with damped natural frequency $\omega_d = 0.628$ rad/s and damping $\xi = 0.625$.

Full vector feedback gains of positions and velocities are synthesized using Theorem 27 with $\alpha = 0.9$ and $\mu = 1$, yielding a guaranteed upper bound $\gamma = 1.462$. The true upper bound of the augmented system in closed-loop computed using Theorem 25 is $\gamma = 0.1058$. Controller gains are

$$G_v = \begin{bmatrix} -0.3734 & -0.1702 & 0.0028 \end{bmatrix}, \quad G_p = \begin{bmatrix} -0.1951 & -0.1029 & -0.0096 \end{bmatrix}$$

Bode plots of the closed-loop, open-loop and reference systems are depicted in Fig.9.4a. A good agreement between the closed-loop and reference model is noticeable. The chosen reference model indirectly impose some damping of the tower fore-aft displacement by trying to reduce the difference in magnitude between open-loop and reference model at the tower natural frequency. Step responses of the controlled and reference systems are compared in Fig.9.4b, showing a good correspondence.

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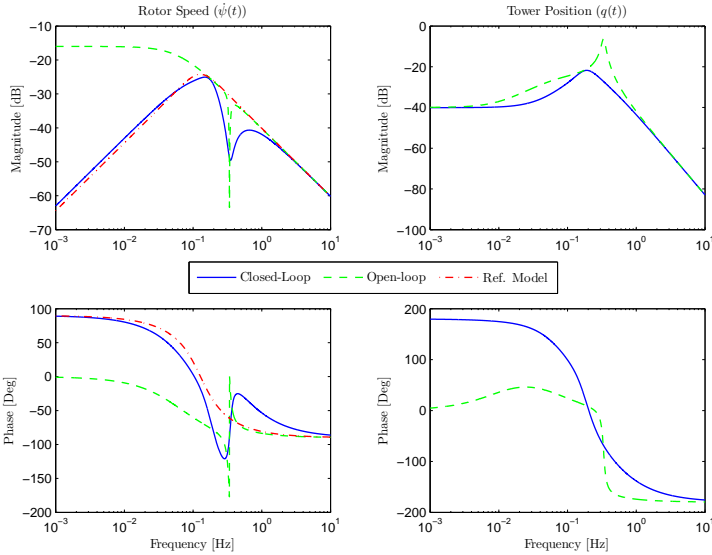
(a) Bode Plots
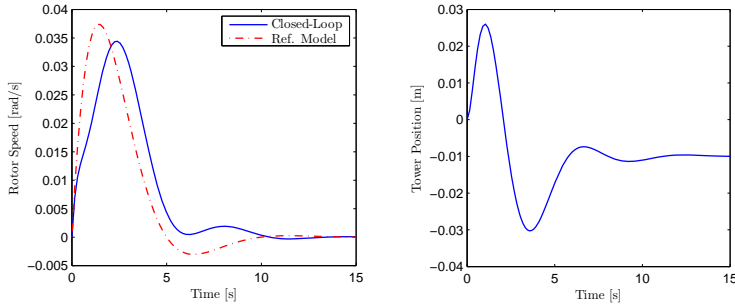
(b) Step Response

Figure 9.4: $\mathcal{H}_\infty$ model matching control of a simplified wind turbine model.

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Appendix: A Remark on Algebraic Duality

One of the most convenient features of the input-state-output framework of linear systems theory is the notion of duality. It is a well known fact that a minimal, controllable and observable first-order state-space form (9.2) admits the dual representation [30]

$$\dot{x}'(t) = A^T x'(t) + C^T u'(t) \tag{9.81a}$$

$$y'(t) = B^T x'(t) + D^T u'(t). \tag{9.81b}$$

The dual system has also a frequency domain interpretation. Define a transfer matrix as $G(s) := C(sI - A)^{-1}B + D$. The transpose of a transfer matrix $G(s)$ or the dual system is defined as [Zhou]

$$G(s) \mapsto G(s)^T = B^T \left(sI - A^T\right)^{-1} C^T + D^T.$$

The transfer matrix (9.49) of a vector second-order system is repeated here for convenience

$$H_{zw}(s) := \left(s^2 U + sV + X\right)\left(s^2 M + sC + K\right)^{-1} F_w + D_{zw}.$$

The definition of the dual of the transfer matrix above follows a similar rationale. The transpose of the transfer matrix $H_{zw}(s)$ or the dual vector second-order system is defined as

$$H_{zw}(s) \mapsto H_{zw}(s)^T = F_w^T \left(s^2 M^T + sC^T + K^T\right)^{-1} \left(s^2 U^T + sV^T + X^T\right) + D_{zw}^T. \tag{9.82}$$

or equivalently

$$M^T \ddot{q}'(t) + C^T \dot{q}'(t) + K^T q'(t) = U^T \ddot{w}'(t) + V^T \dot{w}'(t) + X^T w'(t) \tag{9.83a}$$

$$z'(t) = F^T q'(t) + D_{zw}^T w'(t), \qquad q'(0), \ \dot{q}'(0) = 0. \tag{9.83b}$$

Some observations about the dual system and the original system can be drawn. The dimensions of $q(t)$ and $q'(t)$ are the same, as well as the dimensions of $w(t)$ and $z'(t)$, and those of $z(t)$ and $w'(t)$. The dependence of the system dynamics on the derivatives of the dual disturbance input $w'(t)$ brings difficulties on the application of the methods described on this manuscript. Analysis and synthesis conditions derived from (9.83) can be a subject of future work.

Paper G

Linear Matrix Inequalities Conditions for Simultaneous Plant-Controller Design

Fabiano Daher Adegas

Abstract

This paper presents sufficient extended LMI conditions to the simultaneous plant-controller design problem. We introduce the notion of linearizing change-of-variables between the multipliers and plant parameters to be optimized. Integral quadratic constraints offer the possibility of designing linear systems with guaranteed $\mathcal{L}_2$-norm performance, passivity properties, and sector bounds on input/output signals.

1 Introduction

Motivated by the need of designing light-weight spacecraft systems, the first efforts to develop simultaneous plant-controller design (SPCD) methods dates from the 1980's. The solid theory in state-space and optimal control influenced the choice of linear feedback control laws and quadratic performance indices. The resulting complex non-convex constrained optimization problems had major drawbacks such as high computational cost, no guaranteed convergence to a global solution, and lack of robustness in face of uncertainties on the plant.

In order to overcome the computational difficulty, some authors propose a sequential redesign of plant and controller by adding a constraint on the closed-loop system while optimizing the plant [1]. The constraint changes the problem to a convex optimization and can either be (i) preservation of the closed-loop system matrix or (ii) preservation of the closed-loop covariance matrix. This sequential procedure is actually a suboptimal problem with proof of convergence to a solution which improves the initial design. Approaches of SPCD followed the advances in robust control theory during the last decades. The SPCD is originally a non-convex Bilinear Matrix Inequality (BMI) problem when formulated in the state-space domain and stability is characterized by Lyapunov theory. Nowadays, the numerical solution of BMI optimization problems does not reach a satisfactory maturity. A natural way to handle the problem is to "split" the BMI into Linear Matrix Inequalities (LMI) optimization problems which typically yields a stationary solution. Several LMI-based algorithms for SPCD have been suggested, taking the advantage of well established LMI approaches for robust optimal control [2, 3]. Most of the proposals consider the state-space matrices as affine functions on the structural optimization parameters, and differ from each other on how the sequential redesign is derived from the original non-convex BMI problem. The approach in [4] iterates between two different formulations of H_∞ control synthesis, basic characterization and projected characterization, to explore their convexity with respect to different variables. These two convex sub-problems yield to a local optimum and have poor convergence time, since the Lyapunov matrix is kept fixed in the structure redesign step. The algorithm proposed in [5] also iterates between convex sub-problems: a controller is designed for a fixed plant; plant and controller gains are redesigned for a fixed Lyapunov variable. An H_2/H_∞ mixed-performance was possible by using a single Lyapunov variable for both criteria. Reference [6] adds a certain function to render the constraint convex. This "convexifying" function is updated at each iteration until it disappears at a stationary point of the non-convex problem. A homotopy-based procedure was proposed in [7]. In the plant redesign step the variables are slightly perturbed resulting in perturbed matrix inequalities which are approximated to LMIs by ignoring their second and higher order terms.

Different from previous works, the results of the the present paper are not based on sequential LMI-based algorithms. We propose sufficient LMI conditions to the simultaneous plant-controller design subject to integral quadratic constraints. The LMI conditions benefit from the separation between the Lyapunov and system matrices by introducing Lagrange multipliers [8, 9, 10, 11]. We introduce the notion of linearizing change-of-variables between the multipliers and plant parameters to be optimized. Integral quadratic constraints offer the possibility of designing linear systems with guaranteed $\mathcal{L}_2$-norm performance, passivity properties, and sector bounds on input/output signals.

This paper is organized as follows. Some instrumental lemmas are presented in Section II. Main results are given in Section III, initially addressing the plant design problem, that is, the optimal design of plant parameters, and later the simultaneous plant-controller design. Section IV brings conclusions and suggestions for future work.

2 Preliminaries

We resort to a couple of lemmas well known in robust control. The following lemma is originally attributed to Finsler Uhlig [12].

Lemma 11 (Finsler). *Let $x(t) \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$ and $\mathcal{B} \in \mathbb{R}^{m \times n}$ such that $rank(\mathcal{B}) < n$. The following statements are equivalent.*

   i. $x(t)^T \mathcal{Q} x(t) < 0, \quad \forall \, \mathcal{B} x(t) = 0, \quad x(t) \neq 0.$

   ii. $\mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^{\perp} \prec 0.$

   iii. $\exists \mu \in \mathbb{R} \, : \, \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0.$

   iv. $\exists \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^T \mathcal{X}^T \prec 0.$

Finsler's lemma has been used frequently in control theory. Historically, its usage started almost exclusively for the purpose of eliminating variables. For this reason, a more general version of this lemma is even called the Elimination Lemma [2]. In this context, the application of the lemma starts from item iv. to obtain item ii., thus eliminating the variable $\mathcal{X}$. Later the lemma served to the purpose of introducing additional variables [9], starting from item i. and arriving at items iii. and iv. where the Lagrange multipliers are present.

The Elimination lemma [2, 3] is a generalized version of Finsler's lemma.

Lemma 12 (Elimination Lemma). *Let $\mathcal{Q} \in \mathbb{S}^n$, $\mathcal{B} \in \mathbb{R}^{m \times n}$, $\mathcal{C} \in \mathbb{R}^{n \times k}$. The following statements are equivalent.*

   i. $\exists \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{C}^T \mathcal{X} \mathcal{B} + \mathcal{B}^T \mathcal{X}^T \mathcal{C} \prec 0$

   ii. $\qquad \mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^{\perp} \prec 0 \quad$ (10.1a) $\qquad \mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^{\perp} \prec 0 \quad$ (10.1b)

   iii. $\exists \mu \in \mathbb{R} \, : \, \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0, \quad \mathcal{Q} - \mu \mathcal{C}^T \mathcal{C} \prec 0.$

Notice that Elimination Lemma reduces to the Finsler's Lemma when particularized with $\mathcal{C} = I$. In such a case $\mathcal{C}^{\perp} = \{0\}$ and (10.1b) is removed from the statement. A discussion on the relation between these two lemmas can be found in [2, 3].

3 Simultaneous Plant-Controller Design

The plant design problem involves the optimal design of plant parameters according to some performance criteria. Simultaneous plant-controller design includes both plant parameters and controller gains as optimization variables. For clarity, the plant design is addressed first.

Plant Design

Consider the following state-space system for plant design

$$\dot{x}(t) = A(\rho)x(t) + B_w(\rho)w(t) \tag{10.2a}$$
$$z(t) = C_z(\rho)x(t) + D_{zw}(\rho)w(t) \tag{10.2b}$$

where $x \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector, $z(t) \in \mathbb{R}^{n_z}$ is the performance channel, and systems matrices are real valued with appropriate dimensions, and defined as

$$A(\rho) := A_0 + A_\rho \rho, \quad B_w(\rho) := B_{w0} + B_{w\rho}\rho,$$
$$C_z(\rho) := C_{z0} + C_{z\rho}\rho, \quad D_{zw}(\rho) := D_{zw0} + D_{zw\rho}\rho,$$
$$A_\rho := \begin{bmatrix} A_{\rho_1} & A_{\rho_2} & \dots & A_{\rho_N} \end{bmatrix}, \quad B_{w\rho} := \begin{bmatrix} B_{w\rho_1} & B_{w\rho_2} & \dots & B_{w\rho_N} \end{bmatrix},$$
$$C_{z\rho} := \begin{bmatrix} C_{z\rho_1} & C_{z\rho_2} & \dots & C_{z\rho_N} \end{bmatrix}, \quad D_{zw\rho} := \begin{bmatrix} D_{zw\rho_1} & D_{zw\rho_2} & \dots & D_{zw\rho_N} \end{bmatrix},$$
$$\rho := \begin{bmatrix} \rho_1 I_n & \rho_2 I_n & \dots & \rho_N I_n \end{bmatrix}^T, \quad \underline{\rho}_i \leq \rho_i \leq \overline{\rho}_i, \quad i = 1, \dots, N.$$

Note the affine dependence of the above matrices on ρ representing deviations of the plant parameters from the nominal ones. Matrices A_ρ, $B_{w\rho}$, $C_{z\rho}$, $D_{zw\rho}$ define how parameter deviations affect the nominal plant matrices A_0, B_{w0}, C_{z0}, D_{zw0}. These parameters are assumed bounded by a hypercube with lower limit $\underline{\rho}_i$ and upper limit $\overline{\rho}_i$.

An equivalent reformulation of (10.2)-(10.3) is better suited for the derivations that follows. We work with the dual representation

$$\dot{x}'(t) = A^T(\rho)x'(t) + C_z^T(\rho)w'(t) \tag{10.4a}$$
$$z'(t) = B_w^T(\rho)x'(t) + D_{zw}^T(\rho)w'(t). \tag{10.4b}$$

Working with the dual system is not vital to the plant design, but will facilitate the simultaneous plant-controller design later in this manuscript. The influence of the plant parameters on the dynamics of the nominal system is represented by separate channels $p_x'(t)$ and $p_w'(t)$ leading to the system of equations

$$\dot{x}'(t) = A_0^T x'(t) + C_{z0}^T w'(t) + p_x'(t) + p_w'(t) \tag{10.5a}$$
$$z'(t) = B_{w0}^T x'(t) + D_{zw0}^T w'(t) + B_{w\rho}^T p_x'(t) + D_{zw\rho}^T p_w'(t) \tag{10.5b}$$
$$p_x'(t) := \rho^T A_\rho^T x'(t), \quad p_w'(t) := \rho^T C_{z\rho}^T w'(t). \tag{10.5c}$$

To characterize stability of the above system, define the quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(x'(t)) := x'(t)^T P x'(t) \tag{10.6}$$

where $P \in \mathbb{S}^n$ and its time derivative yields

$$\dot{V}(x'(t), \dot{x}'(t)) := \dot{x}'(t)^T P x'(t) + x'(t)^T P \dot{x}'(t). \tag{10.7}$$

Also consider a integral quadratic constraint (ICQ) on the input and output signals

$$\int_0^\infty \begin{pmatrix} z'(t) \\ w'(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z'(t) \\ w'(t) \end{pmatrix} \geq 0 \tag{10.8}$$

where $Q \in \mathbb{S}^{n_z}$, $R \in \mathbb{S}^{n_w}$, $S \in \mathbb{R}^{n_z \times n_w}$, $R \succ 0$. The above constraint yields specialized quadratic performance criteria depending on the choice of Q, S, R. Assign

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad \leftarrow \quad \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

to obtain the $\mathcal{L}_2$ performance criteria also known as *bounded real lemma*. A natural design objective within this particular constraint is the minimization of the performance level γ^2. One may also want to design a linear system with *passivity* properties, by selecting

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad \leftarrow \quad \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix},$$

condition also known as *positive real lemma*. Sector bounds on the signals $z(t)$ and $w(t)$ may also be a design objective by choosing

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad \leftarrow \quad \begin{bmatrix} I & -\frac{1}{2}(\alpha + \beta)I \\ -\frac{1}{2}(\alpha + \beta)I & -\alpha\beta I \end{bmatrix}.$$

Resorting to Lyapunov theory and the S-procedure [9], system (10.5) is asymptotically stable and satisfy the ICQ constraint (10.8) if, and only if, there exists $V(x(t)) > 0$, $\forall x(t) \neq 0$ such that

$$\dot{V}(x'(t), \dot{x}'(t)) < - \begin{pmatrix} z'(t) \\ w'(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z'(t) \\ w'(t) \end{pmatrix}, \quad \forall(x'(t), \dot{x}'(t), z'(t), w'(t)) \neq 0$$

satisfying (10.5). $\tag{10.9}$

The dependence of the ICQ on the vector $z(t)$ can be turned into an explicitly dependence in terms of $x(t)$, $p_x(t)$ and $p_w(t)$ that yields

$$\dot{V}(x^{'}(t), \dot{x}^{'}(t)) < - \begin{pmatrix} x^{'}(t) \\ p^{'}_x(t) \\ p^{'}_w(t) \\ w^{'}(t) \end{pmatrix}^T M \begin{pmatrix} x^{'}(t) \\ p^{'}_x(t) \\ p^{'}_w(t) \\ w^{'}(t) \end{pmatrix}, \tag{10.10a}$$

$$M := \begin{bmatrix} B_{w0}QB_{w0}^T & B_{w0}QB_{w\rho}^T & B_{w0}QD_{zw\rho}^T & B_{w0}SD_{zw0} \\ \star & B_{w\rho}QB_{w\rho}^T & B_{w\rho}QD_{zw\rho}^T & B_{w\rho}SD_{zw0}^T \\ \star & \star & D_{zw\rho}QD_{zw\rho}^T & D_{zw\rho}SD_{zw0}^T \\ \star & \star & \star & \bar{R} \end{bmatrix} \tag{10.10b}$$

$$\bar{R} := R + D_{zw0}QD_{zw0}^T + D_{zw0}S + S^T D_{zw0}^T \tag{10.10c}$$

$$\forall (x^{'}(t), \dot{x}^{'}(t), p^{'}_x(t), p^{'}_w(t), w(t)) \neq 0 \quad \text{satisfying} \tag{10.10d}$$

$$\dot{x}^{'}(t) = A_0^T x^{'}(t) + C_{z0}w^{'}(t) + p^{'}_x(t) + p^{'}_w(t) \tag{10.10e}$$

$$p^{'}_x(t) := \rho^T A_\rho^T x^{'}(t), \quad p^{'}_w(t) := \rho^T C_{z\rho}^T w^{'}(t). \tag{10.10f}$$

An LMI condition equivalent to non-emptiness of the set (10.10) is obtained by the use of Finsler's Lemma.

Theorem 28 (Integral Quadratic Constraint)**.** *The set* (10.10) *is not empty if, and only if,* $\exists\, P \in \mathbb{S}^n$, Φ_1, Φ_2, Φ_3, Γ_1, Γ_2, Γ_3, Λ_1, Λ_2, Λ_3, Π_1, Π_2, $\Pi_3 \in \mathbb{R}^{n \times n}$, Ξ_1, Ξ_2, $\Xi_3 \in \mathbb{R}^{n \times n_z}$:

$$\mathcal{J} + \mathcal{H}^T + \mathcal{H} \prec 0, \qquad P \succ 0, \tag{10.11}$$

$$\mathcal{J} := \begin{bmatrix} B_{w0}QB_{w0}^T & P & B_{w0}QB_{w\rho}^T & B_{w0}QD_{zw\rho}^T & B_{w0}SD_{zw0} \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & B_{w\rho}QB_{w\rho}^T & B_{w\rho}QD_{zw\rho}^T & B_{w\rho}SD_{zw0}^T \\ \star & \star & \star & D_{zw\rho}QD_{zw\rho}^T & D_{zw\rho}SD_{zw0}^T \\ \star & \star & \star & \star & \bar{R} \end{bmatrix}, \tag{10.12}$$

$$\mathcal{H} := \begin{bmatrix} A_0\Phi_1 + A_\rho\rho\Phi_2 & A_0\Gamma_1 + A_\rho\rho\Gamma_2 & A_0\Lambda_1 + A_\rho\rho\Lambda_2 & A_0\Pi_1 + A_\rho\rho\Pi_2 \\ -\Phi_1 & -\Gamma_1 & -\Lambda_1 & -\Pi_1 \\ \Phi_1 - \Phi_2 & \Gamma_1 - \Gamma_2 & \Lambda_1 - \Lambda_2 & \Pi_1 - \Pi_2 \\ \Phi_1 - \Phi_3 & \Gamma_1 - \Gamma_3 & \Lambda_1 - \Lambda_3 & \Pi_1 - \Pi_3 \\ C_{z0}\Phi_1 + C_{z\rho}\rho\Phi_3 & C_{z0}\Gamma_1 + C_{z\rho}\rho\Gamma_3 & C_{z0}\Lambda_1 + C_{z\rho}\rho\Lambda_3 & C_{z0}\Pi_1 + C_{z\rho}\rho\Pi_3 \end{bmatrix}$$

$$\begin{matrix} A_0\Xi_1 + A_\rho\rho\Xi_2 \\ -\Xi_1 \\ \Xi_1 - \Xi_2 \\ \Xi_1 - \Xi_3 \\ C_{z0}\Xi_1 + C_{z\rho}\rho\Xi_3 \end{matrix} \Bigg]. \tag{10.13}$$

Proof. Assign

$$x(t) \leftarrow \begin{pmatrix} x^{'}(t) \\ \dot{x}^{'}(t) \\ p^{'}_x(t) \\ p^{'}_w(t) \\ w^{'}(t) \end{pmatrix}, \; \mathcal{Q} \leftarrow \mathcal{J}, \; \mathcal{B}^T \leftarrow \begin{bmatrix} A_0 & A_\rho\rho & 0 \\ -I & 0 & 0 \\ I & -I & 0 \\ I & 0 & -I \\ C_{z0} & 0 & C_{z\rho}\rho \end{bmatrix}, \; \mathcal{X} \leftarrow \begin{bmatrix} \Phi_1^T & \Phi_2^T & \Phi_3^T \\ \Gamma_1^T & \Gamma_2^T & \Gamma_3^T \\ \Lambda_1^T & \Lambda_2^T & \Lambda_3^T \\ \Pi_1^T & \Pi_2^T & \Pi_3^T \\ \Xi_1^T & \Xi_2^T & \Xi_3^T \end{bmatrix}$$

and apply Lemma 11 to the constrained Lyapunov problem (10.10) with $P \succ 0$. □

The matrix inequality above is a function of several multipliers. Would it be possible to constrain or eliminate multipliers without loss of generality? The Elimination Lemma [2, 3] will serve for the purpose of removing multipliers without adding conservatism to the solution.

Lemma 13 (Elimination of Multipliers)**.** *The following constraints on the multipliers*

$$\Gamma_1 := \alpha\Phi_1, \quad \Gamma_2 := \alpha\Phi_2, \quad \Gamma_2 := \alpha\Phi_3, \quad \Xi_1 := 0, \quad \Xi_2 := 0, \quad \Xi_3 := 0$$

can be enforced without loss of generality, where $\alpha > 0$ is a real scalar variable.

Proof. Assign

$$\mathcal{Q} \leftarrow \mathcal{J}, \mathcal{B}^T \leftarrow \begin{bmatrix} A_0 & A_\rho\rho & 0 \\ -I & 0 & 0 \\ I & -I & 0 \\ I & 0 & -I \\ C_{z0} & 0 & C_{z\rho}\rho \end{bmatrix}, \mathcal{C}^T \leftarrow \begin{bmatrix} \alpha I & 0 \\ -I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}^T, \mathcal{X} \leftarrow \begin{bmatrix} \Phi_1^T & \Phi_2^T & \Phi_3^T \\ \Gamma_1^T & \Gamma_2^T & \Gamma_3^T \\ \Lambda_1^T & \Lambda_2^T & \Lambda_3^T \\ \Pi_1^T & \Pi_2^T & \Pi_3^T \\ \Xi_1^T & \Xi_2^T & \Xi_3^T \end{bmatrix}$$

and apply the Elimination Lemma with $P \succ 0$. The chosen $\mathcal{C}^{\perp}$ does not introduce conservativeness to the condition. To see this expand (10.1b)

$$\mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^{\perp} = \begin{bmatrix} \alpha^2 B_{w0} Q B_{w0}^T - 2\alpha P & \alpha B_{w0} S D_{zw0} \\ \star & \bar{R} \end{bmatrix} \prec 0. \tag{10.14}$$

The above is equivalent to

$$\bar{R} \prec 0, \quad P \succ \frac{1}{2}\alpha \left(B_{w0} Q B_{w0}^T - B_{w0} S D_{zw0} \bar{R}^{-1} D_{zw0}^T S^T B_{w0}^T \right) \succeq 0 \tag{10.15}$$

where the second inequality arises from a Schur complement argument. Therefore, no conservatism is introduced provided that $\alpha > 0$ is considered a variable in the LMI. □

The ICQ condition with fewer multipliers is computationally less costly and more revealing to the purpose of synthesis. Notice that the matrix of plant parameters ρ has products with the multipliers Φ_2, Φ_3, Λ_2, Λ_3, Π_2, Π_3. The first step towards a condition for plant design is to enforce the following constraints on the multipliers

$$\Phi_2 := \Phi_3 := \Lambda_2 := \Lambda_3 := \Pi_2 := \Pi_3 := \Psi, \quad \Psi := \psi I_n. \tag{10.16}$$

where $\psi \in \mathbb{R}$ are scalar variables. Some conservatism to the condition is introduced by these constraints. The second step is to define the following nonlinear change-of-variables

$$\hat{\rho} := \rho\psi I_n, \quad \hat{\rho} = \begin{bmatrix} \hat{\rho}_1 I_n & \hat{\rho}_2 I_n & \dots & \hat{\rho}_N I_n \end{bmatrix}^T \tag{10.17}$$

involving ρ_i and ψ. Whenever $\psi \neq 0$, the original plant parameters can be reconstructed according to $\rho_i = \hat{\rho}_i \psi^{-1}$. With these definitions at hand, a condition for plant design subject to ICQ constraints can be stated.

Theorem 29 (Plant Synthesis). *There exists plant parameters ρ_i, $i = 1, \ldots, N$ such that the set* (10.10) *is not empty if* $\exists P \in \mathbb{S}^n$, $\hat{\rho}_i$, $i = 1, \ldots, N$, ψ, $\alpha \in \mathbb{R}$, $\Phi_1, \Gamma_1, \Lambda_1, \Pi_1, \in \mathbb{R}^{n \times n}$:

$$\mathcal{J} + \mathcal{H}^T + \mathcal{H} \prec 0, \qquad P \succ 0, \tag{10.18}$$

$$\mathcal{J} := \begin{bmatrix} B_{w0}QB_{w0}^T & P & B_{w0}QB_{w\rho}^T & B_{w0}QD_{zw\rho}^T & B_{w0}SD_{zw0} \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & B_{w\rho}QB_{w\rho}^T & B_{w\rho}QD_{zw\rho}^T & B_{w\rho}SD_{zw0}^T \\ \star & \star & \star & D_{zw\rho}QD_{zw\rho}^T & D_{zw\rho}SD_{zw0}^T \\ \star & \star & \star & \star & \bar{R} \end{bmatrix}, \tag{10.19}$$

$$\mathcal{H} := \begin{bmatrix} A_0\Phi_1 + A_\rho\hat{\rho} & \alpha(A_0\Phi_1 + A_\rho\hat{\rho}) & A_0\Lambda_1 + A_\rho\hat{\rho} & A_0\Pi_1 + A_\rho\hat{\rho} & 0 \\ -\Phi_1 & -\alpha\Phi_1 & -\Lambda_1 & -\Pi_1 & 0 \\ \Phi_1 - \psi I_n & \alpha(\Phi_1 - \psi I_n) & \Lambda_1 - \psi I_n & \Pi_1 - \psi I_n & 0 \\ \Phi_1 - \psi I_n & \alpha(\Phi_1 - \psi I_n) & \Lambda_1 - \psi I_n & \Pi_1 - \psi I_n & 0 \\ C_{z0}\Phi_1 + C_{z\rho}\hat{\rho} & \alpha(C_{z0}\Phi_1 + C_{z\rho}\hat{\rho}) & C_{z0}\Lambda_1 + C_{z\rho}\hat{\rho} & C_{z0}\Pi_1 + C_{z\rho}\hat{\rho} & 0 \end{bmatrix}, \tag{10.20}$$

$$\underline{\rho}_i\psi \geq \hat{\rho}_i \geq \overline{\rho}_i\psi. \tag{10.21}$$

and if the solution yields ψ non-singular. In this case, the original plant parameters can be recovered from the auxiliary ones according to $\rho_i = \hat{\rho}_i\psi^{-1}$.

Inequalities (10.18) to (10.20) result from combining Theorem 28 with Lemma 13, multiplier constraints (10.16) and change-of-variables (10.17). Bounds on the plant parameters are respected during synthesis by including (10.21) in the formulation. Notice the assumption of a solution rendering ψ non-singular. Although the structure of the above LMI does not imply non-singularity of ψ, the inclusion of an extra constraint $\psi > \epsilon$ or $\psi < -\epsilon$, $\epsilon > 0$ may render this variable non-singular.

Plant-Controller Design

The results from the previous section are now extended with controller design. Control inputs $u(t) \in \mathbb{R}^{n_u}$ are included in the open-loop system

$$\dot{x} = A(\rho)x(t) + B_u u(t) + B_w(\rho)w(t) \tag{10.22a}$$

$$z(t) = C_z(\rho)x(t) + D_{zu}u(t) + D_{zw}(\rho)w(t) \tag{10.22b}$$

where $B_u \in \mathbb{R}^{n \times n_u}$ and $D_{zu} \in \mathbb{R}^{n_z \times n_u}$ are considered independent of the plant parameters. This assumption can be fulfilled by augmenting the plant with states containing filtered inputs. We consider the problem of designing a full-state feedback controller $u(t) = Kx(t)$ concurrently with plant parameters ρ_i. Similarly to the plant design, the derivations are based on the dual closed-loop system

$$\dot{x}'(t) = (A_0^T + K^T B_u^T)x'(t) + (C_{z0}^T + K^T D_{zu}^T)w'(t) + p_x'(t) + p_w'(t) \tag{10.23a}$$

$$z'(t) = B_{w0}^T x'(t) + D_{zw0}^T w'(t) + B_{w\rho}^T p_x'(t) + D_{zw\rho}^T p_w'(t) \tag{10.23b}$$

$$p_x'(t) := \rho^T A_\rho^T x'(t), \quad p_w'(t) := \rho^T C_{z\rho}^T w'(t). \tag{10.23c}$$

The first step towards a condition for simultaneous plant-controller design is to enforce the constraint

$$\Phi_1 := \Lambda_1 := \Pi_1 := \Phi. \tag{10.24}$$

Some conservatism to the condition is introduced by these constraints. The second step is to resort to the nonlinear change-of-variables between controller and multiplier data [13, 8, 10]

$$\hat{K} := K\Phi. \tag{10.25}$$

Whenever Φ is non-singular, the original controller gains can be reconstructed from the auxiliary ones according to $K = \hat{K}\Phi^{-1}$. With these definitions at hand, a condition for simultaneous plant-controller synthesis subject to ICQ constraints can be stated.

Theorem 30 (Simultaneous Plant-Controller Synthesis). *There exist plant parameters* ρ_i, $i = 1, \ldots, N$ *and a controller gain K such that the set* (10.10) *is not empty if* $\exists\, P \in \mathbb{S}^n$, $\hat{\rho}_i$, $i = 1, \ldots, N$, ψ, $\alpha \in \mathbb{R}$, Φ_1, Γ_1, Λ_1, Π_1, $\in \mathbb{R}^{n \times n}$:

$$\mathcal{J} + \mathcal{H}^T + \mathcal{H} \prec 0, \qquad P \succ 0, \tag{10.26}$$

$$\mathcal{J} := \begin{bmatrix} B_{w0}QB_{w0}^T & P & B_{w0}QB_{w\rho}^T & B_{w0}QD_{zw\rho}^T & B_{w0}SD_{zw0}^T \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & B_{w\rho}QB_{w\rho}^T & B_{w\rho}QD_{zw\rho}^T & B_{w\rho}SD_{zw0}^T \\ \star & \star & \star & D_{zw\rho}QD_{zw\rho}^T & D_{zw\rho}SD_{zw0}^T \\ \star & \star & \star & \star & \bar{R} \end{bmatrix}, \tag{10.27}$$

$$\mathcal{H} := \begin{bmatrix} A_0\Phi + B_u\hat{K} + A_\rho\hat{\rho} & \alpha(A_0\Phi + B_u\hat{K} + A_\rho\hat{\rho}) & A_0\Phi + B_u\hat{K} + A_\rho\hat{\rho} \\ -\Phi & -\alpha\Phi & -\Phi \\ \Phi - \psi I_n & \alpha(\Phi - \psi I_n) & \Phi - \psi I_n \\ \Phi - \psi I_n & \alpha(\Phi - \psi I_n) & \Phi - \psi I_n \\ C_{z0}\Phi + D_{zu}\hat{K} + C_{z\rho}\hat{\rho} & \alpha(C_{z0}\Phi + D_{zu}\hat{K} + C_{z\rho}\hat{\rho}) & C_{z0}\Phi + D_{zu}\hat{K} + C_{z\rho}\hat{\rho} \end{bmatrix}$$

$$\begin{matrix} A_0\Phi + B_u\hat{K} + A_\rho\hat{\rho} & 0 \\ -\Phi & 0 \\ \Phi - \psi I_n & 0 \\ \Phi - \psi I_n & 0 \\ C_{z0}\Phi + D_{zu}\hat{K} + C_{z\rho}\hat{\rho} & 0 \end{matrix} \Bigg] \tag{10.28}$$

$$\underline{\rho}_i\psi \geq \hat{\rho}_i \geq \overline{\rho}_i\psi. \tag{10.29}$$

Proof. The above inequalities are trivially obtained from Theorem 29 but considering the closed-loop (10.23), multiplier constraints (10.24) and change-of-variables with controller data (10.25). Bounds on the plant parameters are respected during synthesis by the presence of (10.29). If a feasible solution is found, the original plant and controller parameters can always be recovered from the auxiliary ones according to $\rho_i = \hat{\rho}_i\psi^{-1}$ and $K = \hat{K}\Phi^{-1}$, respectively. To realize this, first notice that $-\alpha(\Phi + \Phi^T) \prec 0$ in the (2,2) entry of $\mathcal{J} + \mathcal{H} + \star \prec 0$ with $\alpha > 0$ implies Φ non-singular thus invertible. Moreover, $\psi \neq 0$ is implied by the entry (3,3), i.e. $-\psi I_n \prec -\Phi$ with Φ non-singular. $\qquad\square$

4 Conclusions and Future Work

Sufficient LMI conditions to the simultaneous plant-controller design problem is presented in this paper. The insertion of multipliers in the formulation facilitates a linearizing

change-of-variables involving plant parameters to be optimized. Synthesis is subject to integral quadratic constraints on input and output signals, offering flexibility on the specification of closed-loop performance. The linear dependence of the LMIs on the Lyapunov matrix facilitates the usage of parameter-dependent Lyapunov functions as certificates of stability of uncertain and time-varying systems. Results on simultaneous plant-controller design for uncertain and linear-parameter varying systems have already been derived and will be presented in a separate manuscript.

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Report H

D-Stability Control of Wind Turbines

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Techical Report

Abstract

Tuning a model-based multivariable controller for wind turbines can be a tedious task. In this report, multiobjective output-feedback control via LMI optimization is explored for wind turbine control design. In particular, we use D-stability constraints to increase damping and decay rate of resonant structural modes. In this way, the number of weighting functions and consequently the order of the final controller is reduced, and the control design process is physically more intuitive and meaningful.

1 Introduction

Optimal methods are attractive to wind turbine control, as the controller is the minimizer of a cost function that is chosen a priori. Optimal control also handles multiple-input multiple-output (MIMO) systems naturally. In practice, building the cost function can be just as difficult as tuning a classical controller. Due to mathematical convenience, the cost function is usually defined as a quadratic functional of the states and inputs (LQR control). The selection of the weights composing the cost function to achieve generator speed and power regulation as well as load attenuation can be a tedious task, often relying on trial-and-error procedures. For example, the designer tries to indirectly control mechanical loads by penalizing positions and velocities of the structural degrees of freedom. To achieve asymptotic generator speed regulation, it is common to include a term to penalize the speed error, and another term for the integral of the speed error. An analytical characterization of the weights that should be attributed to each of these terms is not trivial. H_∞-control of wind turbines also suffers from some problems related to weighting, since the weighting functions are usually transfer functions augmented to the design model, increasing the the order and complexity of the controller.

2 D-Stability Control of Wind Turbines

Our interest lies in the design of a dynamic output feedback controller

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{11.1}$$
$$u(t) = C_c x_c(t) + D_c y(t) \tag{11.2}$$

for a wind turbine, such that satisfactory generator speed (and power) regulation as well as mitigation of mechanical vibrations are attained. Consider the closed-loop system

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}w(t) \tag{11.3}$$
$$z(t) = \mathcal{C}x(t) + \mathcal{D}w(t). \tag{11.4}$$

The ideas presented here can be extended to linear-parameter varying (LPV) systems by making plant and controller matrices as functions of some scheduling parameter. Choose input-output operators (transfer matrices) $T_{w \to z}$ of interest. For example, $T_{V \to \Omega_g}$ is the operator from wind disturbance to generator speed and $T_{V \to \dot{\beta}}$ is the operator from wind disturbance to pitch angle velocity. Then, adopt time and frequency domain specifications in closed-loop. These specification can include input-output norms as

- $\mathcal{L}_2$-norm ($\|T_{w \to z}\|_{i,2}^2 < \gamma$): "energy gain" of a system to a worst-case disturbance, e.g. a constant relating the energy of the worst-case wind (wind gust) to the energy on the gen. speed;

- $\mathcal{H}_2$-norm ($\|T_{w \to z}\|_2^2 < \mu$): expected variance of the output when the disturbance is assumed white noise with zero mean;

- Generalized $\mathcal{H}_2$-norm ($\|T_{w \to z}\|_g < \varrho$): energy-to-peak gain of a system, e.g. a constant relating wind gust with maximum pitch angle rates.

The specifications can also take regional pole placement constraints, such as the ones illustrated in Fig. 11.1.

- Conic region: minimum damping for all modes;

- Decay region: minimum decay rate for all modes;

- Circle region: a circle with radius r and center $-c$.
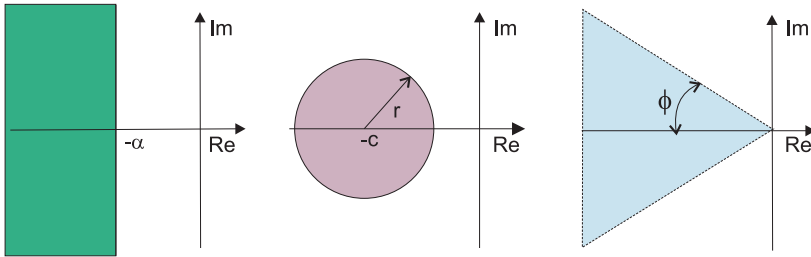
Figure 11.1: Regions of the complex plane for pole placement constraints.

For illustration purposes, we adopt a generic wind turbine model with 9-states composed of the following modes/states:

- Two-mass flexible drive-train;

- Tower fore-aft displacement;

- Second-order pitch actuator;

- First-order generator lag;

- Generator speed integral.

The chosen design criteria is to minimize the performance level γ subject to

- $\|T_{v \to z}\|_{i,2}^2 < \gamma$, $\qquad z(t) = [\Omega_i(t) \quad Q_g(t) \quad \beta(t)]^T$;

- $\left\|T_{v \to \dot{\beta}}\right\|_g < 8$ deg/s;

- $\arg(\lambda(\mathcal{A})) < \phi$, $\mathrm{Re}(\lambda(\mathcal{A})) < \alpha$.
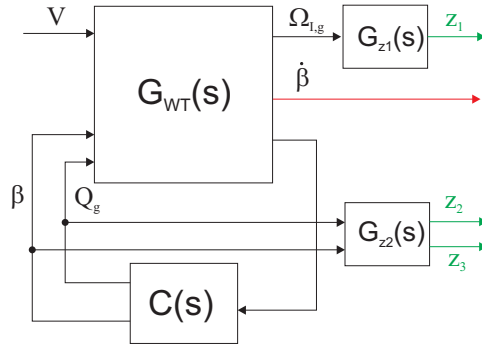
Figure 11.2: Augmented system for synthesis.

where Ω_i is the integral of speed error, Q_g is the generator torque and β is the pitch angle. In words, this performance criteria states that the generator speed "integral square error (ISE)" should be small with minimum control effort while respecting the pitch rate limit, for the worst-case wind disturbance. The augmented system for synthesis purposes is depicted in Fig. 11.2. The weighting functions were chosen frequency independent on the form $G_1(s) := k_1$ and $G_2(s) := \text{diag}(k_2, k_3)$. All modes should be contained in a conic region with internal angle 2ϕ and decay rate of at least α. The adopted multiobjective LMI formulation follows [1]. The LMIs were solved with the semidefinite programming code SeDuMi [2] and parser YALMIP [3].

The drive-train torsional mode is the least damped mode in open-loop. Active vibration control can be obtained by a D-stability constraint on the minimum damping of the closed-loop system. via the conic region constraint. Figure 11.3 depicts the closed-loop system response for a unit step on wind speed disturbance, under various internal angles of the conic region. The location of the closed-loop poles are illustrated in Fig. 11.4. The effect of the damping constraint on the generator speed Ω_g and generator torque Q_g signals is noticeable.

Increased damping in the tower fore-aft direction can be achieved by intersecting the conic region with a minimum decay rate constraint. Figures 11.5 and 11.5 depicts wind disturbance step response and location of the poles in the complex plane, respectively, for various decay rates intersected with a conic region with angle of 87°. Notice that the decay constraint does not influence significantly the torque signal, responsible for damping the drive-train mode. This is due to the differences in terms of damping of these modes in open-loop. The contribution of aerodynamic damping is higher in the tower mode than in the drive-train mode, making the former more damped than the later. Therefore, the decay rate acts on the tower mode without moving the drive-train poles and vice-versa.

All the above simulations considered an energy-to-peak constraint on the pitch rate $\dot{\beta}$ of 10 deg/s. We now access how changes in the pitch rate bounds impacts the response of the system in closed-loop. Wind speed step responses for different bounds on the pitch rate are depicted in (11.7). Notice that both drive-train damping and tower decay rates are kept the same irrespective of the energy-to-peak gain of pitch rate. Response of generator speed regulation expectedly changes, and could be fine tuned by varying the gain k_1.
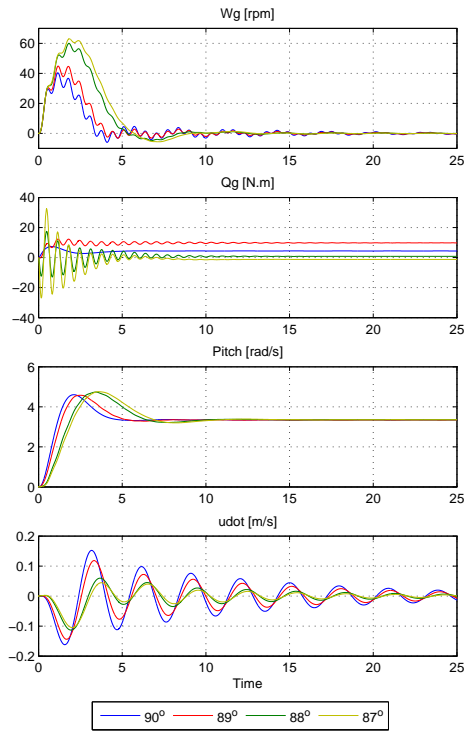
Figure 11.3: Closed-loop response for a unit step on wind speed disturbance for different angles of the conic region.
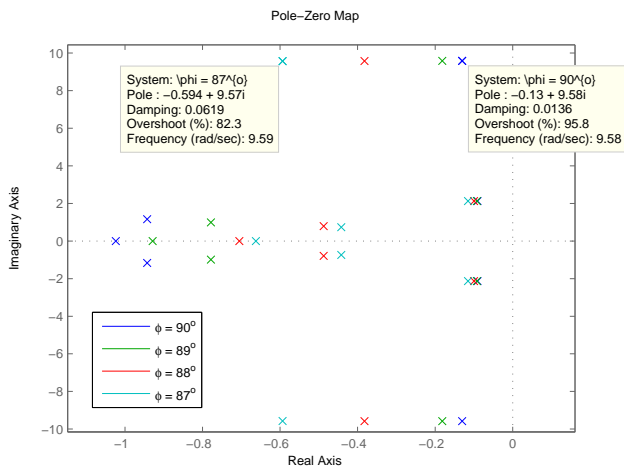

Figure 11.4: Location of the closed-loop poles for different angles of the conic region.
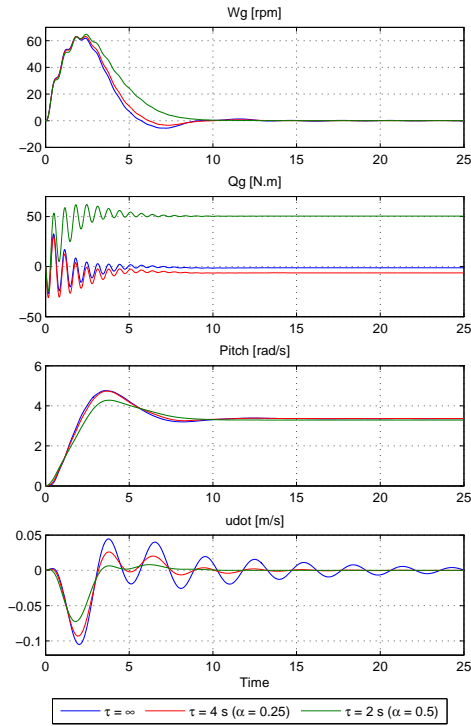
Figure 11.5: Closed-loop response for a unit step on wind speed disturbance for different decay rates and conic region with angle of 87°.
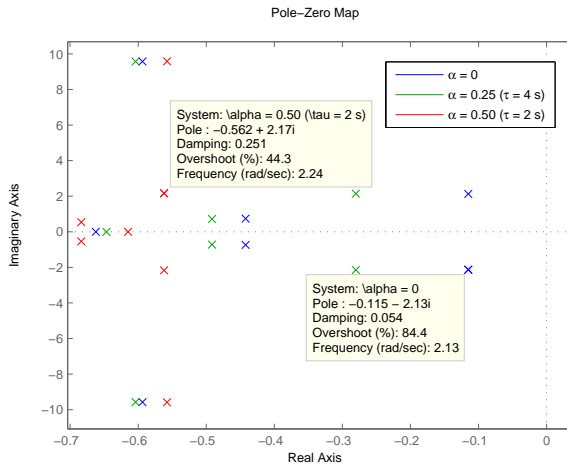

Figure 11.6: Closed-loop poles for different decay rates and conic region with angle of 87°.
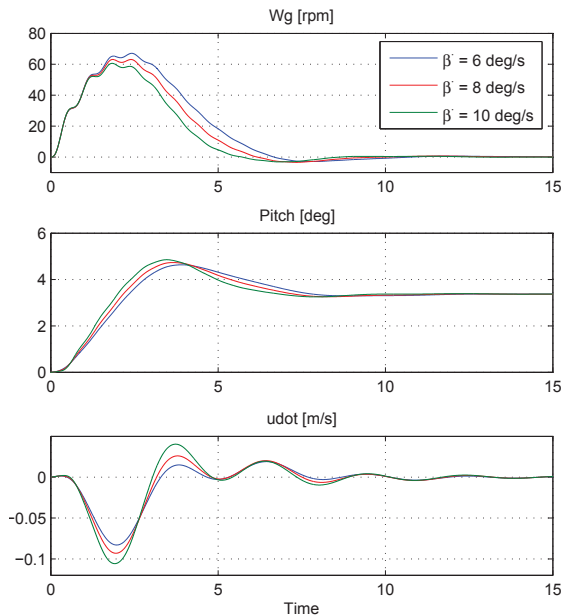
Figure 11.7: closed-loop response for a unit step on wind speed disturbance for different energy-to-peak gains on pitch angle rate, with decay rate of $\tau = 4$s and conic region with angle of 87°.

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