NONLINEAR DISSIPATIVE AND SET-STABILIZING CONTROL

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Abstract

This dissertation treats analysis and state feedback control synthesis of deterministic nonlinear and continuous time systems of ordinary differential equations. In particular, we are investigating the input-output behavior in terms of dissipation in the sense of Willems (1972), and the asymptotic behavior in state space in relation to general compact and invariant sets. Included in this framework is the well-known $\mathcal{H}_\infty$ state feedback control theory, which forces the input-output map to have a bounded $\mathcal{L}_2$ gain, and ensures local asymptotical stability of equilibrium points under zero disturbances.

The theory here presented extends the usual $\mathcal{H}_\infty$ approach significantly. Apart from using general dissipation inequalities to describe the desired input-output performance of the system, we consider stability issues of state trajectories near compact and invariant sets, which represent the desired behavior in state space. Hence, oscillatory and any other non-stationary mode of operation is included in this framework. This set stability approach is not local, but regional (semi-global) insofar as the basin of attraction of the above mentioned invariant set is estimated by compact sets, here called performance envelopes. Moreover, under certain conditions we can guarantee that the attractiveness and/or asymptotic stability of invariant sets in state space is not destroyed by non-zero, but decaying disturbances. Furthermore, smoothness of storage functions, and practical stability of invariant sets with respect to time-persistent, not decaying, but bounded disturbances can be obtained.

Contents

Preface

This dissertation is written and submitted to the Technical University of Denmark in partial fulfillment of the requirements for the degree of Doctor of Philosophy, Graduate Program in Mathematics. The research has been financed by a grant given by the Technical University of Denmark (DTU). The Department of Mathematics (MAT), DTU, has provided all facilities needed, in particular a pleasant working environment and access to powerful computer systems. My Ph.D studies have been supervised by

- Associate Professor Morten Brøns, MAT, main supervisor, and
- Professor Jakob Stoustrup, Department of Control Engineering, University of Aalborg, secondary supervisor.

Associate Professor Morten Brøns works mainly in the area of dynamical systems and topological flows, whereas Professor Jakob Stoustrup has specialized in linear, finite-dimensional control theory. The topic of my research intersects but minimally with their interests, therefore I thank my supervisors for the liberty of choice of research topic and the liberty to structure my research according to my own needs and demands. I thank them also for all constructive discussions and comments during the last three years, which - due to diverging topics - have been mostly on a general level. I thank also my room mate and best friend Ph.D Eric Beran for good working company, and for providing help on the most obscure features of Unix, LaTeX2e and MATLAB. Special thanks goes also to fellow graduate student Uffe Høgsbro Thygesen for proof reading, and for being a perfect stand-in at the ACC97.

Related presentations and publications

The content and main ideas of this dissertation have partially been presented at various national workshops and international conferences during the last three years. The following list states the most important presentations held:

Ikke-lineær kontrolteori, Årlig DCAMM-symposium, Hotel Hvide Hus, Maribo, 13-15/3 1995

Non-linear control and bifurcation, Lund-Lyngby Day on Control, Department of Automatic Control, LTH, Lund, 30/5 1995

Aircraft, bifurcation, and control, EURACO-workshop for nonlinear control, Lagoa, Portugal, 14/5 1996

Surge in compressors and control of Hopf bifurcations by nonlinear $\mathcal{H}_\infty$ state feedback, ECMI conference 1996, DTU, 28/6 1996

On Set-stabilizing $\mathcal{H}_\infty$ Control, presentation at Department of System Sciences, Washington University, St.Louis, Missouri, 19/11 1996

Lyapunov-metoder i ikke-linear kontrol, Årlig DCAMM-symposium, Hotel Ebeltoft Strand, 17-19/3 1997

Nonlinear $\mathcal{H}_\infty$ State Feedback Controllers: Estimation of Valid Region, American Control Conference, Albuquerque, New Mexico, USA, 4-6/5 1997

Semi-global $\mathcal{H}_\infty$ State Feedback Control, European Control Conference, Brussels, Belgium, 1-4/6 1997

New Questions and Answers to Nonlinear $\mathcal{H}_\infty$ Control, 1st Workshop on Industrial Mathematics for Nordic Ph.D.-students, Hillerød, DK, 29-31/8 1997

Aspects of nonlinear Control: a Partial Differential Inequality in $\mathcal{H}_\infty$ control, SNF netværksmøde, MAT, 11/12 1997

Qualitative Investigation of a Partial Differential Inequality in Dissipative System Theory, Danish Symposium on Partial Differential Equations; Analysis and Applications, Department of Mathematics and Computer Science, Odense University, 29/5 1998

I thank the head of our department for partial financing of a large number of travels to European countries and to the U.S.A. associated with these presentations.

This dissertation is essentially composed as a collection of articles which have been printed in, or are submitted to, international proceedings and journals in the field of nonlinear control and dynamic systems. The following contributions have been, or will be, peer reviewed before acceptance.

[CMPP97] Nonlinear $\mathcal{H}_\infty$ State Feedback Controllers: Computation of Valid Region, co-autors Jens Moeller-Pedersen and Martin Pagh Petersen. Printed in: Proceedings of the American Control Conference June 4-6, 1997, Albuquerque, New Mexico, U.S.A.

[CS97] Semiglobal $\mathcal{H}_\infty$ State Feedback Control, co-author Jakob Stoustrup. Printed in: Proceedings of the European Control Conference, 1-4 July 1997, Brussels, Belgium

[Cro98a] On Asymptotic Behavior of Almost Asymptotic Systems. Submitted February 1998

[Cro98c] On Dissipative Systems and Set Stability. Submitted April 1998

I thank my co-authors for their cooperation in preparing these articles.

Some of these papers are published as technical reports in a long, more detailed version. The following can be obtained from the the Department of Mathematics, Technical University of Denmark:

[CMPP96] Nonlinear $\mathcal{H}_\infty$ State Feedback Controllers: Computation of Valid Region, co-authors Jens Moeller-Pedersen and Martin Pagh Petersen, Technical Report 1996-32

[CS96] Semiglobal $\mathcal{H}_\infty$ State Feedback Control, co-author Jakob Stoustrup, Technical Report 1996-32

[Cro96] Surge in Compressors and Control of Hopf Bifurcations by Nonlinear $\mathcal{H}_\infty$ State Feedback, Printed in: Book of Abstracts of the 9th Conference of the European Consortium for Mathematics in Industry, Technical University of Denmark, June 25-29, 1996

[Cro98b] On Asymptotic Behavior of Almost Asymptotic Systems, Technical Report 1998-03

[Cro98d] On Dissipative Systems and Set Stability, Technical Report 1998-07

Educational part of my Ph.D studies

During my research I followed and passed the following graduate courses:

Higher Order Computational Methods for Incompressible Continuae, Institut for Bygningsteknlogi og Strukturer, AUC, Aalborg, 29/8-14/9 1994, Prof. Søren Jensen

Behavioral Systems and Control, Department of Automatic Control, LTH, Lund, 20-24/3 1995, Prof. Jan C. Willems, Mathematical Institute, University of Gronningen, The Nederlands

Symbolic Computation, Matematisk Institut, DTU, 12-30/6 1995, Ph.D. Petr Lisonek

Introduction to Wavelets, 5th International Summer School, University of Jyväskylä, Finland, 31/7-11/8 1995, Prof. D.L. Salinger, Univ. of Leeds

Signal Processing and Multifractal Analysis, 5th International Summer School, University of Jyväskylä, Finland, 31/7-11/8 1995, Prof. D.L. Salinger, Univ. of Leeds

Nonlinear Dynamical Systems, Fractals and Chaos, 5th International Summer School, University of Jyväskylä, Finland, 14-25/8 1995, Prof. J. Norbury, Univ. of Oxford

Advanced Controller Design: Nonlinear Systems, ass. Prof. Jakob Stoustrup, MAT, spring 1995

Seminar Course in Mathematics, Statistics, Operations Research and Numerical Analysis, obligatory graduate course, fall 1995 and spring 1996

Domain Optimization in Linearized Elasticity by the Transformation Method, MAT, 15-22/4 1996, Ph.D Ludwig Holzleitner, Universität Linz, Austria

Advanced Nonlinear Control Design, Department of System Sciences and Mathematics, Washington University, St.Louis, Missouri, Sep. - Dec. 1996, Prof. A. Isidori

C++ og objekt-orienteret programmering, standard DTU-kursus semester F 1997

Ph.D kursus i patentforhold, Forskerakademiet og Patentdirektoriatet, Taastrup, 1-3/12 1997

I thank all those who helped organizing these courses and research schools.

International research stay

In the course of my research in dissipative control I had the possibility to visit Professor Alberto Isidori in two terms of one month length at the Universitá "La Sapienza", Dipartimento di Informatica e Sistemistica, Via Eudossiana 18, 00184 Roma, Italy, in June 1996 and in January 1997. I thank Professor Alberto Isidori and Ph.D Stefano Battilotti for fruitful discussions and suggestions during this stay. My special thank goes also to Barbara Fantoni for help to deal with all practical details, and for showing me Rome, and to Annamaria, Anna Maria and Alessandra Aloe for providing a cosy base during the first stay. I wish also to thank "Det danske institut i Rom", especially Museumsinspektør Dr. Phil. Jan Zahle, Administrator cand. Mag. Karen Ascani, secretary Bente Rasmussen, and Ermano for a really pleasant stay in January 1997. A special thank goes to "H.M. Dronning Ingrids romerske Fond" for financing, and to all fellow inhabitants for providing most interesting discussions during our meals and visits to the countryside of Rome.

I had also the chance to visit Professor Alberto Isidori at the Department of System Sciences and Mathematics, Washington University, St.Louis, Missouri, during three month from September to December 1996. I thank Professor Alberto Isidori for discussions, and for the possibility to follow his graduate course "Advanced Nonlinear Control Design". I thank also secretary Sandra Devereux for being very helpful with the practical arrangements of this stay, and for showing me St.Louis. My special thank goes also to fellow graduate students Alberto Bemporad, Andrea Serrani, Alpay, Ben Schwarz and Matt Kiefer for good company and exchange of ideas, and David Lockman for keeping Unix work.

How to read this dissertation

As mentioned before, this dissertation is composed as a collection of articles which have been printed in, or are submitted to, international proceedings and journals in the field of nonlinear control and dynamic systems. This does not mean that the present text is a bunch of articles duplicated at random. To make the content of this dissertation easily accessible, each paper is placed in a separate chapter devoted its own subject. In the beginning of each chapter the main ideas of the following paper are loosely explained. Then, after each paper, a discussion of important related topics is included, as well as a section with further comments and references. This sequence of papers is preceded by an introductory chapter presenting in general terms the kind of problems investigated. A short chapter on different known numerical approaches to solve a certain kind of nonlinear partial differential inequalities, called Hamilton-Jacobi inequalities, concludes the main part of the dissertation. Finally, a bibliography containing the entries of all papers and all secondary material is found.

A note on handbooks: during the preparation of this thesis I used Nicholas J. Higham's "Handbook of writing for the mathematical sciences"[Hig93], Leslie Lamport's "LaTeX user's guide and reference manual "[Lam94], and "The LaTeX Companion" [GMS94]. Help on the usage of modern English has been found in Cassell's Dictionary [Cas94]. Without these reference books I had probably not succeeded writing my thesis. I can only blame myself for all errors left.

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Summary

This dissertation deals with control issues of nonlinear systems. We focus on problems concerning the input-output behavior, and the asymptotic behavior in state space. In particular, we are investigating the notion of dissipation, the property of set stability, and the choice of control strategy to obtain these goals.

Chapter 1 gives a general overview of deterministic control problems addressed in this thesis. These are formulated in terms of systems of n ordinary differential equations (ODE's). The fundamental questions asked concerning stability, robustness, and performance of such systems - and partially answered in this thesis - are motivated.

Chapter 2 is devoted the subproblem of $\mathcal{H}_\infty$ control on compact sets containing equilibrium points of the system. We are considering the regular case alone, that is, the use of controls is penalized. Unlike in the linear context, local control does not imply global control, therefore we can not from local asymptotic stability of equilibrium points deduce the size of the basin of attraction. The later is estimated by a compact region, called performance envelope. According to the game theoretic approach to $\mathcal{H}_\infty$ theory, we solve a partial differential inequality, called the state-feedback Hamilton-Jacobi inequality (HJI), to obtain a suitable control law and a performance envelope. We show how Luke's approximation scheme can be implemented in the symbolic language MAPLE to solve HJI's near equilibrium points.

In the commentary section following the first paper we give a historic overview focusing on the the game theoretic and the differential geometric approach to $\mathcal{H}_\infty$ control theory. Then, state feedback $\mathcal{H}_\infty$ control techniques for general affine nonlinear systems which are not simplified by the so-called DGKF-conditions are showed. This generalization allows us to treat tracking problems.

In chapter 3 we generalize the regional $\mathcal{H}_\infty$ state control problem described in chapter 2 to stabilize invariant compact sets in state space. This is a new approach which combines elements of La Salle's invariance principle with $\mathcal{H}_\infty$ control and set stability issues. The here described results on semi-global stability and set-stability by $\mathcal{H}_\infty$ control use conditions in terms of the state-feedback Hamilton-Jacobi inequality and the properness of C^1 storage functions, provided that a certain detectability property is satisfied. Usual $\mathcal{H}_\infty$ control techniques are generalized such that oscillating and other non-stationary modes of operation can be dealt with.

The commentary section following the second paper mentions briefly non-differentiable solutions to Hamilton-Jacobi equations, also called weak or viscosity solutions.

Chapter 4 has the purpose to extend the class of disturbances under which attractiveness of invariant sets can be assured. More precisely, we are investigating robustness properties of systems with respect to non-zero, but decaying disturbances. The main tool used in the third paper is the notion of asymptotically autonomous systems, originally developed by L. Markus (1956). We provide an alternative, detailed proof of his main theorem to improve the understanding of asymptotic behavior of asymptotically autonomous systems. Moreover, known results on local asymptotic properties near simple asymptotically stable equilibrium points are generalized to regional (semiglobal) asymptotic properties of general invariant and compact sets. These sets have either to be asymptotically stable, or completely unstable in the sense that they are asymptotically stable under the time-reversed flow of the system.

The commentary section following the third paper applies the above mentioned findings to the $\mathcal{H}_\infty$ control problem with set-stability derived in the previous chapter. A new corollary formalized the robust set-stabilizing $\mathcal{H}_\infty$ control approach, where non-zero, but decaying disturbances are included in the framework of stability analysis.

Chapter 5 presents the author's main contribution to nonlinear dissipation and it's relations to stability issues of general invariant sets. The overall idea is that as much as possible information on the qualitative behavior of a dissipative system should be extracted from the data of the system, the structure of the Hamilton Jacobi inequality, and the storage function found. We start with an introduction to general dissipative analysis in the sense of Willems (1972). The required regularity properties of storage functions are relaxed: In the case that we are interested in the dissipation inequality alone, lower semicontinuity suffices, as pointed out by James (1993). To make investigation of the positive limit sets of state trajectories treatable, we have to consider continuous and locally Lipschitz storage functions. The analysis is performed on an open subset called the reachable set.

A novelty is the definition of four different subsets of the state space in terms of generalized gradients; the relations among these sets are giving valuable information on the (asymptotic) stability properties of undisturbed trajectories. Moreover, the use of strict Hamilton-Jacobi inequalities (these are HJI's which are satisfied strictly negative) reveals attractive properties like properness, or positive definiteness of storage functions, and asymptotic stability of undisturbed trajectories with respect to general invariant sets. The tools used here are inspirated by La Salle's invariance principle (1961), and give also useful performance envelopes.

Then, storage functions are treated as Input-to-State-Stability (ISS) Lyapunov functions in the sense of Sontag and Lin (1990-1995). This point of view provides existence proofs for smooth storage functions, and assures robustness of stability properties with respect to time-persistent, not decaying, but bounded disturbances. Different classes of control systems are compared to each other, and it is pointed out why the existence of smooth storage functions is beneficial in practical applications and numerical approximations.

In the commentary section following the fourth paper the existence of storage functions which are continuous and locally Lipschitz is commented. The problem of state feedback synthesis is taken care of using the interaction between dissipation, stability issues with respect to general invariant sets, and state feedback algorithms based on differential games.

We derive the state feedback Hamiltonian-Jacobi inequality, and discuss asymptotic behavior of controlled systems. It is furthermore shown that the minimizing control u_{min} and the maximizing disturbance w_{max} vanish at positive limit sets, this is a generalization of a property well known in $\mathcal{H}_\infty$ control. The application of the analysis results to accomplish smooth and robust state feedback control are mentioned.

Chapter 6 gives a short review of the efforts which have been made to solve the Hamilton-Jacobi inequalities associated with nonlinear $\mathcal{H}_\infty$ control. This chapter displays no original work of the author, it is mainly included to help the reader in the process of comparing numerical tools.

Sammenfatning

Denne Ph.D-afhandling beskæftiger sig med analyse og regulering af ikke-lineære systemer. Vi interesserer os for input-output opførelsen, og for asymptotisk opførsel i tilstandsrummet. Specielt undersøges begrebet dissipativitet, egenskaben af mængde-stabilitet og valget af bedst mulig reguleringsstrategi for at opnå disse mål.

Kapitel 1 giver et generelt overblik over den type deterministiske reguleringsproblemer, som omhandles i nærværende afhandling. Disse er givet som systemer af n-dimensionale ordinære differentialligninger (ODE's). Vi motiverer de fundamentale spørgsmål om stabilitet, robusthed og ydelse (performance) af systemer, som er delvist løst i afhandlingen.

Kapitel 2 beskæftiger sig med underproblemet om $\mathcal{H}_\infty$-regulering på kompakte mængder, der indeholder systemets kritiske punkter. Kun det regulære tilfælde er omhandlet; det vil sige at brugen af reguleringssignalet bliver altid straffet ved en passende omkostningsfunktion. I modsætning til lineær regulering medfører lokal stabilitet ikke global stabilitet, og en estimering af størrelsen af den tiltrækkende region i tilstandsrummet ved en kompakt mængde - her kaldet performance envelope - er nødvendigt. Når man følger den spilteoretiske indfaldsvinkel til $\mathcal{H}_\infty$-regulering, så skal vi løse en partiel differential-ulighed - her kaldet Hamilton-Jacobi-ulighed (HJI) - til at finde en passende reguleringsstrategi og en performance envelope. Vi viser hvordan Luke's approksimations-skema kan implementeres i det symbolske sprog MAPLE til at løse HJI'er nær kritiske punkter.

I det kommenterende afsnit, som følger den første artikel, gives en kort indføring i den spilteoretiske og den differentialgeometriske indfaldsvinkel til $\mathcal{H}_\infty$-regulering. Derefter vises tilstands-tilbagekobling og $\mathcal{H}_\infty$-regulering for generelle affine og ikke-lineære systemer, der ikke er underlagt forenklende, såkaldte DGKF-betingelser. Denne generalisering tillader os at anvende $\mathcal{H}_\infty$-regulering på tracking problemer.

I kapitel 3 generaliseres det i det forgående kapitel omhandlede regionale $\mathcal{H}_\infty$-reguleringsproblem til også at omfatte stabilisering af kompakte og invariante mængder i tilstandsrummet. Denne nye indfaldsvinkel kombinerer elementer af La Salle's invarians-princip med $\mathcal{H}_\infty$-regulering, og med analyse af stabilitetsforhold af mængder i tilstandsrummet. De i beskrevne resultater om regional mængde-stabilitet forudsætter at systemet er detekterbart på passende måde, og at differentiable og radielt ubegrænsede løsninger til Hamilton-Jacobi-uligheden eksisterer. Sædvanlige $\mathcal{H}_\infty$-reguleringsalgoritmer er udvidet til også at omfatte oscillerende og andre ikke-stationære driftstilstande.

Efter artiklen nævnes kort ikke-differentiable løsninger til Hamilton-Jacobi-ligninger. Disse kaldes også svage, eller viskøse løsninger.

Kapitel 4 har formålet at udvide klassen af forstyrrelser der ikke forhindrer tiltrækning til invariante mængder i tilstandsrummet. Vi undersøger robusthed af systemer med hensyn til ikke-trivielle, men i tiden uddøende forstyrrelser. Værktøjet, som bruges i den tredie artikel, er begrebet asymptotisk autonome systemer, defineret af L. Markus (1956). Vi giver et alternativt bevis for Markus' hovedsætning, først og fremmest for at forbedre forståelsen af den asymptotiske opførsel af asymptotisk autonome systemer. Desuden generaliseres før kendte resultater om lokal asymptotisk opførsel nær asymptotisk stabile kritiske punkter til regional asymptotisk opførsel i forbindelse med kompakte og invariante mængder. Disse mængder skal enten være asymptotisk stabile, eller totalt ustabile i den forstand at de er asymptotisk stabile med hensyn til det tids-inverterede system.

Efter den tredie artikel anvendes de før omtalte resultater om asymptotisk autonome systemer på $\mathcal{H}_\infty$-reguleringsproblemet med mængde-stabilitet fra forrige kapitel. Et nyt korollar formaliserer robust mængde-stabiliserende $\mathcal{H}_\infty$-regulering hvor ikke-trivielle, men i tiden uddøende forstyrrelser er inkluderet i stabilitetsanalysen.

I kapitel 5 præsenterer forfatteren sit væsentligste bidrag til ikke-lineær dissipativitet og dennes relationer til stabilitesforhold af invariante mængder. Den overordnede ide er at man bør udvinde mest mulig information om systemets kvalitative opførsel af systemets data, af Hamilton-Jacobi-ulighedens struktur, og af de såkaldte storage-funktioners struktur. Først introduceres generel dissipativitet i Willems forstand (1972). Storage-funktionen behøver ifølge James (1993) kun at være nedad halvkontinuert, hvis dissipativitet er vores ønske. På den anden side er vi nød til at beskæftige os med kontinuerte og lokal Lipschitz storage-funktioner for at opnå fornuftige udsagn om de positive grænsemængder af tilstandstrajektorier. Analysen begrænses til en åben delmængde af tilstandsrummet, som vi kalder den tilgængelige mængde.

For første gang i dissipativitetsanalysen defineres fire forskellige delmængder af tilstandsrummet ved hjælp af generaliserende gradienter af de før omtalte storage-funktioner; værdifulde informationer om (asymptotiske) stabilitets-forhold af uforstyrrede trajektorier kan udvindes af relationerne imellem disse mængder. Desuden kan brugen af strengt negative Hamilton-Jacobi-uligheder forårsage meget attraktive egenskaber, såsom radial ubegrænsethed og positiv definithed af storage-funktioner med hensyn til generelle, invariante mængder. Beviserne er inspireret af La Salle's invarians-princip; disse giver også brugbare performance envelopes.

Derefter bliver storage-funktionerne betragtet som Input-to-State-Stabilty (ISS) Lyapunov-funktioner, oprindelig defineret af Sontag og Lin (1990-1995). Denne indfaldsvinkel giver simple betingelser under hvilken glatte (uendelig mange gange differentiable) storage-funktioner eksisterer. Også robust stabilitet med hensyn til ikke-uddøende, men begrænsede forstyrrelser er en følge af ovennævnte betingelser. Forskellige klasser af dissipative systemer sammenlignes med hinanden, og der påpeges vigtigheden af eksistensen af glatte storage-funktioner i praktiske anvendelser, og i numeriske approksimationer.

I afsnittet efter den fjerde artikel kommenteres eksistensen af kontinuerte og lokal Lipschitz storage-funktioner. Problemet at syntetisere tilstands-tilbagekoblingsalgoritmer er løst ved at forbinde dissipativitet, stabilitetsforhold med hensyn til invariante mængder og tilstands-tilbagekoblingsalgoritmer baseret på differentiable spil med hinanden. Hamilton-

Jacobi-uligheden svarende til tilstands-tilbagekobling vises, og asymptotisk opførsel af regulerede systemer diskuteres. Desuden vises at den minimerende regulator $u_{\min}$ og den maksimerende forstyrrelse $w_{\max}$ er lige nul på positive grænsemængder af tilstandstrajektorier, et faktum som har været kendt i $\mathcal{H}_\infty$-regulering. Der vises hvordan resultaterne fra analysen af dissipative systemer oversættes til analoge resultater omhandlende glat og robust dissipativ tilstands-tilbagekobling.

Kapitel 6 fortæller kort hvilke algoritmer til numerisk løsning af Hamilton-Jacobi-uligheder er kendte og har været benyttet. Dette kapitel er inkluderet i afhandlingen for at give læseren, der skal implementere numeriske approksimationer, en hjælpende hånd. Her findes ingen original bidrag fra forfatteren.

Chapter 1

Introduction

The purpose of this chapter is to give a general overview over the kind of deterministic control problems addressed in this thesis. Secondly, we want to motivate the fundamental questions asked and investigated here, and to display some basic tools used to answer them.

1 The idea of feedback control

This thesis concerns control and regulation issues of physical systems or plants which can be successfully modelled by systems of n autonomous ordinary differential equations (ODE's) of the form

$$\dot{x} = X(x) \quad , \tag{1}$$

where the states $x(\cdot) : \mathbb{R} \mapsto \mathbb{R}^n$ represent the behavior of the system. It is assumed that the plant is constructed - or pre-compensated - such that the preferred modes of operation are represented by an equilibrium point, a periodic orbit, a union of such trajectories, or more generally, by some compact invariant set $\mathcal{S} \subset \mathbb{R}^n$ (a set is compact if it is closed and bounded, it is invariant if it consists of a union of trajectories defined on $\mathbb{R}$ of (1)).

In case that we can guarantee that the model represents the behavior of the physical plant exactly, and the plant is only started in initial points satisfying $x_0 \in \mathcal{S}$, then we know that the unique trajectory started in x_0 at time t_0, here denoted $x(\cdot, x_0, t_0)$ never leaves $\mathcal{S}$, and perfect behavior of the plant is obtained.

Unfortunately, real world systems do not behave in such an idealized manner. The following problems are most likely to occur:

stability: The initial point x_0 may not be inside $\mathcal{S}$, thus possibly leading to mostly undesired behavior like unboundedness of $x(\cdot, x_0, t_0)$, or instability of the set $\mathcal{S}$. Which stability criteria must be satisfied?

robustness: The model may be oversimplified, and the influence of unknown or unpredictable exterior inputs, here called disturbances, must be taken into account. A better model might be a system of perturbed ODE's of the form

$$\dot{x} = X(x, w) \quad , \tag{2}$$

where the influence of a disturbance signal $w(\cdot) : \mathbb{R} \mapsto \mathbb{R}^l$ is reasonably modelled. But do the desired stability properties still hold for $w(\cdot) \neq 0$?

performance: Other performance criteria like quality, cost or risk have to be improved. That is, in addition to the extended dynamics (2), we are interested to assure "nice" properties of some map $w(\cdot) \mapsto z(\cdot)$, where the performance signal $z(\cdot) : \mathbb{R} \mapsto \mathbb{R}^p$ is generated by the augmented system

$$\begin{aligned} \dot{x} &= X(x, w) \quad , \\ z &= Z(x, w) \quad . \end{aligned} \tag{3}$$

Which side performance criteria can be achieved without destroying stability?

Even if the plant represented in the model (1) has been designed with care to accomplish nice behavior on $\mathcal{S}$, it may be mandatory to modify it's dynamics by an appropriate choice of feedback devices to improve stability, robustness or other performance criteria. In the following it is assumed that the states $x(\cdot)$ are accessible. Therefore, after connecting some actuating devices like motors, valves, or any other devices which use an control input signal $u(\cdot) : \mathbb{R} \mapsto \mathbb{R}^m$ to alter the dynamics of (1), after considering the most significant disturbances, and after defining secondary performance criteria, we have a model of the form

$$\begin{aligned} \dot{x} &= X(x, u, w) \quad , \\ z &= Z(x, u, w) \quad . \end{aligned} \tag{4}$$

It is still assumed that the preferred behavior of our model is represented by a compact invariant set $\mathcal{S} \subset \mathbb{R}^n$ of the uncontrolled and undisturbed dynamic part of the system, that is, of the autonomous sub-system

$$\dot{x} = X(x, 0, 0) \quad . \tag{5}$$

Now we are faced with the fundamental question investigated in this thesis:

How to design a state feedback law $u = u(x)$ which guarantees sufficient stability, robustness, and performance?

Let us take a look on some known basic tools which have been used in the past to solve similar, but less complex problems individually, one by one.

2 Some basic tools

Consider the time-varying, n-dimensional system of ordinary differential equations (ODE's) of the form

$$\dot{x} = X(x, t) \quad , \tag{6}$$

where $X : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ is any sufficient smooth vectorfield. We assume that to each pair of initial conditions $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ there is a unique state trajectory $x(\cdot) : \mathcal{I} \subset \mathbb{R} \mapsto \mathbb{R}^n$ which satisfies $x(t_0) = x_0$. We denote the state by $x(\cdot) = x(\cdot, x_0, t_0)$ when emphasizing on the initial conditions. We notice that the dynamical part of any of the different models (1), (2), (3), or (4) with $u(\cdot) = u(x)$ and $w(\cdot) = w(t)$, can be represented in the form (6).

We assume that there is a compact set $\mathcal{S} \subset \mathbb{R}^n$ which represents the desired modes of operation of the system (6). These might be one or several equilibrium points at which the physical plant operates efficiently, or these may be closed, periodic orbits, or more general, a union of state trajectories with beneficial properties. Hence, if $x(\cdot) \in \mathcal{S}$, we are on the safe side. On the other hand, the initial point x_0 may be unproper set due to disturbances or inaccuracies in the model, it may be outside $\mathcal{S}$. The interesting question is then: what happens if initial conditions $x_0 \neq \mathcal{S}$ are considered?

It turns out that all kind of dramatic behavior might happen. We are urged to consider the continuity properties of $x(\cdot, x_0, t_0)$ with respect to x_0.

To do so, we need some preliminary definitions: Let $|p|_{\mathcal{S}}$ denote the distance between some closed set $\mathcal{S}$ and any point $p \in \mathbb{R}^n$, that is,

$$|p|_{\mathcal{S}} \equiv \min_{q \in \mathcal{S}} |p - q| \quad , \tag{7}$$

where $|\cdot|$ denotes the usual Euclidean vector norm on $\mathbb{R}^n$. We denote the non-negative reals by $\mathbb{R}^+$. A real valued function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ belongs to class $\mathcal{K}$ if it is continuous and strictly increasing, and satisfies $\alpha(0) = 0$. We say α belongs to class $\mathcal{K}_\infty$ if it in addition satisfies $\alpha(r) \to \infty$ as $r \to \infty$. Finally, a real valued function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is of class $\mathcal{KL}$ if it is continuous, and $\beta(r, s)$ is of class $\mathcal{K}$ for each fixed s, and for each fixed r is monotone (not necessarily strict monotone) decreasing to zero as $s \to \infty$. Now we are able to define the following cases of behavior:

The solution $x(\cdot, x_0, t_0)$ is called

bounded (in future) if there exists a constant $c(x_0, t_0)$ such that $|x(t, x_0, t_0)| \leq c(x_0, t_0)$ for all $t \geq t_0$, or equivalently, if there exists a compact set $\Omega(x_0, t_0) \subset \mathbb{R}^n$ such that $x(t, x_0, t_0) \in \Omega(x_0, t_0)$ for all $t \geq t_0$;

unbounded if not bounded.

The closed set $\mathcal{S} \subset \mathbb{R}^n$ is called

stable if for each $\varepsilon > 0$ there is a $\delta(\varepsilon, t_0) > 0$ such that $|x_0|_{\mathcal{S}} \leq \delta$ implies $|x(t, x_0, t_0)|_{\mathcal{S}} \leq \varepsilon$ for all $t \geq t_0$, or equivalently, if there exist a $\mathcal{K}$ function α_{t_0} such that

$$|x(t, x_0, t_0)|_{\mathcal{S}} \leq \alpha_{t_0}(|x_0|_{\mathcal{S}})$$

for all $t \geq t_0$ and any x_0;

attractive if there exists an $r(t_0) > 0$ and, for each $\varepsilon > 0$, a $T(\varepsilon, t_0) \geq t_0$ such that $|x_0|_{\mathcal{S}} \leq r$ implies $|x(t, x_0, t_0)|_{\mathcal{S}} \leq \varepsilon$ for all $t \geq T(\varepsilon, t_0)$ (we write shortly $x(\cdot) \to \mathcal{S}$);

asymptotically stable if it is stable and attractive with respect to $\mathcal{S}$, or equivalently, if there exist a $\mathcal{KL}$ function β_{t_0} such that

$$|x(t, x_0, t_0)|_{\mathcal{S}} \leq \beta_{t_0}(|x_0|_{\mathcal{S}}, t - t_0)$$

for all $t \geq t_0$ and any x_0; and finally,

unstable if it is not stable.

In case that the above properties are not dependent on the initial time t_0, that is, the constants can be chosen as $c(x_0)$, $\delta(\varepsilon)$, and r, or equivalently, the compact set and the comparison functions are given by $\Omega(x_0)$, α and β, independently of t_0, the properties are called **uniform**. This will often be the case in the following chapters.

Any attractive closed set $\mathcal{S}$, and in particular, any asymptotically stable closed set $\mathcal{S}$ has a **basin of attraction**, denoted $\boldsymbol{\mathcal{A}}^+$, that is, the set of initial points x_0 such that $x(\cdot, x_0, t_0) \to \mathcal{S}$. Whenever we refer to a stability property for which the basin of attraction is estimated by some compact set $\Omega \subset \mathcal{A}^+$, we call the stability property **regional**. In case that the stability property of concern holds on the entire space $\mathbb{R}^n$, it is called **global**, and in case that the basin of attraction is not estimated, we call the stability property **local**.

Which of these stability properties are essential in practical applications? More or less, all of them!

To be more precise, boundedness of all state trajectories $x(\cdot)$ can not be discarded, since real world systems only do sustain finite stress, and will certainly be destroyed if some unbounded trajectory $x(t, x_0, t_0) \to \infty$ for $t \to \infty$ exists. Hence, boundedness of all state trajectories inside some compact set $\Omega(x_0, t_0) \subset \mathbb{R}^n$ has our primary interest, and the compact set Ω has often to be independent of the initial conditions x_0 and t_0. Of course, we must then be able to guarantee that the initial point satisfies $x_0 \in \Omega$. Often the strength of the physical system at hand - the maximal allowable forces, temperatures, or pressures which can be sustained within some safety margin - define a **performance envelope**, that is, a compact set in state space which has to be positive invariant (we call a compact set $\Omega \subset \mathbb{R}^n$ positive invariant if for each initial point $x_0 \in \Omega$ there holds $x(t, x_0, t_0) \in \Omega$ for all $t \geq t_0$).

Moreover, there are some preferred modes of operation, which are represented by the compact set $\mathcal{S}$. There may be many objectives such as quality of production, fuel consumption, smoothness of a ride in vehicles, generation of preferred signals, or any other preferential qualities which can be represented by the compact set $\mathcal{S}$. Common for all of them is that we do not want due to reasons of quality, cost or safety, that state trajectories with initial point inside $\mathcal{S}$, or arbitrarily close to $\mathcal{S}$, drift around such that, after some time, $x(t)$ can be found any place in Ω. It follows that stability of the set $\mathcal{S}$ must be our next target.

Still, stability alone opens the possibility that the system is performing for all times in a suboptimal manner. But, in many applications where the difference between suboptimal behavior and optimal behavior causes loss of resources, we are urged to ensure the attraction of the set $\mathcal{S}$, and hence, asymptotical stability is our goal.

To be of practical interest, stability conditions must not require that we explicitly solve (6) for all possible initial data x_0 and t_0. Fortunately, there exists a vast variation of Lyapunov function techniques to establish those stability conditions, see for example [SL61, Yos66, Kha96] for details. In the following subsection we recall four different Lyapunov-like theorems and a game-theoretic idea, which had great impact on the course of the investigations presented in this thesis.

2.1 Boundedness and attraction

For the purpose of this thesis it is beneficial to recall boundedness and attraction properties of smooth **autonomous systems** of the form

$$\dot{x} = X(x) \quad . \tag{8}$$

A set $\mathcal{S} \subset \mathbb{R}^m$ is called **invariant** if all trajectories starting in $\mathcal{S}$ are defined on $\mathbb{R}$, and evolve entirely inside $\mathcal{S}$ in past and future, or equivalently, if it consists of a union of trajectories defined on $\mathbb{R}$. A set Ω is called **positive invariant** if all trajectories of (8) with initial point $x_0 \in \Omega$ are never leaving Ω for all $t \geq t_0$.

We have the following theorem on autonomous systems and invariant sets, which has been published in the early sixties by La Salle and Lefschetz [SL61]. This theorem is often referred to as "the La Salle's invariance principle".

2.1 Theorem (La Salle and Lefschetz) *Let $V : \mathbb{R}^m \mapsto \mathbb{R}$ be a C^1 function, and let Ω denote a component of the preimage $V^{-1}(]-\infty, c])$ for some $c > 0$. Assume that Ω is connected and bounded, and that*

$$\tfrac{d}{dt}V = \tfrac{\partial V}{\partial x}X(x) \leq 0 \tag{9}$$

within Ω along any trajectory of the autonomous system (8). Let $\mathcal{V} \subset \Omega$ be the largest set where $\frac{d}{dt}V = 0$, and let $\mathcal{S}$ be the largest invariant set contained in $\mathcal{V}$.

Then Ω is positive invariant, and every solution in Ω converges to $\mathcal{S}$ as $t \to \infty$.

In other words: Ω is an estimate of the basin of attraction for the attractive invariant set $\mathcal{S}$, and the theorem assures boundedness of state trajectories inside Ω and regional attraction of the invariant set $\mathcal{S}$.

2.2 Robustness and stability

Assume we have a time dependent system of ODE's - here called a **disturbed**, or **perturbed system** - of the form

$$\dot{x} = X(x, w) \quad , \tag{10}$$

where $x(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is called the state, $w(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^l$ the exogenous input, also called disturbance, which is assumed to be an measurable signal taking values in a compact set $\mathcal{W} \in \mathbb{R}^l$, shortly $w(\cdot) \in \mathcal{M}_{\mathcal{W}}$. The symbol $X(x, w)$ denotes a continuous vector field $\mathbb{R}^n \times \mathcal{W} \mapsto \mathbb{R}^n$ which is locally Lipschitz in x, uniformly on $\mathcal{W}$.

We state the following converse robust Lyapunov theorem which is a slightly simplified version of a theorem by Lin, Sontag, and Wang. The proof is omitted here, it can be found in [LSW96].

We assume that there exists some compact set $\mathcal{A} \subset \mathbb{R}^n$ which is positive $\mathcal{W}$-invariant (that is for all initial conditions $x_0 \in \mathcal{A}$ and all $w(\cdot) \in \mathcal{M}_{\mathcal{W}}$ the resulting unique trajectory $x(\cdot)$ is defined in the future and $x(t, x_0, w(\cdot)) \in \mathcal{A}$ for all $t \geq 0$).

The system (10) is called **robust global asymptotically stable** (RGAS) with respect to a compact positive $\mathcal{W}$-invariant set $\mathcal{A} \subset \mathbb{R}^n$ and a compact value set $\mathcal{W} \subset \mathbb{R}^l$ if there exist a $\mathcal{KL}$ function β such that

$$|x(t, x_0, w(\cdot))|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t) \tag{11}$$

for all $t \geq 0$, any x_0, and any $w(\cdot) \in \mathcal{M}_{\mathcal{W}}$.

A **robust Lyapunov function** for the system (10) with respect to the compact positive $\mathcal{W}$-invariant set $\mathcal{A} \subset \mathbb{R}^n$ is a function $V : \mathbb{R}^n \mapsto \mathbb{R}$ which is smooth (C^∞) on $\mathbb{R}^n \backslash \mathcal{A}$, and which is such that there exists two $\mathcal{K}_\infty$ functions α_1, α_2 satisfying

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad ,$$

and a $\mathcal{K}$ function α_3 such that for any $x \in \mathbb{R}^n \backslash \mathcal{A}$ and any $w(\cdot) \in \mathcal{M}_{\mathcal{W}}$

$$\tfrac{d}{dt} V = \tfrac{\partial V}{\partial x}(x) X(x, w) \leq -\alpha_3(|x|_{\mathcal{A}}) \quad \text{on } \mathbb{R}^n \backslash \mathcal{A} \quad . \tag{12}$$

The robust Lyapunov function is called **smooth** if it is smooth (infinitely differentiable) on $\mathbb{R}^n$.

It follows from the definition that a robust Lyapunov function is continuous on $\mathbb{R}^n$, and that $V : \mathbb{R}^n \overset{\text{onto}}{\mapsto} \mathbb{R}^+$. Moreover, $x \in \mathcal{A} \Longleftrightarrow V(x) = 0$.

2.2 Theorem (Smooth converse robust Lyapunov theorem) *[LSW96] Assume that the system (10) has a nonempty, compact, and positive $\mathcal{W}$-invariant set $\mathcal{A} \subset \mathbb{R}^n$. Then the system is RGAS with respect to $\mathcal{A}$ if and only if there exists a smooth robust Lyapunov function with respect to $\mathcal{A}$.*

It has been showed in [LSW96] by a counterexample that the assumption of compactness of the disturbance value set $\mathcal{W}$ is crucial. Also, the existence of a robust Lyapunov function for the system (10) with respect to $\mathcal{A}$ implies the existence of a smooth robust Lyapunov function.

Notice also that we use the expressions positive invariant and robust global asymptotic stable where [LSW96] uses invariant and uniform global asymptotic stable.

2.3 Performance criteria

The additional performance criteria, which we will allow for **uncontrolled systems** of the form

$$\begin{aligned} \dot{x} &= X(x, w) \\ z &= Z(x, w) \quad , \end{aligned} \tag{13}$$

are for example **passivity**, that is, the requirement

$$-\int_0^T w(t)^T z(t)\; dt \leq K(x_0)$$

holds along the trajectories of (13) for all $T > 0$. Another criteria might be an $\mathcal{L}_2$ **gain** less than or equal $\gamma > 0$, that is, any trajectory of (13) is such that

$$\int_0^T |z(t)|^2\; dt \leq \gamma^2 \int_0^T |w(t)|^2\; dt + K(x_0)$$

is satisfied for all $T > 0$. In both cases K is a constant only depending on the initial point.

More generally, the theory of dissipative dynamical systems, which has been developed by Jan C. Willems [Wil72a] (see also [HM80b]), gives the generalization of additional performance criteria which we want to impose besides stability: Assume that an uncontrolled system is given together with a real valued function $s : \mathbb{R}^l \times \mathbb{R}^p \mapsto \mathbb{R}$, called the **supply rate**. Then, the uncontrolled system is called **dissipative** if there exists a nonnegative locally bounded function $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ satisfying $\min_{x\in\mathbb{R}^n} V(x) = 0$, called the **storage function**, such that

$$V(x_T) - V(x_0) \leq \int_0^T s(w(t), z(t))\, dt \tag{14}$$

for all initial points x_0, exogenous inputs $w(\cdot)$, and times $T \geq 0$, where the final point x_T is $x_T = x(T, 0, x_0, w(\cdot))$. The above inequality is called **dissipation inequality**.

It has been shown in [Wil72a] that passivity is equivalent to dissipation with respect to the supply rate $s(w, z) \equiv w^T z$, and the $\mathcal{L}_2$ gain condition is given by dissipation with respect to the supply rate $s(w, z) = \gamma^2|w|^2 - |z|^2$.

Associated to those performance criteria is a concept of generalized total energy stored inside the system, which is expressed in differential form by the **pre-Hamiltonian** function

$$\mathsf{H}(x, p, w) \equiv pX(x, w) - s(w, z) \quad . \tag{15}$$

In case that there exists a maximizing exogenous input for the pre-Hamiltonian (15)

$$w_{\max}(x, p) \equiv \arg\ \max_w \mathsf{H}(x, p, w) \quad , \tag{16}$$

we define the **Hamiltonian** function

$$\mathsf{H}^*(x, p) \equiv \mathsf{H}(x, p, w_{\max}) = pX(x, w_{\max}(x, p)) - s(w_{\max}(x), Z(x, w_{\max}(x, p))) \quad . \tag{17}$$

Then we have the following relation between a partial differential inequality - commonly called **Hamilton-Jacobi inequality** - and the associated performance criteria:

2.3 Theorem *Assume that there is a nonnegative C^1 solution $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ to the Hamilton-Jacobi inequality*

$$\mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \leq 0 \tag{18}$$

such that $\min_{x\in\mathbb{R}^n} V(x) = 0$, then the dissipation inequality (14) holds, and V is a storage function.

Proof: The Hamilton-Jacobi inequality (18) causes $\mathsf{H}(x, \frac{\partial V}{\partial x}, w) \leq 0$ to hold for all $w(\cdot)$, and a simple integration with boundary condition $\min_{x \in \mathbb{R}^n} V(x) = 0$ implies that the dissipation inequality (14) holds. $\square$

2.4 Control Lyaponov functions

This thesis is about control design, our goal is to construct state feedback laws such that - at least - the required stability properties hold. Therefore, we have to consider the extension of the Lyapunov function tools to a concept called control Lyapunov function. We define that $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ is positive definite with respect to some compact set $\mathcal{S} \subset \mathbb{R}^n$ if $V(x) = 0$ for all $x \in \mathcal{S}$ and $V(x) > 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{S}$, or equivalently, if there exists two functions $\underline{\alpha}_V$ and $\overline{\alpha}_V$ of class $\mathcal{K}$ such that

$$\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}}) \quad .$$

We say that $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ is proper if $V(x) \to \infty$ for $|x| \to \infty$, or equivalently, if $\underline{\alpha}_V$ and $\overline{\alpha}_V$ are of class $\mathcal{K}_\infty$.

Let us assume that we are given the **controlled**, but **undisturbed system**

$$\dot{x} = X(x, u) \quad , \tag{19}$$

where $u(\cdot) : \mathbb{R} \mapsto \mathbb{R}^m$ represents the control strategy (control signal). We want to construct a state feedback control law $u(\cdot) = u(x)$ such that some compact set $\mathcal{S}$ which is invariant to $\dot{x} = X(x, 0)$ is asymptotically stabilized, that is, is an asymptotically stable set of the **controlled**, or **closed loop dynamics**

$$\dot{x} = X(x, u(x)) \quad . \tag{20}$$

We can pick an candidate Lyaponov function and require that its derivative along the solutions of (20) is decreasing. Therefore, we say that a continuous, positive definite and radially unbounded function $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ is a **control Lyapunov function** if in addition

$$\inf_{q \in \mathbb{R}^m} \frac{\partial V}{\partial x} X(x, q) < 0 \quad \text{for all} \quad x \notin \mathcal{S} \quad . \tag{21}$$

The results of Artstein, Lin and Sontag show the following conclusion (see [Art83, Son89c, Son90, LS95]).

2.4 Theorem *The existence of a smooth (C^∞) control Lyapunov function implies the existence of a globally stabilizing state feedback $u = u(x)$ which is smooth on $\mathbb{R}^n \backslash \mathcal{S}$ and continuous on $\mathbb{R}^n$.*

2.5 Differential games

As we have seen in the preceding subsections, we can regard the control $u(\cdot)$ as the signal which has to assure that all the desired and beneficial properties hold, whereas the disturbance $w(\cdot)$ acts to worsen things, maybe to destroy stability properties. It must therefore be our goal to find a balance between the influence of the disturbance and the control, and the theory of differential games is a useful approach to this problem. We consider a **controlled and disturbed system** of n ODE's of the form

$$\dot{x} = X(x, u, w) \tag{22}$$

with initial condition x_0 and **payoff**

$$P(x_0, u(\cdot), w(\cdot)) = g(x_T) + \int_0^T h(x, u(t), y(t))\, dt \quad , \tag{23}$$

where T is the terminal time of the play, and $u(\cdot) = u(t)$ is the minimizing player strategy, whereas $w(\cdot) = w(t)$ is the maximizing player strategy. We assume in the following that the vector field X is "sufficient nice", that is, locally Lipschitz and bounded from above (see A.Friedmann [Fri71] for details), and that g and h are continuous and locally Lipschitz functions.

Suppose that there exists strategies $u_{\min}(\cdot)$ and $w_{\max}(\cdot)$ such that the saddle point property

$$P(x_0, u_{\min}(\cdot), w(\cdot)) \leq P(x_0, u_{\min}(\cdot), w_{\max}(\cdot)) \leq P(x_0, u(\cdot), w_{\max}(\cdot)) \tag{24}$$

holds for all other strategies $u(\cdot)$, $w(\cdot)$, than we say that the play given by (22), (23), and (24), has the value $V(T, x_0) = P(u_{\min}, w_{\max})$. Obviously, if each player assumes that the other acts sensible and does not make foolish decisions, the best result obtainable is the value of the game, and the best possible strategy is to use $u_{\min}$, or $w_{\max}$.

It can be showed that the value V can be presumed to be continuous and locally Lipschitz on $[0, T] \times \mathbb{R}^n$, and satisfies $V(T, x) = g(x)$. Isaacs had heuristically derived an equation [Isa51b], now called the Isaacs equation (see also [Isa51a]):

$$\tfrac{\partial}{\partial t} V(t, x) + \max_w \min_u \left\{ \tfrac{\partial V}{\partial x}(t, x) X(x, u, w) + h(x, u, w) \right\} = 0 \quad . \tag{25}$$

This equation can be interpreted as the PDE formulation of the differential game:

2.5 Theorem *[Fri71] Denote by $V(t, x)$ the value of the differential game (22), (23), and (24), then $V(T, x) = g(x)$, and V satisfies the Isaacs equation (25) almost everywhere.*

3 Combining these tools

Given a controlled and disturbed system of the type (4) where it is assumed that the preferred behavior of our model is a compact invariant set $\mathcal{S} \subset \mathbb{R}^n$ of the uncontrolled and undisturbed dynamics (5), it is the believe of the author that the choice of control strategy shall be made to accomplish the following five fundamental tasks:

asymptotically stabilization of mode of operation,

estimation of basin of attraction,

improve other dissipation performance criteria, achieve robustness with respect to disturbances,

and finally, determine the optimal control strategy.

As we have seen in the preceding section, these problems have been solved partially - one by one - in the past. Common to all these approaches is that a Lyapunov-like auxiliary function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is related to a partial differential inequality, or equality. These PDI's or PDE's are expressing sufficient conditions in theorems 2.1, 2.3, and 2.4; necessary conditions in theorem 2.5, and both sufficient and necessary conditions in theorem 2.2. Moreover, we see that the use of the time derivative $\frac{d}{dt}V = \frac{\partial V}{\partial x}X$ plays an important role in all these formulations, and the auxiliary functions have similar interpretations as abstract energy concept. It seems therefore reasonable to assume that these different formulations can be unified in one formula.

Indeed, in the extensive literature on dissipative control, and $\mathcal{H}_\infty$ control in particular, some of these concepts have been gathered together in one formulation: in the context of $\mathcal{H}_\infty$ control by state feedback for example, the theory of differential games has been used to identify the optimal control needed to render the controlled system dissipative with respect to the supply rate $\gamma^2|w|^2 - |z|^2$, but stability investigations have so far been restricted to local stability issues of the undisturbed system $\dot{x} = X(x, u, 0)$ with respect to the equilibrium point $x = 0$. Neither stability issues of disturbed systems or stability with respect to general invariant sets have attracted interest before.

It is the aim of this thesis to show the progress made during the process of solving some of the fundamental puzzles and riddles: Given a **feedback system** of the form

$$\begin{aligned} \dot{x} &= X(x, u, w) \\ z &= Z(x, u, w) \quad , \end{aligned} \tag{26}$$

does there exists a smooth feedback law $u = u(x)$ which can be found from a saddlepoint property of some pre-Hamiltonian, and some smooth storage function?

Can the properties of the Lyapunov function, the robust Lyapunov function, the control Lyapunov function, the storage function, and the game value function be unified in one and the same function V?

Will such a unified V be smooth?

Is there a partial differential inequality involving the supply rate and the time derivative $\frac{d}{dt}V = \frac{\partial V}{\partial x}X$ which constitutes an equivalent formulation of the problem?

How is V related to a solution to this partial differential inequality?

Given such a solution, what can we say about the boundedness of state trajectories, the invariantness of certain subsets $\mathcal{S} \subset \mathbb{R}^n$, robust stability under the influence of disturbances, dissipativety, and the construction of a (smooth?) control law?

These general problems have not been entirely solved yet. It is the aim of this thesis to describe some major milestones found on the road to definite answers.

Chapter 2

Regional Nonlinear $\mathcal{H}_\infty$ State Feedback Control

Linear asymptotically stable systems of ODE's of the form

$$\dot{x} = Ax$$

are exponentially stable, and therefore local asymptotically stability implies global asymptotically stability. Hence, in a linear context, local control implies global control.

This property does not hold for nonlinear autonomous systems of the form

$$\dot{x} = X(x) \quad ,$$

and most stability results obtained in nonlinear control so far are local only. Especially in the field of $\mathcal{H}_\infty$ control the asymptotic stability of equilibrium points has been confined to a neighborhood of the critical point of question, which may be arbitrarily small if the system is ill-constructed. Moreover, no estimates of the size of the basin of attraction of asymptotically stable equilibrium points have yet been provided.

From an engineering point of view this is a very unsatisfactory state of art, since designers of real world control systems have to be sure that the designed regulator works as intended on some given compact region. It is therefore mandatory to consider control on compact sets here called **performance envelopes** (in the paper these are called valid region), and to confine the closed loop state trajectories to these compact sets, a requirement which we denote **regional control**. Finally, all trajectories which are confined to some given performance envelope, and are generated by zero disturbances, are shown to converge to the equilibrium point of concern for $t \to \infty$.

In this chapter the well known game theoretic approach to $\mathcal{H}_\infty$ theory is extended with the use of the storage function as a Lyapunov function to prove a result similar to the La Salle's invariance principle described in the preceding chapter.

To solve the state-feedback Hamilton-Jacobi inequality we show how Luke's approximation scheme can be implemented in MAPLE. A couple of simple examples are provided to show that nonlinear state feedback control is an interesting alternative to linear control techniques, even in the case that the linear control problem can be solved locally. The use of nonlinear, polynomial control strategies is shown to improve robustness against disturbances, and to improve the size and shape of the performance envelope (valid region).

The main ideas of this chapter have been published in

> Marc Cromme, Jens Møller-Pedersen, and Martin Pagh Petersen. Nonlinear $\mathcal{H}_\infty$ state feedback controllers: Computation of valid region. In *Proceedings of the 1997 American Control Conference*, Albuquerque, New Mexico, June 1997.

The following paper is included here exactly as it was printed, except for a changed graphical layout.

Nonlinear $\mathcal{H}_\infty$ state feedback controllers: computation of valid region

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regional (semi-global) stabilizing nonlinear $\mathcal{H}_\infty$ control, Luke's approximation scheme, implementation in MAPLE

Abstract

"From a general point of view the state feedback $\mathcal{H}_\infty$ suboptimal control problem is reasonable well-understood. Important problems remain with regard to a priori information of the size of the neighborhood where the local state feedback $\mathcal{H}_\infty$ problem is solvable, and with regard to the nature of solutions V to the Hamilton-Jacobi inequalities ... such as properness of V as a candidate Lyapunov function" Citation van der Schaft, 1992 [vdS92a].

The first of these problems is solved regionally (semi-globally) in this paper, and the obtained control laws are implemented in `MAPLE`.

1 Introduction

The local nonlinear state feedback $\mathcal{H}_\infty$ control problem has been solved in the early nineties. Van der Schaft [vdS92a] describes the solution process for affine systems using the theory of dissipative systems first introduced by Willems [Wil72a]. Isidori [Isi92], and Isidori and Astolfi [IA92b], [IA92a] approach the problem by the theory of differential games. The problem has been solved for more general nonlinear plants by Isidori and Kang [IK95], and Ball, Helton, and Walker [BHW93]. Lukes approximation scheme [Luk69] has been used to compute solutions of the Hamilton-Jacobi inequalities of the local nonlinear state feedback $\mathcal{H}_\infty$ control problem [IK95], [MP95].

From an engineering point of view all these results are insufficient: local control laws are found without any knowledge of boundedness of the closed loop state trajectories, or the size and shape of the neighborhood where state feedback control works as intended.

State boundedness is an indispensable property of controlled systems for two reasons: state blow-up leads usually to plant or control system failure or damaging, and local control is certainly not applicable when the bound on state trajectories is not known a priory.

The size and shape of the neighborhood where the implemented control law is meaningful are important design parameters for any practical control purpose. If they do not cover the intended performance envelope of the plant, another control strategy must be chosen.

Finally, any practical oriented control law should allow for inaccurate setting of initial conditions. This kind of initial value robustness may be very important when calculating the secure performance envelope of a closed loop control system.

In the following, given a particular solution V of the Hamilton-Jacobi inequality of concern, any compact neighborhood $\Omega \subset \mathbb{R}^m$ of the equilibrium point which has these beneficial properties is called a **valid region**. We assume without loss of generality that the equilibrium point of concern is at the origin. Moreover, a class of disturbances such that some given Ω is a valid region, is called **valid disturbance set**, and it is denoted $\mathcal{W}^\epsilon$. A set of initial conditions, denoted $\Omega^\epsilon \subset \Omega$, is called **valid initial set** if all state trajectories driven by valid disturbances renders Ω a valid region. The problem addressed here is:

1.1 Problem Formulation *Given a nonlinear state feedback $\mathcal{H}_\infty$ control problem and a formal solution V to the associated Hamilton-Jacobi inequality, find a valid region Ω, a valid disturbance set $\mathcal{W}^\epsilon$, and a valid initial set $\Omega^\epsilon \subset \Omega$ such that every state trajectory $x(\cdot)$ with initial condition $x_0 \in \Omega^\epsilon$ subject to disturbances $w(\cdot) \in \mathcal{W}^\epsilon$ satisfies an L_2 gain less than or equal γ, and approaches the origin as time goes to infinity.*

2 Local $\mathcal{H}_\infty$ state feedback control

Let $\mathbb{R}^+$ denote the real positive closed time axis $[0, \infty[$. In general we consider the plant

$$\dot{x} = X(x, u, w) \quad , \quad z = Z(x, u) \tag{1}$$

where $x(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^m$ is called the state, $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^l$ the input, $w(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^n$ the exogenous input, also called disturbance, and $z(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$ the performance, or to be controlled signal. The symbol $X(x, u, w)$ denotes a smooth vector field on $\mathbb{R}^m$, and the vector valued function $Z(x, u)$ specifies smoothly the performance measure. We assume the equilibrium conditions $X(0, 0, 0) = 0$ and $Z(0, 0) = 0$ to hold.

The open loop system $\dot{x} = X^{\text{open}}(x, w) \equiv X(x, 0, w)$ subject $w(\cdot) = 0$ is autonomous, and by assumption the origin is an an equilibrium point, hence an invariant set. The static feedback control is given by some vector valued function $a : \mathbb{R}^m \mapsto \mathbb{R}^l$, $u = a(x)$, where $a(0) = 0$ is assumed in order to preserve the equilibrium point zero. The closed loop system is given by the equations

$$\dot{x} = X(x, a(x), w) \quad , \quad z = Z(x, a(x)) \quad . \tag{2}$$

Whenever convenient, we use the notation $x(\cdot)$ for the signal $x(\cdot, t_0, x_0, u(\cdot), w(\cdot))$. It is assumed that all signals are L_2^{loc}, and that the state exist uniquely for all input, and is a C^1 signal.

Let $|\cdot|$ denote the usual Euclidean norm on the Banach space $\mathbb{R}^p$, then the L_2 norm of any locally square integrable signal $y(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$, $p \in \mathbb{N}$ is for all $T \in \mathbb{R}^+$ defined

$$\|y\|_T^2 \equiv \int_0^T |y(t)|^2 \, dt \quad , \tag{3}$$

and the closed loop system (2) has by definition local L_2 gain less than or equal $\gamma \geq 0$ if

$$\|z\|_T^2 \leq \gamma^2 \|w\|_T^2 + V_\mathrm{a}(x_0) \tag{4}$$

for all $w(\cdot), z(\cdot) \in L_2^{\mathrm{loc}}$, all $T \in \mathbb{R}^+$ and all initial conditions $x_0 \in \mathbb{R}^m$ such that the state trajectories never leave Ω. Here the available storage $V_\mathrm{a} : \mathbb{R}^m \mapsto [0, \infty[$ is a nonnegative and bounded function with minimum $V_\mathrm{a}(0) = 0$ at the origin [vdS92a], [IA92a].

It is known [vdS92a] that the local L_2 gain condition is implied by the existence of a non-negative, bounded C^1 storage function $V : \Omega \mapsto [0, \infty[$ satisfying the closed loop differential dissipation inequality

$$\begin{aligned} \mathsf{H}(u, w) &\equiv \tfrac{d}{dt}V - (\gamma^2 |w|^2 - |z|^2) \\ &= \tfrac{\partial V}{\partial x} X(x, u, w) - \gamma^2 |w|^2 + |Z(x, u)|^2 \\ &\leq 0 \end{aligned} \tag{5}$$

for all $t \in \mathbb{R}^+$, where the pre-Hamiltonian function H is defined by equation (5). Assuming that $Z(x, u)$ is such that $\frac{\partial Z}{\partial u}(0, 0)$ has rank l, it is known [IK95] that H has a unique saddle point $(u_\mathrm{min}, w_\mathrm{max})$ for all x and all $\frac{\partial V}{\partial x}$ near zero, and the extremal functions $u_\mathrm{min}(x, \frac{\partial V}{\partial x})$, $w_\mathrm{max}(x, \frac{\partial V}{\partial x})$ are characterized by the equations

$$\begin{aligned} &\tfrac{\partial \mathsf{H}}{\partial u}(u_\mathrm{min}, w_\mathrm{max}) = 0 \quad , \quad u_\mathrm{min}(0, 0) = 0 \\ &\tfrac{\partial \mathsf{H}}{\partial w}(u_\mathrm{min}, w_\mathrm{max}) = 0 \quad , \quad w_\mathrm{max}(0, 0) = 0 \quad . \end{aligned} \tag{6}$$

We deduce by the saddle point property (6) that u_min and w_max are the best possible state feedback law and the worst possible disturbance respectively.

Hence, we seek for a sufficient small $\gamma \geq 0$, and a C^1 storage function V defined on a sufficient big neighborhood Ω around the origin satisfying the Hamilton-Jacobi inequality [IK95]

$$\begin{aligned} &\mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) \\ &\quad = \tfrac{\partial V}{\partial x} X(x, u_\mathrm{min}(x, \tfrac{\partial V}{\partial x}), w_\mathrm{max}(x, \tfrac{\partial V}{\partial x})) - \gamma^2 |w_\mathrm{max}(x, \tfrac{\partial V}{\partial x})|^2 + |Z(x, u_\mathrm{min}(x, \tfrac{\partial V}{\partial x}))|^2 \\ &\quad \leq 0 \quad \text{for all } x \in \Omega \quad . \end{aligned} \tag{7}$$

This problem can be solved locally by an polynomial expansion algorithm due to Lukes [Luk69].

In the case of input affine systems

$$\begin{aligned} \dot{x} &= A(x) + B_u(x)u + B_w(x)w \\ z &= C(x) + D(x)u \quad , \end{aligned} \tag{8}$$

(here we have $A(0) = 0$ and $C(0) = 0$) satisfying $D^T D > 0$ for all x, the worst case disturbance and the minimizing input can be found completing the squares, they are for all x and $\frac{\partial V}{\partial x}$ given by the expressions [Isi92]

$$\begin{aligned} u_{\min} &= -(D^T D)^{-1}(\tfrac{1}{2}\, B_u^T \tfrac{\partial V}{\partial x}^T + D^T C) \quad , \\ w_{\max} &= \tfrac{1}{2\gamma^2}\, B_w^T \tfrac{\partial V}{\partial x}^T \quad . \end{aligned} \tag{9}$$

Here we seek for a sufficient small $\gamma \geq 0$, and a C^1 storage function V defined on a sufficient big neighborhood Ω around the origin satisfying the Hamilton-Jacobi inequality [Isi92]

$$\begin{aligned} \mathsf{H}^{**}(\tfrac{\partial V}{\partial x}, x) &= \tfrac{\partial V}{\partial x} Q(x) \tfrac{\partial V}{\partial x}^T + \tfrac{\partial V}{\partial x} L(x) + K(x) \\ &\leq 0 \quad \text{for all } x \in \Omega \quad , \end{aligned} \tag{10}$$

where the quadratic term $Q(x)$, the linear term $L(x)$, and the constant term $K(x)$ are defined

$$\begin{aligned} Q(x) &\equiv \tfrac{1}{4\gamma^2} B_w B_w^T - \tfrac{1}{4} B_u (D^T D)^{-1} B_u^T \\ L(x) &\equiv A - B_u (D^T D)^{-1} D^T C \\ K(x) &\equiv C^T (I - D(D^T D)^{-1} D^T)\, C \quad . \end{aligned} \tag{11}$$

The existence of a C^1 storage function satisfying (7) or (10) locally guarantees that the input-output map of the closed loop system has L_2 gain less than or equal γ as defined in equation (4) *if and only if every closed loop trajectory is bounded inside* Ω. Unfortunately, we do not have any priory estimates on the boundedness or on the asymptotic behavior of the state.

We will impose in the following a detectability assumption on the system: The control system (1) is **zero-detectable** if all bounded trajectories $x(\cdot)$ subject $u(\cdot) = 0$, $w(\cdot) = 0$ generating the zero-output $z(\cdot) = 0$ are approaching the origin as $t \to \infty$.

3 Regional $\mathcal{H}_\infty$ control

A set $\mathcal{M} \subset \mathbb{R}^m$ is **positive invariant** with respect to the autonomous system (2) if all trajectories starting in $\mathcal{M}$ are defined in the future and never leave $\mathcal{M}$ as time increases. It is **invariant** if $x(\cdot)$ defined in future and past, and evolves entirely in $\mathcal{M}$.

It is our purpose to use a formal solution to the Hamilton-Jacobi inequality as a Lyapunov function in order to establish regional stability properties of the $\mathcal{H}_\infty$ state feedback problem. Our new theorem is inspired on the proof of the well known La Salle's invariance principle for autonomous systems [SL61], which is connecting the existence of a C^1 Lyapunov function $V : \mathbb{R}^m \mapsto \mathbb{R}$ with bounded and connected pre-image $\Omega \equiv V^{-1}(]-\infty, c])$ satisfying $\frac{\partial}{\partial t} V \leq 0$, with the positive invariance of Ω and the asymptotic stability property of largest invariant set contained in the set where $\frac{\partial}{\partial t} V = 0$.

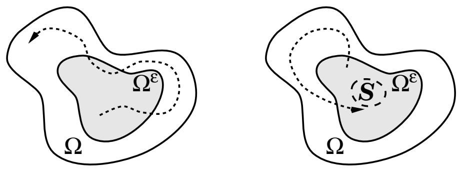

Figure 1: Boundedness of state, asymptotic stability

In order to prove the boundedness of state trajectories we have to restrict ourselves to the class of disturbances

$$\mathcal{W}^\epsilon \equiv \left\{ w(\cdot) \in L_2(\mathbb{R}^+) \;\middle|\; \gamma^2 \left\|w\right\|_2^2 \leq \epsilon \right\} \quad . \tag{12}$$

Given some solution V to the standard $\mathcal{H}_\infty$ Hamilton-Jacobi inequality, the following new lemma will help us to construct the valid sets (left part of figure 1). The lemma is proved in [CMPP96].

3.1 Lemma *Given a formal C^1 solution V of the Hamilton-Jacobi inequality (7) or (10), assume that some component of $V^{-1}(]-\infty, c_\Omega])$, $c_\Omega \in \mathbb{R}$, denoted Ω, is connected and bounded.*

Then Ω is compact and closed loop positive invariant by use of the state feedback law $a(x) = u_{min}(x)$ subject to the condition $w(\cdot) = 0$.

Pick some $\epsilon < c_\Omega$, then the appropriate subset $\Omega^\epsilon \subset \Omega$ of $V^{-1}([-\infty, c_\Omega - \epsilon])$ is such that any closed loop trajectory $x(\cdot)$ with initial condition $x_0 \in \Omega^\epsilon$ is bounded inside Ω if driven by the state feedback law $a(x) = u_{min}(x)$, and by any disturbance $w(\cdot) \in \mathcal{W}^\epsilon$.

Following the principal idea of the paper [IK95] as outlined in section 2, we conclude that any C^1 function V satisfying the Hamilton-Jacobi inequality (7) will also satisfy the L_2 gain (4) in case that the state is bounded inside Ω. We take advantage of lemma 3.1 to state the following theorem, and to follow the above explained ideas to proof it. The proof is found in appendix A.

3.2 Theorem *Assume that some C^1 solution $V : \Omega \mapsto \mathbb{R}$ of the Hamilton-Jacobi inequality (7) is defined on a bounded and connected component Ω of $V^{-1}(]-\infty, c_\Omega])$, $c_\Omega \in \mathbb{R}$. Assume furthermore that $\frac{\partial Z}{\partial u}(0,0)$ has rank l*[1]*.*

Then all closed loop trajectories $x(\cdot)$ subject $a(x) = u_{min}(x)$ with initial condition $x_0 \in \Omega^\epsilon$ do not leave Ω if driven by some $w(\cdot) \in \mathcal{W}^\epsilon$, and the system has L_2 gain less than or equal to γ.

Moreover, all such $x(\cdot)$ generated by $w(\cdot) \in \mathcal{W}^\epsilon$ which are identically zero for all times $t > T$, $T \in \mathbb{R}$, approach the biggest closed loop invariant set $\mathcal{M}$ contained in the null set

$$\mathcal{N} \equiv \left\{ x \in \Omega \;\middle|\; \mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) = 0 \right\} \quad .$$

[1]To be more precise, the rank condition is necessary, but not sufficient. We must also assume that the saddlepoint property (6) holds for all $x \in \Omega$, which implies that $\mathsf{H}(u_{\min}, w) \leq \mathsf{H}^{**} \leq \mathsf{H}(u, w_{\max})$ is satisfied on Ω for all u and all w.

Assume furthermore that the control system (1) is zero-detectable, then $x(\cdot)$ approaches the origin as $t \to \infty$.

In other words, given a formal solution V to the associated Hamilton-Jacobi inequality, the valid region Ω is given by some bounded (and connected) component of the pre-image $V^{-1}(]-\infty, c)$, the valid disturbance set $\mathcal{W}^\epsilon$ is the set of signals which L_2 norm is bounded by ϵ, and the valid initial set Ω^ϵ is then given by the appropriate component of the pre-image $V^{-1}(]-\infty, c-\epsilon)$.

4 Lukes approximation

Local solutions of the Hamilton-Jacobi inequality can be obtained by use of an approximation scheme originally developed by Lukes [Luk69] for quadratic cost functions [IK95]. For computational ease, we discuss in the following sections only affine systems although Lukes approximation method can be used for general nonlinear systems. We use the `MAPLE` implementation [MP95].

Consider the perturbed Hamilton-Jacobi *equation*

$$\begin{aligned} \mathsf{H}^{**}(\tfrac{\partial V}{\partial x}, x) &= \tfrac{\partial V}{\partial x} Q(x) \tfrac{\partial V}{\partial x}^T + \tfrac{\partial V}{\partial x} L(x) + K(x) \\ &= -\Phi(x) \quad \text{for all } x \in \Omega \quad , \end{aligned} \tag{13}$$

where $\Phi(x) : \Omega \mapsto \mathrm{IR}^+$ is some positive definite perturbation function which we will use as design parameter to style the size and shape of the valid region.

It is easily seen by (8) (11) that $L(x)$ is of least first order, and $K(x)$, $\Phi(x)$ and $V(x)$ of least second. We make the following analytic data expansions

$$\begin{aligned} V(x) &= x^T V^{[2]} x + \sum_{k=3}^{\infty} V^{[k]}(x) \quad , \\ Q(x) &= Q^{[0]} + \sum_{k=1}^{\infty} Q^{[k]}(x) \quad , \\ L(x) &= L^{[1]} x + \sum_{k=2}^{\infty} L^{[k]}(x) \quad , \\ K(x) &= x^T K^{[2]} x + \sum_{k=3}^{\infty} K^{[k]}(x) \quad , \\ \Phi(x) &= x^T \Phi^{[2]} x + \sum_{k=3}^{\infty} \Phi^{[k]}(x) \quad , \end{aligned}$$

where $(\cdot)^{[k]}$ denotes k-th order monomials. Then the perturbed Hamilton Jacobi equality (13) can be rewritten

$$0 = \sum_{m=2}^{\infty}\left[\sum_{k=1}^{m-1}\frac{\partial V}{\partial x}^{[m-k+1]}L^{[k]} + K^{[m]} + \Phi^{[m]} + \left(\sum_{k=0}^{m-2}\sum_{l=1}^{m-k-1}\frac{\partial V}{\partial x}^{[m-k-l+1]}Q^{[k]}\frac{\partial V}{\partial x}^{[l+1]}\right)\right]. \quad (14)$$

Note that $\frac{\partial V}{\partial x}^{[m]} \equiv \frac{\partial (V^{[m]})}{\partial x}$ are of order $m-1$.

Isolating the second order terms we find the usual Riccati equation of the linearized and perturbed $\mathcal{H}_\infty$ control problem

$$0 \;=\; L^{[1]}V^{[2]} + V^{[2]}L^{[1]^T} + V^{[2]}Q^{[0]}V^{[2]} + K^{[2]} + \Phi^{[2]} \;. \quad (15)$$

In case that $V^{[2]}$ is a solution of the Riccati equation (15), the second order terms of the perturbed Hamilton-Jacobi equation (14) vanish, and the m order terms can be rearranged

$$\begin{aligned} &-\frac{\partial V}{\partial x}^{[m]}\left(L^{[1]} + Q^{[0]}V^{[2]}\right)x \\ &\quad = \sum_{k=2}^{m-1}\frac{\partial V}{\partial x}^{[m-k+1]}L^{[k]} + K^{[m]} + \Phi^{[m]} + \sum_{k=1}^{m-2}\sum_{l=1}^{m-k-1}\left(\frac{\partial V}{\partial x}^{[m-k-l+1]}Q^{[k]}\frac{\partial V}{\partial x}^{[l+1]}\right) \\ &\qquad\qquad \text{for} \quad m = 3, 4, 5, \ldots \quad . \end{aligned} \quad (16)$$

Assuming that the linearized problem (15) has a stabilizing solution $V^{[2]}$, the matrix $F = L^{[1]} + Q^{[0]}V^{[2]}$ is positive definite and has therefore an inverse. Consequently, the equations (16) can be solved recursively.

5 Example

The theory developed in the former sections is now illustrated by a simplified two-dimensional example arising in the field of robotics. We consider the plant

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \sin(x_1) - 2x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w \\ z &= \begin{bmatrix} x_2 + x_2^3 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \end{aligned}$$

with equilibrium point zero.

Note that the linearized system is stabilizable by linear feedback, and we want to compare the quality of a linear controller with a third order controller. The optimal feedback L_2 gain is near 1.2, but we use the suboptimal value $\gamma = 2$.

Solving the unperturbed Riccati equation (15) (that is $\Phi = 0$), we find the formal second order storage function

$$V_{\text{lin}}(x) = 0.710x_1^2 + 3.097x_1x_2 + 4.037x_2^2$$

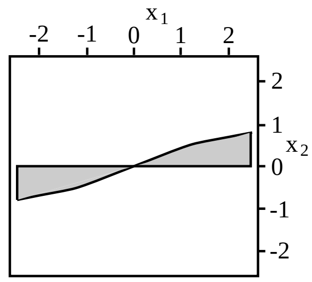

Figure 2: Unperturbed solution

which gives the linear feedback

$$u_{\mathrm{lin}}(x) = -1.421x_1 - 3.097x_2 \quad .$$

Unfortunately, this storage function satisfies not the Hamilton-Jacobi inequality (7) on a sufficient large area: in figure 2 the set

$$\mathcal{A} \equiv \left\{ x \;\middle|\; \mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) < 0 \right\}$$

is depicted grey.

Now perturbing the Riccati equation (15) with $\Phi(x) = 2x_1^2 + x_2^2$ we calculate the formal second order storage function

$$V_{\mathrm{lin,per}}(x) = 0.818x_1^2 + 3.277x_1x_2 + 4.470x_2^2$$

which gives the linear feedback

$$u_{\mathrm{lin,per}}(x) = -1.636x_1 - 3.277x_2 \quad .$$

Now (7) is satisfied on a sufficiently large area around zero. The set $\mathcal{A}$ is displayed in figure 3 (left) (white). The pre-images $\Omega = V^{-1}([0,2])$ (grey) and $\Omega^\epsilon = V^{-1}([0,1])$ with $\epsilon = 1$ (dark grey) are both inside the set $\mathcal{A}$.

Finally, we find a fourth order Lukes approximation for the Hamilton-Jacobi equation (14), using the same perturbation function as in the linear feedback case. We compute the formal storage function

$$\begin{aligned} V_{\mathrm{4th,per}}(x) = {} & 0.818x_1^2 + 3.277x_1x_2 + 4.470x_2^2 \\ & + 0.128x_1^2x_2^2 + 0.3113x_1x_2^3 + 0.025x_1^3x_2 \\ & + 0.002x_1^4 + 0.4124x_2^4 \end{aligned}$$

which gives the third order feedback

$$\begin{aligned} u_{\mathrm{3rd,per}}(x) = & - 1.636x_1 - 3.277x_2 - 0.076x_1^2x_2 \\ & - 0.256x_1x_2^2 - 0.008x_1^3 - 0.311x_2^3 \quad . \end{aligned}$$

Figure 3 (right) shows a slightly improved set $\mathcal{A}$ (white). Note that the valid region $\Omega = V^{-1}([0,4])$ (grey) is enlarged considerably. The set of valid initial conditions $\Omega^\epsilon = V^{-1}([0,1])$ (dark grey) is approximatively the same as for the linear design. Note also that we have chosen $\epsilon = 3$, thereby allowing for valid disturbances of three times larger energy that in the linear case.

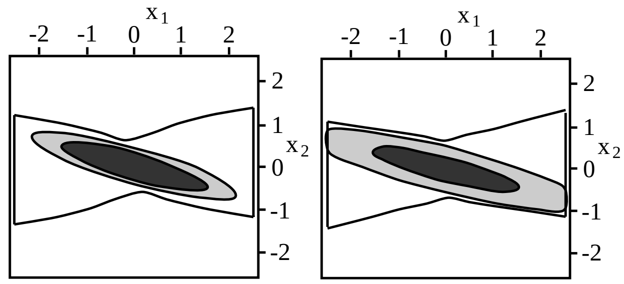

Figure 3: Linear and third order control

6 Conclusion

In this paper it is shown that state feedback problems involving the regional stabilization of the origin can successfully be recast as generalized formulations of nonlinear local state feedback $\mathcal{H}_\infty$ control problems. Given a formal solution V to a certain Hamilton-Jacobi inequality, the generalized problem is solved regionally provided V is such that some connected component of the pre-image $V^{-1}(]-\infty, c)$ for some $c \in \mathbb{R}$ is bounded and includes the to be stabilized origin. The plant is assumed to have the standard zero-detectability assumption. Sets of allowable initial conditions and disturbance classes are specified.

Performance is guaranteed in a range of operational conditions, in contrast to local $\mathcal{H}_\infty$ control. Numerical and symbolic computation methods which apply to local $\mathcal{H}_\infty$ theory can without problem be applied in a regional context.

Lukes approximation scheme is explained and implemented in the symbolic language `MAPLE`, and a formal, local storage function is computed. An example shows that the theory developed in this paper can be used to estimate the valid performance region (performance envelope) of linear as well as nonlinear state feedback controllers.

A Proof of theorem 3.2

By lemma 3.1 all $x(\cdot)$ are bounded inside Ω. Therefore, the L_2 gain (4) is satisfied.

We show now that all such $x(\cdot)$ generated by $w(\cdot) \in \mathcal{W}^\epsilon$ which are identically zero for all times $t > T$, $T \in \mathbb{R}$, approach the biggest closed loop invariant set $\mathcal{M}$ contained in the null set $\mathcal{N}$. By state boundedness, and time invariance of the system, we can assume without loss of generality that $w(\cdot) = 0$ for all $t \in \mathbb{R}^+$. The saddle point property (6) implies that the C^1 solution V serves as a Lyapunov function for the closed loop dynamics. We have

$$\mathsf{H}(u_{\min}, w) \le \mathsf{H}(u_{\min}, w_{\min}) = \mathsf{H}^{**} \le 0 \tag{17}$$

for all $w(\cdot) \in \mathcal{W}^\epsilon$. Choosing $w(\cdot) = 0$ then gives with (5)

$$\tfrac{d}{dt}V - \gamma^2 \left|0\right|^2 + \left|z\right|^2 \le 0 \tag{18}$$

for all such $x(\cdot)$. Hence we have $\frac{d}{dt}V < 0$ for all $x(\cdot)$ evolving on $\Omega/\mathcal{N}$. Trajectories on $\mathcal{N}$ are satisfying $\frac{d}{dt}V = 0$ if and only if $\left|z\right|^2 = \left|Z(x, u_{\min}(x))\right|^2 = 0$, and $\frac{d}{dt}V < 0$ else.

Now observe that $V(x)$ by assumption is continuous and defined on a bounded set, hence $V(x)$ is bounded from below. Given some particular state $x(\cdot)$, the storage function $V(\cdot)$ is decreasing and bounded from below, hence approaches some minimal value, say $c_\Gamma \in \mathrm{I\!R}$, as $t \to \infty$. By continuity we conclude that $V(x) = c_\Gamma$ on the positive limit set Γ^+, and consequently $\frac{d}{dt}V = 0$ on Γ^+. Rearranging (17) and (18) then shows that

$$0 \leq +\left|z\right|^2 \leq \mathsf{H}^{**} \leq 0 \quad , \tag{19}$$

therefore we conclude that Γ^+ is a non-empty subset of the null set $\mathcal{N}$. But Γ^+ is an invariant set, hence contained in the maximal invariant set $\mathcal{M}$, and consequently any trajectory $x(\cdot)$ is approaching $\mathcal{M}$ as $t \to \infty$.

We show finally that zero-detectability implies that $x(\cdot)$ approaches the origin as $t \to \infty$. Clearly any trajectory evolving entirely on Γ^+ satisfies by (19) that $z(\cdot) = 0$, hence by zero-detectability the origin is approached. Finally, any trajectory with the same limit set $\Gamma^+ = \{0\}$ is by continuity of the closed loop dynamics forced to approach zero as $t \to \infty$.

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Comments and References

The theory of $\mathcal{H}_\infty$ state feedback and measurement feedback control is recently developed. A major contribution was the state space solution to linear $\mathcal{H}_\infty$ control measurement feedback problems first published by J. Doyle, K. Glover, P. Khargonekar and B.A. Francis in 1989 [DGKF89]. Linear $\mathcal{H}_\infty$ control matured in a couple of years to a well-understood mathematical theory, see for example the textbook [ZDG95].

Historically seen, the nonlinear theory of $\mathcal{H}_\infty$ state feedback and measurement feedback control is a very young discipline, developed in the nineties, even if the fundamental system theoretic point of view has been developed twenty years earlier. Among the leading articles describing the input-output structure and stability issues of nonlinear systems, we must remember the work of Jan C. Willems. A very good and throughout study of general dissipation is found in [Wil72a], and the basic properties developed there are applied to linear control systems with quadratic supply rates in [Wil72b]. The general structure of feedback systems is investigated in the textbook [Wil71b]. The issues of stability in the context of input-output maps and in state-space representation (there called Lyapunov approach) are treated in [Wil71a], and later in [Wil76]. Finally, the system theoretic background for modelling and analyzing physical systems with the tools of differential geometry has been discussed in [Wil79].

Three major contributions to the understanding of the $\mathcal{H}_\infty$ control problem has been the work of A. J. van der Schaft, Alberto Isidori and Alessandro Astolfi, and J.A. Ball, J.W. Helton and M.L. Walker. These contributions are discussed in two separate subsections because they are of different philosophical nature: the game theoretic approach is used in the work of Isidori, Astolfi, Ball, Helton and Walker, whereas the differential geometric approach is mainly due to professor A. J. van der Schaft.

The game theoretic approach to $\mathcal{H}_\infty$ control

This approach is based on two fundamental concepts in system analysis: the notion of dissipation and of differential games. The notion of dissipation shows via the Bounded Real Lemma that a stable, uncontrolled linear system has an $\mathcal{L}_2$ gain less than or equal to $\gamma > 0$ if and only if the system is dissipative with respect to the supply rate $s(w, z) = \gamma^2|w|^2 - |z|^2$. The theory of differential games shows that the problem of minimizing the $\mathcal{L}_2$ gain of a controlled linear system can be viewed as two person, zero sum differential game, where the disturbance $w_{\max}(\cdot)$ acts as maximizing player, whereas the control $u_{\min}(\cdot)$ represents the minimizing player. We defer the details of the analysis of the game theoretic approach to chapter 5, where the notion of general dissipation is used.

Linear $\mathcal{H}_\infty$ control has used the theory of differential games (see [Fri71] for a throughout exposition) in the research paper [DGKF89], and an excellent exposition of this approach is found in the textbook [BB95] and the Ph.D. thesis [Sch90] and [Sto90], both focusing on a state space approach to $\mathcal{H}_\infty$ control.

The linear approach has been ported to nonlinear systems in the research paper [BH89], and further investigated in [PAJ91]. Isidori and Astolfi [Isi92, IA92b, IA92a] approach the problem of $\mathcal{H}_\infty$ measurement feedback control of input affine systems by the theory of differential games. See also the summarizing exposition [Isi94] and an application for a special class of composite and/or interconnected nonlinear systems [IT93]. The problem has later been solved for more general nonlinear plants: Ball, Helton, and Walker [BHW93] consider systems which are affine only in the exogenous input $w(\cdot)$, whereas Isidori and Kang [IK95] consider general nonlinear plants without any form of affine structure. We remark however, that all these approaches to nonlinear $\mathcal{H}_\infty$ control consider only systems where the asymptotic stability of state trajectories is confined to a neighborhood of unknown size around the to-be-stabilized equilibrium point, and moreover, only the stability issues of undisturbed motions, that is, state trajectories $x(\cdot)$ subject $w(\cdot) = 0$ are considered. These restrictions are indeed odd, since they make nonlinear $\mathcal{H}_\infty$ theory essentially worthless for practical applications.

The differential geometric approach to $\mathcal{H}_\infty$ control

In the research paper [vdS89] A. J. van der Schaft describes 1989 a system theoretic approach to mechanics, where Hamiltonian systems are regarded as conservative "mechanical m-ports". It is illustrated in three particular cases how the Hamiltonian structure of a system can be profitably used for control purposes. More precisely, the Euler-Lagrange equations

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot q}(q,\dot q,u) - \tfrac{\partial L}{\partial q}(q,\dot q,u) = 0 \quad , \tag{20}$$

are assumed to be controlled by the variables u. Here, $q \in \mathbb{R}^n$ are the generalized coordinates, and $L(q,\dot q,u)$ is the Lagrangian, that is, the difference of potential and kinetic energy stored in the system. Notice that the form of the above controlled equations allows a control input which not necessarily appear as external forces, but which are allowed to act on the potential or the kinetic energy directly. The Euler-Lagrange equations transform into the Hamiltonian equations of motions given by

$$\begin{aligned} \dot q &= \tfrac{\partial \mathsf{H}}{\partial p}^T(q,p,u) \\ \dot p &= -\tfrac{\partial \mathsf{H}}{\partial q}(q,p,u) \\ y &= -\tfrac{\partial \mathsf{H}}{\partial u}(q,p,u) \quad , \end{aligned} \tag{21}$$

where $\mathsf{H} \equiv p\dot q - L$, and $p \in (\mathbb{R}^n)^*$ are the generalized momenta. We use the convention that q is a column vector living in the tangent space $\mathbb{R}^n$, whereas p is a row vector living in the co-tangent space $(\mathbb{R}^n)^*$. The last equation defines the "natural" outputs, hence a Hamiltonian input-output system is defined. It is then show in [vdS89] that the Bolza optimal control problem, which requires minimization of the cost functional

$$J(x_0,u(\cdot)) \equiv K(x(T)) + \int_0^T L(x(t),u(t))\,dt \tag{22}$$

under the dynamical constraints

$$\dot x = X(x,u) \quad , \quad x(0) = x_0 \quad , \tag{23}$$

can be recast in terms of the Hamiltonian system (21): the Maximum Principle tells us to consider the pre-Hamiltonian $\mathsf{H}(x,p,u) \equiv pX(x,u) - L(x,u)$, with p the co-state. A necessary condition for the control signal $u_{\text{opt}}(\cdot)$ on $[0,T]$ to be optimal is that for every $t \in [0,T]$, $\mathsf{H}(x(t),p(t),u_{\text{opt}}(t)) = \max_u \mathsf{H}(x(t),p(t),u)$, where $(x(t),p(t))$ is the unique solution to (21) with initial values $x(0) = x_0$ and $p(T) = -\frac{\partial}{\partial x}K(x(T))$. So we are led to the following problem: Find for every (x,p) a $u_{\text{opt}} = u_{\text{opt}}(x,p)$ such that $\mathsf{H}(x,p,u_{\text{opt}}) = \max_u \mathsf{H}(x,p,u)$, which implies that $\frac{\partial \mathsf{H}}{\partial u}(x,p,u) = 0$. Hence, the Maximum Principle leads in a natural way to the Hamiltonian control system (21) with $q = x$, and a necessary condition for $u_{\text{opt}}(\cdot)$ to be optimal is that that the outputs y of this system equal zero. This Hamiltonian framework is constrained to conservative systems, and can therefore not include the action of dissipative forces like friction and other kind of energy loss. Also the optimizing control u_{opt} of the above sketched optimal control problem has been identified as a function of the gradient $u_{\text{opt}}(x) = u_{\text{opt}}(x, \frac{\partial}{\partial x}V_O(x))$ of the optimal value function $V_O(x)$, and the map $(x,p) = (x, \frac{\partial}{\partial x}V_O(x))$ represents the unique stable and invariant n-dimensional sub-manifold of the Hamiltonian system (21) through the point $(0,0)$ [vdS91a] (that is, the set of all initial points $(x_0,p_0) \in \mathbb{R}^n \times (\mathbb{R}^n)^*$ which are attracted to $(0,0)$ under the dynamics of (21) subject $y = 0$).

As it has been done previously in the field of linear control, the Hamiltonian approach to nonlinear optimal control problems has been generalized to $\mathcal{H}_\infty$ control problems. First, in the research papers [vdS91b, vdS91a], the state feedback $\mathcal{H}_\infty$ control problem is linked to the solvability of the linearized, algebraic Riccati equation. In contrast to the uniqueness of the value function belonging to the optimal control problem, there exist in general infinitely many storage functions which solve the $\mathcal{H}_\infty$ control problem. Then, the differential geometric interpretation of the optimal cost $V_O(x)$ has in the papers [vdS91b, vdS91c] been transferred to any of the possible storage functions $V(x)$ associated with affine $\mathcal{H}_\infty$ control problems. More precisely, let us assume that some given $\gamma > 0$ is a suboptimal value of the $\mathcal{L}_2$ gain of the $\mathcal{H}_\infty$ problem at hand. It has then been shown that any solution $V(x)$ to the HJI (10) which is such that the control $u_{\min}(x) = u_{\min}(x, \frac{\partial V}{\partial x}(x))$ asymptotically stabilizes the equilibrium $x = 0$ of the closed loop system

$$\dot{x} = X(x, u_{\min}(x), w_{\max}(x)) \quad , \tag{24}$$

represents an invariant and n-dimensional Lagrangian sub-manifold $(x,p) = (x, \frac{\partial V}{\partial x}(x))$ through $(0,0)$ of the Hamiltonian vectorfield

$$\begin{aligned} \dot{x} &= \frac{\partial \mathsf{H}^{**}}{\partial p}^T(x,p) \\ \dot{p} &= -\frac{\partial \mathsf{H}^{**}}{\partial x}(x,p) \end{aligned} \tag{25}$$

on $T^*M = \mathbb{R}^n \times (\mathbb{R}^n)^*$, where $\mathsf{H}^{**}(x,p) \equiv \mathsf{H}(x,p,u_{\min},w_{\max})$. The Hamiltonian vectorfield is then hyperbolic at $(0,0)$, that is, its linearization at $(0,0)$ has n eigenvalues in the left half plane and n in the right half plane (counted with multiplicity). However, the Lagrangian sub-manifold as such need not to be an asymptotically stable invariant set under the dynamics of (25). The refinements in [vdS92c] state that the available storage represents the unique stable invariant and n-dimensional Lagrangian sub-manifold $(x,p) = (x, \frac{\partial}{\partial x}V_O(x))$, whereas the required supply represents the unstable invariant and n-dimensional Lagrangian sub-manifold $(x,p) = (x, \frac{\partial}{\partial x}V_R(x))$. Moreover, the restriction of the Hamiltonian vectorfield to any of its Lagrangian invariant manifolds is given by equation (24).

Without loss of generality we may assume that $V(0) = 0$, and it can be shown that the map $x \mapsto V(x)$ which represents an invariant Lagrangian sub-manifold $(x, p) = (x, \frac{\partial V}{\partial x}(x))$ of (25) is the unique solution to the HJI (10) with boundary conditions

$$V(0) = 0 \quad , \quad \tfrac{\partial V}{\partial x}(0) = 0 \;\text{ and }\; \tfrac{\partial^2 V}{\partial x^2}(0) = P \quad , \tag{26}$$

where $P : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the linear map such that the generalized eigenspace of the matrix

$$\begin{bmatrix} I \\ P \end{bmatrix}$$

equals the tangent space of $(x, p) = (x, \frac{\partial V}{\partial x}(x))$ at $(0, 0)$. The same P is then also a stabilizing solution to the algebraic Riccati equation

$$P^T \tfrac{\partial^2 \mathsf{H}}{\partial x \partial p}(0,0) + \tfrac{\partial^2 \mathsf{H}}{\partial x \partial p}(0,0) P + P^T \tfrac{\partial^2 \mathsf{H}}{\partial p^2}(0,0) P + \tfrac{\partial^2 \mathsf{H}}{\partial x^2}(0,0) \;=\; 0$$

belonging to the linearized $\mathcal{H}_\infty$ problem, and since there are in general infinitely many such solutions P, it follows that infinitely many storage functions $V(x)$ can be found. In case that we are considering the optimal value $\gamma_0 < \gamma$ of the $\mathcal{H}_\infty$ control problem, the Hamiltonian vectorfield (25) is not hyperbolic at $(0, 0)$, and the resulting dynamics restricted on the invariant center manifold of (25) may or may not destroy the stability property of $(0, 0)$.

There is a very important, but often overseen property associated to the Hamiltonian approach which has been pointed out in [vdS92a]: the regularity of the available storage V_A equals the regularity of the data of the system. More precisely, assume that the data $A(x)$, $B_u(x)$, $B_w(x)$ and $C(x)$ of the (input-affine) system is C^k with $k \geq 2$, and that a suboptimal value $\gamma > 0$ is considered. Then it follows that the Hamiltonian vectorfield (25) is C^{k-1}. Hence, the stable and the unstable invariant Lagrangian manifold are C^{k-1} locally near $(0, 0)$ as well, and it follows directly that the available storage $V_A(x)$ is bounded for all $x \in \mathbb{R}^n$, and is C^k near $(0, 0)$. If $\mathbb{R}^n$ is reachable from $x = 0$ it follows that the required supply $V_R(x)$ exists and is C^k near $(0, 0)$, and bounded for all $x \in \mathbb{R}^n$. Unfortunately, the solvability of the linearized algebraic Riccati equation implies only that the above mentioned Lagrangian manifolds are C^k near $(0, 0)$, there is no indication of the size of the neighborhood of $(0, 0)$ where regularity can be guaranteed. Only a truly nonlinear analysis can tell *if* and *where* problems with the parameterization of invariant Lagrangian manifolds by the x-coordinates arise.

The $\mathcal{H}_\infty$ control problem has then in [vdS92b] been extended to output feedback control (see also the exposition [vdS92a]). Necessary conditions for $\mathcal{H}_\infty$ control by dynamic output feedback control have been derived for input-affine systems, and some sufficient conditions have been given. Unfortunately, the dynamic measurement feedback problem is closely linked to the nonlinear observer problem, both can not (in general) be solved globally by finite-dimensional dynamic feedback. In [vdS93] van der Schaft describes a subclass of input affine systems, where $B_u(x)$ and $B_w(x)$ are constant in x, and $y = C_y(x)$ is linear in x. These restrictions allows the use of finite-dimensional dynamic controllers [vdS94].

There is also an interesting interpretation of the steady state response of a nonlinear system due to A. Isidori, which has been developed in the context of output regulation by use of differential geometric tools in the papers [IB89, IB90], and has been transferred to the case of $\mathcal{H}_\infty$ control in [IA92a]. Assuming that the system

$$\dot{x} = X(x, u_{\min}(x), w) \tag{27}$$

has a locally exponentially stable equilibrium $(x, w) = (0, 0)$, that is, the eigenvalues of the Jacobian matrix

$$\frac{\partial X(x, u_{\min}(x), w)}{\partial x}(0, 0)$$

are all in the left half plane, and that the exogenous inputs are generated by a nonlinear system

$$\dot{w} = W(w)$$

in which all trajectories are periodic of a given period T_0, the composed system

$$\begin{aligned} \dot{x} &= X(x, u_{\min}(x), w) \\ \dot{w} &= W(w) \end{aligned} \tag{28}$$

has a locally attractive center manifold which can be parameterized by the graph

$$\mathcal{M} = \left\{ (x, w) \;\middle|\; x = \pi(w) \right\}$$

of a suitable C^1 mapping $\pi : \mathcal{W} \mapsto \mathbb{R}^n$ defined on a neighborhood $\mathcal{W} \subset \mathbb{R}^l$ of the point $w = 0$. Since $\mathcal{M}$ is locally attractive, it follows for all periodic $w(\cdot) \in \mathcal{W}$ that the integral curve of the composed system (28) through the point (x_0, w_0) converges as $t \to \infty$ to the integral curve $x_{w_0}(\cdot)$ of (28) through the point $(\pi(w_0), w_0)$ in case that $|x_0 - \pi(w_0)|$ is small enough. Moreover, since the center manifold of (28) is invariant, $x_{w_0}(t) = \pi(w(t)$ holds for all t, and it follows that the trajectory $x_{w_0}(\cdot)$ and the performance output $z_{w_0}(\cdot) = Z(x_{w_0}(\cdot))$ are periodic with period T_0 as well. Hence, we can view the unique state trajectory $x_{w_0}(\cdot)$ as the steady state response of all other trajectories with initial condition x_0 satisfying that $|x_0 - \pi(w_0)|$ is sufficient small. Evaluating (4) for $t_0 \to \infty$ yields then

$$\frac{1}{T_0^2} \|z\|_{T_0}^2 \leq \gamma^2 \frac{1}{T_0^2} \|w\|_{T_0}^2$$

which can be regarded as the RMS value attenuation of the periodic steady state response of the nonlinear and locally exponentially stable system (27). Notice though that the maximal allowable sizes of the neighborhood $\mathcal{W}$ and the maximal distance $|x_0 - \pi(w_0)|$ are not known. Moreover, system (27) can be designed such that $\mathcal{W}$ and $|x_0 - \pi(w_0)|$ must be arbitrarily small to make the above sketched central manifold analysis work.

The output feedback regulation results of the papers [IB89, IB90] have also been used to prove the existence of a feedback law (under suitable conditions) that solves the problem of robust regulation for a nonlinear system in presence of an $\mathcal{L}_2$ gain bounded, but otherwise unknown dynamic uncertainty system [IT95], thus extending the well known linear approach of robust regulation to nonlinear systems by the differential geometric tools sketched above.

State feedback $\mathcal{H}_\infty$ control for affine nonlinear tracking problems

Let us take a look on general input affine systems with feed-through from $w(\cdot)$ to $z(\cdot)$, that is systems of the form

$$\begin{aligned} \dot{x} &= A(x) + B_u(x)u + B_w(x)w \\ z &= C(x) + D_u(x)u + D_w(x)w \quad , \end{aligned} \tag{29}$$

satisfying $A(0) = 0$, $C(0) = 0$, and $D_u^T(x)D_u(x) > 0$ for all x. Usually, the simplifying conditions

$$\begin{aligned} D_w(x) &= 0 \\ C^T(x)D_u(x) &= 0 \\ D_u^T(x)D_u(x) &= I \\ D_u(x)B_w^T(x) &= 0 \quad , \end{aligned} \tag{30}$$

which are the nonlinear equivalent of the so-called DGKF-conditions [DGKF89], are assumed to hold. While often imposed in nonlinear control, they are not needed, and restrict the class of input affine systems considerably. For example, nonlinear tracking problems can not be handled in this restricted $\mathcal{H}_\infty$ setting. Therefore it is mandatory to examine the structure of non-linear, but input-affine, control systems not restricted by the equations (30). We derive here the computations not found in the common literature on $\mathcal{H}_\infty$ control.

We concentrate on the regular case $D_u^T D_u > 0$ for all x, which essentially means that all controls are penalized (The singular case $D_u^T(x)D_u(x) \geq 0$ has for input affine systems been solved by W.C.A. Maas and A.J. van der Schaft [MvdS94]). It turns out that the important feature of the generalized problem at hand is again the saddlepoint property of the Hamiltonian. We recall that the pre-Hamiltonian function H is defined by

$$\begin{aligned} \mathsf{H}(u,w) &\equiv \tfrac{d}{dt}V - (\gamma^2|w|^2 - |z|^2) \\ &= \tfrac{\partial V}{\partial x}(A(x) + B_u(x)u + B_w(x)w) - \gamma^2|w|^2 + |C(x) + D_u(x)u + D_w(x)w|^2 \\ &= \tfrac{\partial V}{\partial x}A + \tfrac{\partial V}{\partial x}\begin{bmatrix} B_u & B_w \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix} + 2\begin{bmatrix} C^T D_u & C^T D_w \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix} + C^T C \\ &\qquad + \begin{bmatrix} u^T & w^T \end{bmatrix}\begin{bmatrix} D_u^T D_u & D_w^T D_u \\ D_u^T D_w & D_w^T D_w - \gamma^2 I \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix} \end{aligned} \tag{31}$$

for all $t \in \mathbb{R}^+$, hence proceeding as outlined before, we are searching for a saddlepoint in terms of u and w of the pre-Hamiltonian (31). Since $D_u^T D_u > 0$, and since there is a $\gamma_0 > 0$ such that

$$(D_w^T D_w - \gamma^2 I) - D_w^T D_u (D_u^T D_u)^{-1} D_u^T D_w = D_w^T (I - D_u (D_u^T D_u)^{-1} D_u^T) D_w - \gamma^2 I \ < \ 0$$

for all $\gamma > \gamma_0$, it follows from the Schur complement formula that the Hessian

$$H \equiv \begin{bmatrix} D_u^T D_u & D_w^T D_u \\ D_u^T D_w & D_w^T D_w - \gamma^2 I \end{bmatrix} \tag{32}$$

is regular and defines a saddlepoint in (u, w) for all $\gamma \geq \gamma_0$. It follows that H has a unique saddle point $(w_{\max}, u_{\min})$ for all x and all $\frac{\partial V}{\partial x}$, and that the extremal functions $u_{\min}(x, \frac{\partial V}{\partial x})$, $w_{\max}(x, \frac{\partial V}{\partial x})$ are characterized by the equations

$$\begin{aligned} \tfrac{\partial \mathsf{H}}{\partial u}(u_{\min}, w_{\max}) = 0 \quad &, \quad u_{\min}(0,0) = 0 \\ \tfrac{\partial \mathsf{H}}{\partial w}(u_{\min}, w_{\max}) = 0 \quad &, \quad w_{\max}(0,0) = 0 \ . \end{aligned} \tag{33}$$

We compute explicitly the derivatives

$$\begin{bmatrix} \frac{\partial \mathsf{H}}{\partial u}^T \\ \\ \frac{\partial \mathsf{H}}{\partial w}^T \end{bmatrix} = \begin{bmatrix} B_u^T \frac{\partial V}{\partial x}^T + 2D_u^T C \\ \\ B_w^T \frac{\partial V}{\partial x}^T + 2D_w^T C \end{bmatrix} + 2 \begin{bmatrix} D_u^T D_u & D_w^T D_u \\ \\ D_u^T D_w & D_w^T D_w - \gamma^2 I \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \ . \tag{34}$$

Consequently, the extremal functions are given by setting (34) equal zero, which yields

$$\begin{bmatrix} u_{\min} \\ w_{\max} \end{bmatrix} = -H^{-1} \begin{bmatrix} \frac{1}{2} B_u^T \frac{\partial V}{\partial x}^T + D_u^T C \\ \\ \frac{1}{2} B_w^T \frac{\partial V}{\partial x}^T + D_w^T C \end{bmatrix} \ . \tag{35}$$

It follows that the Hamiltonian is given by

$$\begin{aligned} \mathsf{H}^{**} = &-\tfrac{1}{4} \tfrac{\partial V}{\partial x} \begin{bmatrix} B_u & B_w \end{bmatrix} H^{-1} \begin{bmatrix} B_u^T \\ \\ B_w^T \end{bmatrix} \tfrac{\partial V}{\partial x}^T + \tfrac{\partial V}{\partial x} A \\ &- \tfrac{1}{2} \tfrac{\partial V}{\partial x} \begin{bmatrix} B_u & B_w \end{bmatrix} H^{-1} \begin{bmatrix} D_u^T C \\ \\ D_w^T C \end{bmatrix} - \tfrac{1}{2} \begin{bmatrix} C^T D_u & C^T D_w \end{bmatrix} H^{-1} \begin{bmatrix} B_u^T \\ \\ B_w^T \end{bmatrix} \tfrac{\partial V}{\partial x}^T \\ &+ C^T C - \begin{bmatrix} C^T D_u & C^T D_w \end{bmatrix} H^{-1} \begin{bmatrix} D_u^T C \\ \\ D_w^T C \end{bmatrix} \ . \end{aligned} \tag{36}$$

Hence, the Hamilton-Jacobi inequality can be rewritten

$$\mathsf{H}^{**}(x) = \tfrac{\partial V}{\partial x} Q(x) \tfrac{\partial V}{\partial x}^T + \tfrac{\partial V}{\partial x} L(x) + K(x) \leq 0 \tag{37}$$

where

$$\begin{aligned} Q(x) &= -\frac{1}{4} \begin{bmatrix} B_u & B_w \end{bmatrix} H^{-1} \begin{bmatrix} B_u^T \\ \\ B_w^T \end{bmatrix} \\ L(x) &= A - \begin{bmatrix} B_u & B_w \end{bmatrix} H^{-1} \begin{bmatrix} D_u^T C \\ \\ D_w^T C \end{bmatrix} \\ K(x) &= C^T C - \begin{bmatrix} C^T D_u & C^T D_w \end{bmatrix} H^{-1} \begin{bmatrix} D_u^T C \\ \\ D_w^T C \end{bmatrix} \ . \end{aligned} \tag{38}$$

A comparison with (11) shows that the problem at hand has the same structure as the HJI (10), hence the rest of the paper applies without changes except for formulas of higher complexity.

Output tracking of reference signals

While tracking control problems of nonlinear systems also can be solved by geometric, predictive, and $\mathcal{H}_2$ tools (see [Lu94, Lu96]), we want to emphasize the use of $\mathcal{H}_\infty$ control techniques to accomplish this task.

Let us assume that we wish to track a reference signal $r(\cdot)$ with some pre-defined output function $C_e(x)$ subject to unknown disturbances $d(\cdot)$. This task can be reformulated as a $\mathcal{H}_\infty$ state feedback control problem of the form (29): Let the to-be-controlled signal $z(\cdot)$ include some penalty signal $p(\cdot)$ which essentially penalizes the use of control $u(\cdot)$ and the error signal $e(\cdot) \equiv r(\cdot) - C_e(x(\cdot))$, that is, let the signals $w(\cdot)$ and $z(\cdot)$ be divided as follows

$$w(\cdot) \equiv \begin{bmatrix} r(\cdot) \\ d(\cdot) \end{bmatrix} \quad \text{and} \quad z(\cdot) \equiv \begin{bmatrix} e(\cdot) \\ p(\cdot) \end{bmatrix} \quad .$$

Then the control system (29) is subdivided, it is given by

$$\begin{aligned} \dot{x} &= A(x) + B_u(x)u + \begin{bmatrix} 0 & B_d(x) \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix} \\ z &= \begin{bmatrix} e \\ p \end{bmatrix} = \begin{bmatrix} C_e(x) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ D_p(x) \end{bmatrix} u + \begin{bmatrix} -I_r & 0 \\ 0 & D_d(x) \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix} \quad , \end{aligned} \tag{39}$$

where $A(0) = 0$, $C_e(0) = 0$, and $D_p^T(x) D_p(x) > 0$ for all x. Since $D_u^T D_u = D_p^T D_p > 0$, and since there is a $\gamma_0 > 0$ such that

$$\begin{aligned} &(D_w^T D_w - \gamma^2 I) - D_w^T D_u (D_u^T D_u)^{-1} D_u^T D_w \\ &\qquad = \begin{bmatrix} -I_r & 0 \\ 0 & D_d^T \end{bmatrix} \begin{bmatrix} I_e & 0 \\ 0 & I_p - D_p (D_p^T D_p)^{-1} D_p^T \end{bmatrix} \begin{bmatrix} -I_r & 0 \\ 0 & D_d \end{bmatrix} - \gamma^2 I_w \ < \ 0 \end{aligned}$$

for all $\gamma \geq \gamma_0$, it follows again from the Schur complement formula that the Hessian (32) is regular and that the Hamiltonian (36) has the needed saddlepoint property for all $\gamma \geq \gamma_0$. Hence, the output tracking problem sketched above is solved by substitution of the matrices

$$B_w(x) = \begin{bmatrix} 0 & B_d \end{bmatrix} \quad , \quad C(x) = \begin{bmatrix} C_e & 0 \end{bmatrix} \quad D_u(x) = \begin{bmatrix} 0 \\ D_p \end{bmatrix} \quad \text{and} \quad D_w(x) = \begin{bmatrix} -I_r & 0 \\ 0 & D_d \end{bmatrix}$$

into the equations (38), and thereafter solving the HJI (37), and by applying the resulting feedback law $u_{\min}$ given by equations (35).

Proof of lemma 3.1

The proof, which is found in [CMPP96], is displayed here for the convenience of the reader:

Since V is continuous it follows that Ω is closed, hence compact. Moreover (7) implies that V satisfies the dissipation inequality

$$V(x(T)) - V(x_0) \leq \int_0^T \gamma^2 |w(t)|^2 - |z(t)|^2 \, dt \quad ,$$

therefore any trajectory $x(\cdot)$ with initial condition $x_0 \in \Omega$ subject to $w(\cdot) = 0$ fulfills

$$V(x_T) \leq V(x_0) - \|z\|_T^2 \leq c_\Omega$$

for all $T \in \mathbb{R}^+$ (Note that then $\|z\|_T^2 \leq V(0) \leq c_\Omega$ is always true). Therefore $x(T) \in \Omega$ for all $T \in \mathbb{R}^+$, and the trajectory can not leave Ω.

Now consider components of the sets $V^{-1}([-\infty, c])$ with $c < c_\Omega$, which are subsets of Ω. These are clearly closed subsets of Ω, hence compact. Let $x(\cdot)$ be any closed loop trajectory with initial condition $x_0 \in \Omega^\epsilon \subset V^{-1}([-\infty, c_\Omega - \epsilon])$, and assume $w(\cdot) \in \mathcal{W}^\epsilon$. Then we have

$$V(x_T) \leq V(x_0) - \|z\|_T^2 + \gamma^2 \|w\|_T^2 \leq c_\Omega - \epsilon + \epsilon$$

for all $T \in \mathbb{R}^+$ (Note that in this case $\|z\|_T^2 \leq V(0) + \epsilon \leq c_\Omega$ is always true). Therefore $x(\cdot)$ does not leave Ω.

Chapter 3

Semi-global $\mathcal{H}_\infty$ State Feedback Control With Set-Stability

In this chapter we are including a new concept to the regional $\mathcal{H}_\infty$ state control problem described in chapter 2: The wish to stabilize invariant compact sets, as motivated in the introduction. The here presented approach is a combination of several different tools presented in the introductory chapter 1. More precisely, we are combining the functionality of the following tools:

$\mathcal{H}_\infty$ analysis theorem 2.3,

Game theory theorem 2.5,

La Salle's invariance principle and performance envelopes theorem 2.1,

Set-stability theorem 2.2, and

Control Lyapunov functions theorem 2.4.

Unfortunately, the combination is yet incomplete, several features of these tools are not perfectly unified. The most important and most bothering deficiencies at this point are three:

First of all, the set-stabilizing power of theorem 2.2 is only partially exploited. We are considering the asymptotic properties of compact invariant sets for undisturbed motions, that is, for trajectories $x(\cdot)$ subject to $w(\cdot) = 0$, only. As we see in the paper, this approach can without loss of generality be applied to disturbances of finite $\mathcal{L}_2$ norm which are equal zero for all times $t \geq T \in \mathbb{R}$. Also, the stability of those compact invariant sets with respect to undisturbed motions is not proven here. It follows that we have only results on boundedness of state trajectories, and attraction of sets, no results on asymptotic stability of compact invariant sets are given. The main tool used to provide attraction of sets is a very natural, but new, generalization of the well known notion of zero-detectability.

Secondly, only sufficient conditions in terms of the state feedback Hamilton-Jacobi inequality and the properness of C^1 storage functions are provided ($V : \mathbb{R}^n \mapsto \mathbb{R}^+$ is called proper if $V(x) \to \infty$ for all $x \to \infty$). This is in contrast to the robust stability theorem 2.2 mentioned in chapter 1, which states sufficient and necessary conditions. Therefore, two important questions remain unanswered at this stage of our investigations: Do C^1 storage functions exist? And do proper storage functions exist? These questions are answered in chapter 5.

Finally, the possibilities of theorem 2.3 are only partially used, since we are only considering the special supply rate $s(w, z) = \gamma^2|w|^2 - |z|^2$ which belongs to $\mathcal{H}_\infty$ control problems. As it is showed in chapter 5, this is only a minor problem which can be resolved easily.

The main part of this chapter consists of a reprint of the paper

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Semi-global $\mathcal{H}_\infty$ state feedback control

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Abstract

Semi-global set-stabilizing $\mathcal{H}_\infty$ control is local $\mathcal{H}_\infty$ control within some given compact set Ω such that all state trajectories are bounded inside Ω, and are approaching an open loop invariant set $\mathcal{S} \subset \Omega$ as $t \to \infty$. Sufficient conditions for the existence of a continuous state feedback law are given, based on a new theorem.

1 Introduction

The standard formulation of local state feedback $\mathcal{H}_\infty$ control is mainly based on the theory of dissipative systems first introduced by Willems [Wil72a]. In this paper we will approach the problem by the theory of differential games as outlined in the papers by Isidori [Isi92], and Isidori and Astolfi [IA92b, IA92a], but we allow for non-zero initial conditions following van der Schaft [vdS92a]. Recently, the local nonlinear state feedback $\mathcal{H}_\infty$ control problem has been solved for general nonlinear plants by Isidori and Kang [IK95], and Ball, Helton, and Walker [BHW93]. The standard nonlinear $\mathcal{H}_\infty$ control theory is briefly summarized in Section 3.1.

From an applied point of view the theory of local $\mathcal{H}_\infty$ control has a severe drawback: It does not give a bound on the state trajectories, but merely states that it is valid for bounded trajectories. In fact, a linear controller based on the linearization in an equilibrium point might even do better in practice than a local nonlinear $\mathcal{H}_\infty$ controller. Moreover, it can be argued that the real motivation for nonlinear control theory are applications where the plant is operating in a significant range of operating points. Otherwise, linear control theory will work in most cases.

On the other hand, to compute a global nonlinear $\mathcal{H}_\infty$ control is not realistic in most practical cases since it basically requires finding an analytical expression for a global solution to a Hamilton-Jacobi equation or inequality.

This is the main motivation for the present paper which presents a method to design $\mathcal{H}_\infty$ controllers constraining state trajectories to a region of the state space rather than operating with local results without knowledge of boundedness of the state. The regions are specified in terms of invariant sets, and the results are generalizations of local $\mathcal{H}_\infty$ results. Moreover, the computational methods that apply to local $\mathcal{H}_\infty$ control extend directly to the obtained semi-global $\mathcal{H}_\infty$ results. This constitutes a much more practical theory for nonlinear control systems where also oscillating and other non-stationary modes of operation can be dealt with.

It is described in Section 3.2 how semi-global stability has been obtained for autonomous systems. The main idea of this paper is based on the proof of La Salle's invariance principle [SL61], here restated in Theorem 3.2.

The new contribution to the theory of semi-global stability and set-stability by $\mathcal{H}_\infty$ control is found in Section 4. In order to prove the boundedness of state trajectories we have to restrict to a certain class of disturbances denoted $\mathcal{W}^\epsilon$. Given some solution V to the standard $\mathcal{H}_\infty$ Hamilton-Jacobi inequality, a new lemma shows how to compute the region of boundedness Ω, and the region of allowed initial conditions Ω^ϵ. A new theorem, based on La Salle's invariance principle, is the cornerstone of semi-global stability and set-stability by $\mathcal{H}_\infty$ control provided that a certain detectability property is satisfied.

2 Problem formulation

Let $\mathbb{R}^+$ denote the real positive closed time-axis $[0, \infty[$. We consider the smooth, continuous time system

$$\dot{x} = X(x, u, w) \quad , \quad z = Z(x, u) \tag{1}$$

where $x(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is called the state, $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^m$ the input, $w(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^l$ the exogenous input, also called disturbance, and $z(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$ the performance, or to-be-controlled signal.

The open loop system

$$\dot{x} = X^{\text{open}}(x) \equiv X(x, 0, w) \tag{2}$$

with constant disturbance $w(\cdot) = 0$ is autonomous, and it's dynamic is therefore naturally assumed to have at least one connected, non-empty invariant set such as a closed periodic orbit or an equilibrium point.

The static state feedback used here is some vector valued function $a : \mathbb{R}^n \mapsto \mathbb{R}^m$

$$u = a(x) \quad , \tag{3}$$

thus the closed loop system is given by the equations

$$\begin{aligned} \dot{x} &= X^a(x, w) \equiv X(x, a(x), w) \\ z &= Z^a(x) \equiv Z(x, a(x)) \quad . \end{aligned} \tag{4}$$

Whenever convenient, we use the notation $x(\cdot)$ for the unique signal $x(\cdot, t_0, x_0, u(\cdot), w(\cdot))$ generated by the inputs $u(\cdot), w(\cdot)$, where the initial condition at time t_0 is x_0. It is assumed that all signals are L_2^{loc}, and that the state exist uniquely for all inputs, and is a C^1 signal except on a set of measure zero.

Define the L_2 norm for any locally square integrable signal $y(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$ for all $T \in \mathbb{R}^+$ by

$$\|y\|_T^2 \equiv \int_0^T |y(t)|^2 \, dt \quad , \tag{5}$$

where $|\cdot|$ is the usual Euclidean vector norm. By definition, the open or closed loop system (2) or (4) has local L_2 gain less than or equal to $\gamma \geq 0$ if there exists a neighborhood $\Omega \subset \mathbb{R}^n$ around the origin, and a nonnegative and bounded function $V_\text{a} : \mathbb{R}^n \mapsto [0, \infty[$, called available storage, depending only on the initial condition x_0, such that

$$\|z\|_T^2 \leq \gamma^2 \|w\|_T^2 + V_\text{a}(x_0) \tag{6}$$

for all $T \in \mathbb{R}^+$, all initial conditions $x_0 \in \Omega$, and all $w(\cdot), z(\cdot) \in L_2^{\text{loc}}$ such that the state trajectories never leave Ω [vdS92a, IA92a].

To allow for oscillatory or other non-stationary modes of operation we adopt the notion of set-stability introduced in [Lin92], more precisely, we are interested in asymptotically stabilization of some open loop invariant set $\mathcal{S}$ such that the motions on $\mathcal{S}$ are unaltered by feedback.

2.1 Problem Formulation *Given a plant (1) whose open loop dynamics (2) subject to $w(\cdot) = 0$ has a nonempty invariant set $\mathcal{M}$ (e.g. a collection of closed orbits and equilibria), pick a to be stabilized union $\mathcal{S}$ of some components of $\mathcal{M}$, and a $\gamma > 0$. Find, if possible, a nonempty compact set Ω containing $\mathcal{S}$, and a state feedback law (3) such that the closed loop L_2 gain (6) is less than or equal to γ, and such that the closed loop system (4) subject to $w(\cdot) = 0$ asymptotically stabilizes[1] the open loop invariant set $\mathcal{S}$.*

Find also a class of disturbances $\mathcal{W}^\epsilon$ such that the state trajectories never leave Ω if started inside some $\Omega^\epsilon \subset \Omega$, and such that all trajectories generated by $w(\cdot) \in \mathcal{W}^\epsilon$ are approaching the closed loop positive invariant set $\mathcal{S}$.

In other words: we want to solve a local $\mathcal{H}_\infty$ control problem in such a way that all trajectories are bounded inside some compact Ω, and that Ω is a basin of attraction for the to-be-stabilized, hence closed loop positive invariant set $\mathcal{S}$.

3 Background

3.1 Local $\mathcal{H}_\infty$ state feedback

The aim of standard local nonlinear $\mathcal{H}_\infty$ control is to design a controller (3), and to find a sufficient small $\gamma \geq 0$ such that the L_2 gain (6) is satisfied locally on a neighborhood

[1] The use of the expression "asymptotic stability" in this paper does not match the definition given in chapter 1. To be more precise, we show only boundedness of trajectories, and attraction of the set $\mathcal{S}$ here.

$\Omega \subset \mathbb{R}^n$ around the origin. In this subsection the equilibrium condition $X(0,0,0) = 0$ is assumed to hold.

It is known [vdS92a, vdS92c] that the local L_2 gain condition is implied by (equivalence is given subject a reachability condition [Wil72a]) the existence of a non-negative, bounded storage function $V : \Omega \mapsto [0, \infty[$ satisfying the dissipation inequality

$$\begin{aligned} V(x_T) - V(x_0) \;\leq\; \int_0^T (\gamma^2 \left|w(t)\right|^2 - \left|z(t)\right|^2)\, dt \;=\; \gamma^2 \left\|w\right\|_T^2 - \left\|z\right\|_T^2 \quad , \\ V(0) \;=\; 0 \quad , \end{aligned} \tag{7}$$

where $x_T = x(T)$. Whenever convenient we denote in the following the value of V along a given path $x(\cdot)$ by the abuse of notation $V(\cdot) = V(x(\cdot, t_0, x_0, u(\cdot), w(\cdot)))$.

In case that V is continuously differentiable almost everywhere, it satisfies the closed loop differential inequality

$$\mathsf{H}(u,w) \;\equiv\; \tfrac{d}{dt}V - (\gamma^2 \left|w\right|^2 - \left|z\right|^2) \;=\; \tfrac{\partial V}{\partial x} X(x,u,w) - \gamma^2 \left|w\right|^2 + \left|Z(x,u)\right|^2 \;\leq\; 0 \tag{8}$$

for all $t \in \mathbb{R}^+$, where the Hamiltonian function H is defined by equation (8). Assuming that $Z(x,u)$ is such that $\frac{\partial Z}{\partial u}(0,0)$ has rank m, it is known [IK95] that H has a unique saddle point $(w_{\max}, u_{\min})$ for all x and all $\frac{\partial V}{\partial x}$ near zero, and the extremal functions $u_{\min}(x, \frac{\partial V}{\partial x})$, $w_{\max}(x, \frac{\partial V}{\partial x})$ are characterized by the equations

$$\begin{aligned} \tfrac{\partial \mathsf{H}}{\partial u}(u_{\min}, w_{\max}) = 0 \\ \tfrac{\partial \mathsf{H}}{\partial w}(u_{\min}, w_{\max}) = 0 \end{aligned} \tag{9}$$

$$\begin{aligned} u_{\min}(0,0) = 0 \\ w_{\max}(0,0) = 0 \quad . \end{aligned} \tag{10}$$

Clearly, $a(x) = u_{\min}(x) \equiv u_{\min}(x, \frac{\partial V}{\partial x}(x))$ is the best possible state feedback law, and $w_{\max}(x) \equiv w_{\max}(x, \frac{\partial V}{\partial x}(x))$ is the worst possible disturbance. Note that $u_{\min}$ and $w_{\max}$ vanish at the origin, hence the autonomous closed loop systems

$$\begin{aligned} \dot{x} = X^*(x) \equiv X(x, u_{\min}(x), 0) \quad \text{and} \\ \dot{x} = X^{**}(x) \equiv X(x, u_{\min}(x), w_{\max}(x)) \end{aligned} \tag{11}$$

do preserve the equilibrium point 0.

Thus, we seek a sufficient small $\gamma \geq 0$, and a C^1 storage function V defined on a sufficiently large neighborhood Ω around the origin satisfying the Hamilton-Jacobi inequality [IK95]

$$\begin{aligned} &\mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) \\ &= \tfrac{\partial V}{\partial x} X(x, u_{\min}(x, \tfrac{\partial V}{\partial x}), w_{\max}(x, \tfrac{\partial V}{\partial x})) - \gamma^2 \left|w_{\max}(x, \tfrac{\partial V}{\partial x})\right|^2 + \left|Z(x, u_{\min}(x, \tfrac{\partial V}{\partial x})\right|^2 \\ &\qquad\qquad \leq\; 0 \quad \text{for all } x \in \Omega \quad . \end{aligned} \tag{12}$$

In case that the locally linearized problem is solvable, it can easily be seen that any $\gamma > \gamma_*$ can be used, where γ_* is some sub-optimal gain of the linearized $\mathcal{H}_\infty$ control problem. See van der Schaft [vdS91a, vdS92c] for further information.

The existence of a C^1 storage function satisfying (12) locally guarantees that the closed loop system is dissipative in the sense of (7), and the input-output map of the closed loop system has L_2 gain less than or equal to γ as defined in equation (6) *if and only if every closed loop state trajectory is bounded inside* Ω. Unfortunately, local theory does not give any a priori estimates on the boundedness of the state.

3.2 Set-stability

The basic idea of this paper is that the storage function V satisfying (12) shall serve as a Lyapunov function to determine the stability properties of the closed loop trajectories $x(\cdot)$ not only locally, but semi-globally.

For this purpose it is beneficial to recall boundedness and invariance properties of smooth autonomous systems of the form

$$\dot{x} = X(x) \quad . \tag{13}$$

We assume that the integral curves of (13) are uniquely given on some suitable set, and we denote them $x(\cdot) = x(\cdot, t_0, x_0)$.

3.1 Definition *A set $\mathcal{M} \subset \mathbb{R}^m$ is called **invariant** if all trajectories starting in $\mathcal{M}$ are defined in the future and in the past, and evolve entirely inside $\mathcal{M}$.*

*The set is called **positive invariant** if all trajectories starting in $\mathcal{M}$ are defined in the future and never leave $\mathcal{M}$ as time increases.*

Note that invariance is a stronger property of a set than positive invariance.

It is our purpose to use a formal solution to the Hamilton-Jacobi inequality as a Lyapunov function in order to establish semi-global stability properties of the $\mathcal{H}_\infty$ state feedback problem. Our theorem in the next section will be based on a result published in the early sixties by La Salle and Lefschetz [SL61].

3.2 Theorem (La Salle and Lefschetz) *Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^1 function and let Ω denote a connected component of the pre-image $V^{-1}(]-\infty, c])$, $c \in \mathbb{R}$. Assume that Ω is bounded, and that*

$$\tfrac{d}{dt} V \leq 0 \tag{14}$$

within Ω along any trajectory of the autonomous system (13). Let $\mathcal{R} \subset \Omega$ be the largest set where $\frac{d}{dt}V = 0$, and let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{R}$.

Then Ω is positive invariant and every solution in Ω tends to $\mathcal{M}$ as $t \to \infty$.

In other words: Ω is a basin of attraction for the stable invariant set $\mathcal{M}$. This is in fact a semi-global stability property of the type we want to establish for the $\mathcal{H}_\infty$ state feedback problem. Note that the original proof of Theorem 3.2 shows that any such C^1 function V satisfying $\frac{d}{dt}V \leq 0$ is not assumed to be positive definite. Every component of $\mathcal{M}$ is merely a local minimum of the function $V(x)$.

4 Set-stability in $\mathcal{H}_\infty$ control

This section contains the new contribution to the theory of regional (semi-global) stability and set-stability by $\mathcal{H}_\infty$ control. We want to modify Theorem 3.2 such that the property of set-stability can be used in $\mathcal{H}_\infty$ control. We have to use condition (12) instead of (14), thereby ensuring the L_2 gain (6) to hold.

In order to prove the boundedness of state trajectories we have to restrict ourselves to the class of disturbances

$$\mathcal{W}^\epsilon \equiv \left\{ w(\cdot) \in L_2(\mathbb{R}^+) \;\middle|\; \gamma^2 \left\|w\right\|_2^2 \leq \epsilon \right\} \quad . \tag{15}$$

Given some solution V to the standard $\mathcal{H}_\infty$ Hamilton-Jacobi inequality, the following new lemma will help us to construct some appropriate region of boundedness, denoted Ω, and the region of allowed initial conditions Ω^ϵ (see figure 1).

Given a formal C^1 solution V of the Hamilton-Jacobi inequality (12), pick some $c \in \mathbb{R}$ such that some connected component of the pre-image $V^{-1}(]-\infty, c])$, denoted Ω, is bounded. Since V is continuous it follows that Ω is closed, hence compact. Moreover (12) implies that V satisfies the dissipation inequality (7), therefore any trajectory $x(\cdot)$ with initial condition $x_0 \in \Omega$ subject to $w(\cdot) = 0$ fulfills

$$V(x_T) \leq V(x_0) - \left\|z\right\|_T^2 \leq c$$

for all $T \in \mathbb{R}^+$ (Note that then $\left\|z\right\|_T^2 \leq V(0) \leq c$ is always true). Therefore $x(T) \in \Omega$ for all $T \in \mathbb{R}^+$, and the trajectory can not leave Ω.

Now consider components of the sets $V^{-1}(]-\infty, c-\epsilon])$ with $\epsilon > 0$ which are subsets of Ω. These are clearly closed subsets of Ω, hence compact. Let $x(\cdot)$ be any closed loop trajectory with initial condition $x_0 \in \Omega^\epsilon \subset V^{-1}(]-\infty, c-\epsilon])$, and assume $w(\cdot) \in \mathcal{W}^\epsilon$. Then we have

$$V(x_T) \leq V(x_0) - \left\|z\right\|_T^2 + \gamma^2 \left\|w\right\|_T^2 \leq c - \epsilon + \epsilon$$

for all $T \in \mathbb{R}^+$ (Note that in this case $\left\|z\right\|_T^2 \leq V(0) + \epsilon \leq c$ is always true). We conclude that $x(\cdot)$ is bounded inside Ω. Formally we can restate our observations in the following lemma:

4.1 Lemma *[CMPP97] Given a formal C^1 solution V of the Hamilton-Jacobi inequality (12), pick some $c \in \mathbb{R}$ such that some component of $V^{-1}(]-\infty, c])$, denoted Ω, is connected and bounded.*

Then Ω is compact and closed loop positive invariant by use of the state feedback law $a(x) = u_{min}(x)$ subject to the condition $w(\cdot) = 0$.

Pick some $\epsilon > 0$, then the appropriate subset $\Omega^\epsilon \subset \Omega$ of $V^{-1}(]-\infty, c-\epsilon])$ is such that any closed loop trajectory $x(\cdot)$ with initial condition $x_0 \in \Omega^\epsilon$ is bounded inside Ω if driven by the state feedback law $a(x) = u_{min}(x)$, and by any disturbance $w(\cdot) \in \mathcal{W}^\epsilon$.

Note that the formal solution V may be such that the pre-image $V^{-1}(]-\infty, c])$ never has a bounded component, in which case the approach proposed here is not applicable. Moreover, picking $\epsilon \geq 0$ too large may result in $\Omega^\epsilon = \emptyset$.

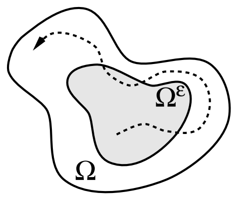

Figure 1: Boundedness of trajectories

Having taken care of the boundedness of state trajectories, we proceed the discussion leading to the new theorem, which will be the cornerstone of semi-global set-stability by $\mathcal{H}_\infty$ control.

Assume that the autonomous open loop system (2) subject to $w(\cdot) = 0$ has an invariant set $\mathcal{M} \subset \Omega$ consisting of a collection of disjoint components (for example periodic orbits and equilibrium points). If we wish to stabilize the motions on an invariant set $\mathcal{S} \subset \mathcal{M}$ consisting of some components of $\mathcal{M}$ without change of the motions on $\mathcal{S}$ (see figure 2), we have to use a feedback law $a(x)$ such that

$$a(x)|_{x \in \mathcal{S}} = 0 \quad .$$

In case that we want to destroy the open loop motions on $\mathcal{M}/\mathcal{S}$, we must have in addition that

$$a(x)|_{x \in \mathcal{M}/\mathcal{S}} \neq 0 \quad .$$

Moreover, in order to be able to fulfill the L_2 gain (6) for all desired motion of the open loop system (2), the performance measure $Z(x,u)$ must satisfy

$$Z(x,0)|_{x \in \mathcal{S}} = 0 \quad .$$

Observability of the state trajectory on $\mathcal{S}$, that is $Z(x,0)|_{x \in \Omega/\mathcal{S}} \neq 0$, may be too severe an assumption. Instead we will impose a weaker detectability assumption on the system:

4.2 Definition *Given some invariant set $\mathcal{S}$ of the open loop system (2) subject to $u(\cdot) = 0$, $w(\cdot) = 0$, the control system (1) is called* ***$\mathcal{S}$-detectable*** *if all bounded trajectories*

$$x(\cdot) = x(\cdot, t_0, x_0, u(\cdot), 0)$$

(subject to $w(\cdot) = 0$) generating the zero-output $z(\cdot) = 0$ are approaching $\mathcal{S}$ as $t \to \infty$.

In case that $\mathcal{S}$ is the origin, we say the control system is ***zero-detectable****.*

Assuming furthermore that $\frac{\partial Z}{\partial u}(x,0)$ has rank m for all $x \in \Omega$, a similar argumentation as in the paper [IK95] shows that H defined in (8) has a unique saddle point $(w_{\max}, u_{\min})$ for all x in Ω and all $\frac{\partial V}{\partial x}$ near zero, and the extremal functions $u_{\min}(x, \frac{\partial V}{\partial x})$, $w_{\max}(x, \frac{\partial V}{\partial x})$ are characterized by the equations (9) and

$$\begin{aligned} u_{\min}(x,0)|_{x \in \mathcal{S}} &= 0 \\ w_{\max}(x,0)|_{x \in \mathcal{S}} &= 0 \quad . \end{aligned} \tag{16}$$

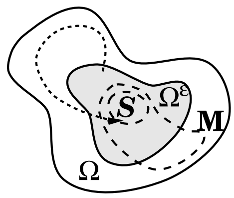

Figure 2: Set-stability

Hence following the principal idea of the paper [IK95] as outlined in Section 3.1, we conclude that any C^1 function V satisfying the Hamilton-Jacobi inequality (12) will also satisfy the dissipation inequality (7), and therefore the L_2 gain (6) in case that the state is bounded inside Ω. We take advantage of Lemma 4.1 to state the following theorem, and to follow the main idea of Theorem 3.2 to prove it.

4.3 Theorem *Assume that some C^1 solution $V : \Omega \mapsto \mathbb{R}$ of the Hamilton-Jacobi inequality (12) is defined on a bounded and connected component Ω of $V^{-1}(]-\infty, c])$, $c \in \mathbb{R}$. Assume furthermore that $\frac{\partial Z}{\partial u}(x,0)$ has rank m for all $x \in \Omega$*[2]*.*

Then all closed loop trajectories $x(\cdot)$ subject to $a(x) = u_{min}(x)$ with initial condition $x_0 \in \Omega^\epsilon$ do not leave Ω if driven by some $w(\cdot) \in \mathcal{W}^\epsilon$, and consequently the system has L_2 gain less than or equal to γ.

Moreover, all such $x(\cdot)$ generated by $w(\cdot) \in \mathcal{W}^\epsilon$ which are identically zero for all times $t > t^*$, $t^* \in \mathbb{R}$, approach the biggest closed loop invariant set $\mathcal{A}$ contained in the null set

$$\mathcal{N} \equiv \left\{ x \in \Omega \;\middle|\; \mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) = 0 \right\} \quad .$$

Assume furthermore that the control system (1) is $\mathcal{S}$-detectable, where $\mathcal{S}$ is a collection of components of the maximal open loop autonomous invariant set $\mathcal{M} \subset \Omega$, then $x(\cdot)$ approaches $\mathcal{S}$ as $t \to \infty$.

Proof. By Lemma 4.1 all state trajectories $x(\cdot)$ are bounded inside Ω. Therefore, as outlined in the discussion before Theorem 4.3, the dissipation inequality (7) and the L_2 gain (6) are satisfied for all trajectories.

We show now that all such $x(\cdot)$ generated by $w(\cdot) \in \mathcal{W}^\epsilon$ which are identically zero for all times $t > t^*$, $t^* \in \mathbb{R}$, approach the biggest closed loop invariant set $\mathcal{A}$ contained in the null set $\mathcal{N}$. By boundedness of state trajectories and time invariance of the system, we can assume without loss of generality that $w(\cdot) = 0$ for all $t \in \mathbb{R}^+$. Then the saddle point property defined by (9) and (16) implies that the C^1 solution V serves as a Lyapunov function for the closed loop dynamics. More explicitly, we have

$$\mathsf{H}(u_{\min}, w) \le \mathsf{H}(u_{\min}, w_{\min}) = \mathsf{H}^{**} \le 0 \tag{17}$$

[2] Again, the rank condition is necessary, but not sufficient. We must also assume that the saddlepoint property (9) holds for all $x \in \Omega$, which implies that $\mathsf{H}(u_{\min}, w) \le \mathsf{H}^{**} \le \mathsf{H}(u, w_{\max})$ is satisfied on Ω for all u and all w.

for all $w(\cdot) \in \mathcal{W}^\epsilon$. Choosing $w(\cdot) = 0$ then gives with (8)

$$\tfrac{d}{dt}V - \gamma^2 \left|0\right|^2 + \left|z\right|^2 \leq 0 \tag{18}$$

for all such trajectories. Hence we have $\frac{d}{dt}V < 0$ for all motions evolving on $\Omega/\mathcal{N}$. Trajectories on $\mathcal{N}$ are satisfying $\frac{d}{dt}V = 0$ if and only if $\left|z\right|^2 = \left|Z(x, u_{\min}(x)\right|^2 = 0$, and $\frac{d}{dt}V < 0$ else.

Now, observe that $V(x)$ by assumption is continuous and defined on the bounded set Ω, hence $V(x)$ is bounded from below. Given some particular state trajectory $x(\cdot)$, the storage function $V(\cdot)$ is decreasing and bounded from below, hence approaches some minimal value, say $c_\Gamma \in \mathbb{R}$, as $t \to \infty$. By continuity we conclude that $V(x) = c_\Gamma$ on the positive limit set Γ^+, and consequently $\frac{d}{dt}V = 0$ on Γ^+. Rearranging the inequalities (17) and (18) then shows that

$$0 \leq \left|z\right|^2 \leq \mathsf{H}^{**} \leq 0 \quad , \tag{19}$$

therefore we must conclude that Γ^+ is a (non-empty by boundedness of $x(\cdot)$) subset of the null set $\mathcal{N}$. But Γ^+ is an invariant set, hence contained in the maximal closed loop invariant set $\mathcal{A}$, and consequently any trajectory $x(\cdot)$ satisfying the conditions of the theorem are approaching $\mathcal{A}$ as $t \to \infty$.

We show finally that $\mathcal{S}$-detectability implies that $x(\cdot)$ approaches $\mathcal{S}$ as $t \to \infty$. Clearly any trajectory evolving entirely on Γ^+ satisfies by inequality (19) that $z(\cdot) = 0$, hence by $\mathcal{S}$-detectability $\mathcal{S}$ is approached. Finally, any trajectory with the same limit set Γ^+ is by continuity of the closed loop dynamics forced to approach $\mathcal{S}$ as $t \to \infty$.

Note, that in this case $\Gamma^+ \subset \mathcal{S}$, and that condition (16) shows that closing the loop with the feedback $a(x) = u_{\min}$ does not change the dynamics on the open loop invariant set $\mathcal{S}$.

Note too, that in the case that $\mathcal{S}$ is not connected (it may consist of several open loop positive limit sets for example), the proof indicates that each component of $\mathcal{S}$ is a local minimum of the function $V(x)$, but the constant value $V(x) = c_\Gamma$ will in general be different from component to component. In case that $\mathcal{S} = \{0\}$ we can always assume without loss of generality that $V(x)$ is positive definite. □

Following the proof of Theorem 4.3, it is clear that every connected component of $\mathcal{S}$ is a local minimum of any solution V of the Hamilton-Jacobi inequality (12), and that $\frac{\partial V}{\partial x}\dot{x} = 0$ along any trajectory evolving inside $\mathcal{S}$.

In case that $\mathcal{S} = \{0\}$, local solutions can be obtained by use of an approximation scheme originally developed by Lukes [Luk69] for quadratic cost functions. It has been used to compute solutions of the Hamilton-Jacobi inequalities associated with the local nonlinear state feedback $\mathcal{H}_\infty$ control problem [IK95]. An implementation in the symbolic language `MAPLE` is available for affine control systems [MP95], see [CMPP96] for a calculated example.

4.1 Extending the class of disturbances

From an engineering point of view, the theory so far developed is not yet entirely adequate for practical control purposes: in real systems the disturbance $w(\cdot)$ is often time persistent, and has therefore no finite L_2 norm. In linear $\mathcal{H}_\infty$ theory standard transformation results automatically translate the L_2 induced norm results into power semi-norm induced or spectral semi-norm induced equivalent results. This kind of equivalence does of course not hold for nonlinear systems.

In general the class of allowed disturbances $\mathcal{W}^\epsilon$ is not conservatively chosen as one might think. However, assuming that $\left|Z(x, u_{\min}(x)\right|$ is a function of class $\mathcal{K}_\infty$, and using the principal ideas of the input-to-state stability property as outlined in [Lin92, Son95c] together with the improvements on $\mathcal{H}_\infty$ control mentioned here, it is possible to allow for input and disturbance signals which are time persistent, but bounded in $\mathcal{L}_\infty$ norm (essentially peak bounded). The price to pay is that asymptotic stability of the invariant set $\mathcal{S}$ only is obtained for $w(\cdot) = 0$, but $\mathcal{L}_\infty$ boundedness of $w(\cdot)$ implies then that the state trajectories are bounded in a neighborhood of $\mathcal{S}$ and $x(\cdot) \to \mathcal{S}$ for $w(\cdot) \to 0$. The proof of a similar theorem involves decay estimates, and will be published later on.

5 Conclusion

In this paper it is shown that state feedback problems involving the stabilization of open loop invariant sets can successfully be recast as generalized formulations of nonlinear local state feedback $\mathcal{H}_\infty$ control problems. Given a formal solution V to a certain Hamilton-Jacobi inequality, the generalized problem is solved regionally (semi-globally) provided V is such that the some connected component of the pre-image $V^{-1}(]-\infty, c)$ for some $c \in \mathbb{R}$ is bounded and includes the to-be-stabilized invariant set. The plant is assumed to have a certain detectability property (which is just the generalization of the standard zero-detectability assumption) to prove asymptotic stability of the obtained control law with respect to the invariant set of concern. Sets of allowable initial conditions and disturbance classes are specified.

Hence, the presented results constitute a natural extension of local $\mathcal{H}_\infty$ control theory which possess most of the advantages of global nonlinear control. In particular, performance is guaranteed in a range of operational conditions, in contrast to local $\mathcal{H}_\infty$ control. Non-stationary modes of operation such as stability of periodic orbits are included in this new theory. Numerical methods which apply to local $\mathcal{H}_\infty$ theory can without problem be applied in a semi-global context.

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Comments and References

As mentioned in the introduction to this chapter, one deficiency of the theories presented so far is that theorem4.3 states only sufficient conditions to solve the problem formulation 2.1, no equivalent conditions in terms of partial differential inequalities are known so far. One of the problems in deriving equivalent formulations is the fact that storage functions in general not need to be C^1, nor even continuous. In fact, there are well-known problems where the existence of C^1 storage functions is not given.

To solve this problem, Joseph A. Ball and J. William Helton considered in 1996 the recently developed notion of viscosity solutions to Hamilton-Jacobi equations [BH96] (A more general result including the special case of $\mathcal{H}_\infty$ control has been published in 1993 by James [Jam93a]). While a general introduction to the theory of generalized gradients and viscosity solutions is out of the scope of this thesis, some of the high-lights of the theory, and it's application to $\mathcal{H}_\infty$ control, or general dissipative control, are sketched in chapter 5. We refer to the articles [CEL84, LS85, CIL90] for concise informations on this matter. The main idea is that the HJI can be understood in a certain weak sense, and then equivalence between weak, or viscosity solutions to HJI and existence of storage functions is proved.

The viscosity approach to $\mathcal{H}_\infty$ theory gives us equivalent formulations in terms of storage functions and weak solutions to HJI's, but the practical use of such an approach is rather limited: continuity alone is a weak regularity condition in case that we want to approximate the weak solution to the HJI numerically. There is hardly any fast working approximation scheme which works satisfactory with nonsmooth solutions.

Therefore, the ultimate goal is to impose some other system theoretic side conditions on the problem which can be checked considering the data of the system alone, and which establish equivalence between the existence of smooth storage functions and smooth solutions to - somehow modified - HJI's. This yet not established approach to $\mathcal{H}_\infty$ control would be a rather powerful theory, which facilitates the use of approximation schemes considerably.

Another important contribution to $\mathcal{H}_\infty$ control theory is the paper [BHW93] by Joseph A. Ball and J. William Helton and Michael L. Walker considering $\mathcal{H}_\infty$ control for general nonlinear systems with output feedback. Here, systems which are only affine in the disturbances are considered. Since the problem of output feedback is blurring the clear view towards set-stabilizing and robust dissipative control, we will not follow this path.

Finally, Declan G. Bates and Anthony M. Holohan compared the properties of linear $\mathcal{H}_\infty$, $\mathcal{H}_2$ and $\mathcal{L}_1$ controllers [BH95]. These results are not comparable to nonlinear related control strategies due to the lack of a frequency domain interpretation, but nevertheless, they give another important view on the properties of $\mathcal{H}_\infty$ controllers.

Chapter 4

Almost Autonomous Systems

In this chapter we provide a new tool which is used to assure certain robustness properties of systems with respect to non-zero disturbances. Although the main ideas have been published by L. Markus in 1956, they have not before been applied to robustness analysis of dissipative systems. Since the proof of the main theorem in Markus article is rather terse, and gives hardly any help understanding the asymptotic behavior of asymptotically autonomous systems, the author of the here reprinted paper

> Marc Cromme. On Asymptotic Behavior of Almost Asymptotic Systems. Submitted February 1998

had chosen to provide an alternative, detailed proof. The novelty is however not the improved level of accuracy in the proof, but a generalization of local asymptotic properties near simple asymptotically stable equilibrium points to regional (semi-global) asymptotic properties of general invariant, compact, sets. These compact and invariant sets have either to be asymptotically stable, or completely unstable in the sense that they are asymptotically stable under the time-reversed flow of the system.

This generalization provides a better tool in the context of nonlinear systems, and can easily be applied to the $\mathcal{H}_\infty$ control problem with set-stability derived in the previous chapter. The commentary section after the included paper describes such an robust set-stabilizing $\mathcal{H}_\infty$ control approach.

On asymptotic behavior of almost autonomous systems

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Keywords:
ODE's, non-linear dynamics, asymptotically autonomous systems, positive limit sets, invariant sets, asymptotically stable sets, completely unstable sets

Abstract

The qualitative behavior of solutions to asymptotically autonomous systems are investigated. These are time-dependent nonlinear ODE's which approach autonomous ODE's as time $t \to \infty$. The asymptotic properties of solutions near an invariant set are described, and the structure of the positive limit sets of solutions are investigated.

The findings of this paper are based on the main theorem of the paper "Asymptotically Autonomous Differential Systems" published by L. Markus in 1956, and frequently quoted (and sometimes misquoted) since. Unfortunately, the proof given there omits rather essential steps.

Therefore, before stating new results, a new proof to L. Markus main theorem is provided, which fills in the gaps of the original proof and gives a more descriptive understanding of the dynamical behavior of such systems. Then, a new theorem on semi-global stability of trajectories belonging to perturbed, asymptotically autonomous systems in relation to general positive limit sets of the limit autonomous system, is stated and proved.

1 Introduction

The motivation to write this paper arises from non-linear $\mathcal{H}_\infty$ control: there [CMPP97, CS97] the use of a minimizing state feedback control law results in a disturbed closed loop system of the form $\dot{x} = X(x, w)$, where the unpredictable and unknown vector signal $w(\cdot)$, called disturbance, perturbes the nominal, autonomous closed loop vector system $\dot{x} = X(x, 0)$. The control is usually designed such that the later system has some asymptotically

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stable equilibrium, or more generally, some asymptotically stable invariant set $\mathcal{S}$ describing the desired motion of trajectories $x(\cdot)$.

It is obvious that the invariant properties of $\mathcal{S}$ are usually destroyed when disturbances $w(\cdot) \neq 0$ are feed into the system. However, there might be the hope that the disturbed system is such that the invariant set $\mathcal{S}$ is approached if the disturbance decreases to zero for time $t \to \infty$. In fact, this is trivially true for asymptotically stable equilibrium points of linear systems. The central questions investigated in this paper are the following:

Under which circumstances does $w(t) \to 0$ for $t \to \infty$ imply $x(t) \to \mathcal{S}$ for general nonlinear disturbed systems?

When do the positive limit sets of the disturbed motions equal the positive limit sets of the autonomous system?

To answer these questions we use the notion of asymptotically autonomous systems covering a slightly broader class of systems than our systems perturbed by an disturbance. In 1956 L. Markus [Mar56] answered our questions partially, namely in the case of asymptotically stable equilibrium points of nonlinear asymptotically autonomous systems. Unfortunately, the proof given in [Mar56] omits rater essential steps, which may be the reason that his main theorem has occasionally been misunderstood and misquoted during the last forty years.

The more elaborated findings of L. Markus paper on essentially periodic orbits of asymptotically autonomous systems, and on behavior of two-dimensional asymptotically autonomous systems of Bendixon-Poincaré type, have recently triggered further research in this interesting area. The work of H.R. Thieme [Thi92, Thi94a, Thi94b] on a Bendixon-Poincaré type limit set trichotomerty for planar systems must be mentioned in this context. While these papers investigate the fine structure of positive limit sets associated to trajectories of asymptotically autonomous systems, yet no stability results for other sets than locally stable equilibria have been provided.

To solve our problem at hand a more descriptive understanding of the dynamical behavior of solutions is needed. Therefore this paper re-proves the main theorem of [Mar56] in a more detailed fashion, filling the gaps of the original proof, before treating the behavior of state trajectories near asymptotically stable invariant sets, or completely unstable invariant sets. Finally, the positive limit sets of disturbed motions are proved to equal the positive limit sets of autonomous systems under certain circumstances. Three simple examples are shown to stress the findings of this paper.

2 Basics

Let $|\cdot|$ denote the usual Euclidean vector norm (or any other equivalent norm) on $\mathbb{R}^n$. Given any closed subset $\mathcal{A} \subset \mathbb{R}^n$, we define the distance between $\mathcal{A}$ and some point $p \in \mathbb{R}^n$ by

$$|p|_{\mathcal{A}} \equiv \min_{q \in \mathcal{A}} |p - q| \quad .$$

Note that the usual Euclidean vector norm then is given by $|\cdot| = |\cdot|_{\{0\}}$. Let $\mathbb{R}^+$ denote the future, that is the nonnegative real axis $[0, \infty)$, and $\mathbb{R}^- \equiv (-\infty, 0]$ the past.

A perturbed, or **disturbed system** is a system of the form

$$\dot{x} = X(x, w) \quad , \tag{1}$$

where $x(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is called the **state**, and $w(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^l$ the exogenous input, also called **disturbance**. The symbol $X(x, w)$ denotes a continuous vector field $X : \mathbb{R}^n \times \mathbb{R}^l \mapsto \mathbb{R}^n$, which is locally Lipschitz in x, uniformly in w. More precisely, for each compact set $\mathcal{K} \subset \mathbb{R}^n$ there is a constant $k > 0$ such that

$$|X(x_1, w) - X(x_2, w)| \leq k|x_1 - x_2|$$

for all $x_1,\ x_2 \in \mathcal{K}$ and all $w \in \mathbb{R}^l$. In case that the local Lipschitz condition can not be satisfied but on some subset $\mathcal{X} \times \mathcal{W}$, the following will remain obviously true whenever the state and the disturbance can be bounded inside the compact set $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^l$.

Applying the theory of ordinary differential equations [BN69, chap. 3] [CL55, chap. 1] on systems of the form (1), it is easily seen that these conditions ensure the existence and uniqueness of state solutions whenever the disturbance is a continuous signal. We use the notation $x(\cdot)$ for the unique signal $x(\cdot, t_0, x_0, w(\cdot))$ generated by the input $w(\cdot)$, where the initial condition at time t_0 is x_0. We denote the value of a signal $x(\cdot) : \mathcal{I} \mapsto \mathbb{R}^n$ at time t by $x(t) \in \mathbb{R}^n$. Here $\mathcal{I} \subset \mathbb{R}$ is the maximal time interval where the signal $x(\cdot)$ is defined. Some point in $\mathbb{R}^n$ is denoted x, and in particular the initial point of some given signal $x(\cdot)$ at initial time t_0 is defined by $x_0 \equiv x(t_0)$.

It is assumed that all signals are $\mathcal{L}_2^{\text{loc}}$, that is, the integral $\int_a^b |y(t)|^2\, dt$ is finite for all $a, b \in \mathcal{I} \subset \mathbb{R}$. Moreover, in case that the disturbance is not a continuous signal, we assume further that the state exist uniquely for all disturbances of concern, and is a C^1 signal but on a set of measure zero. In the following it will be important that the state signal is a continuous function of the triplet (t, t_0, x_0), and the signal space of disturbances must be chosen accordingly to this requirement.

One important class of disturbed systems satisfying the later assumptions is given by the requirements that the disturbance is bounded and piecewise continuous, and $\frac{\partial X}{\partial x}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^l$. Then we can show using the techniques of [CL55, chap. 1 & 2] that the state is a continuous function of the triplet (t, t_0, x_0) which is continuously differentiable on some neighborhood of (t_0, x_0) for each fixed t, and piecewise continuously differentiable in t for all fixed (t_0, x_0).

Other interesting classes of problems meeting the above assumptions may be found using Caratheodory theory [CL55, chap. 2], but for simplicity we will not follow this path.

Given some unique disturbance signal $w(\cdot)$, the disturbed system (1) can be regarded as a time variant differential system, and, in case that the disturbance is constant in time, as an autonomous system. In both cases, we say that the state is **defined in the future** if a solution $x(\cdot, t_0, x_0, w(\cdot))$ exists for all $t \geq t_0$, it is **defined in the past** if it exists for all $t \leq t_0$.

We say that the state is **bounded in the future (bounded in the past, bounded)** if it is defined in the future (past, future and past) and satisfies $|x(t)| \le k,\ k \ge 0$ in the future (past, future and past). In case that the state is not defined in the future, it has a finite escape time $T \in \mathbb{R}$, and $|x(t)| \to \infty$ as $t \to T$ from below.

We say that some given trajectory $\boldsymbol{x(\cdot)}$ **defined in future approaches** $\boldsymbol{\mathcal{S}}$ **as** $\boldsymbol{t \to \infty}$, denoted $\boldsymbol{x(t) \to \mathcal{S}}$, if $|x(t)|_{\mathcal{S}} \to 0$ as $t \to \infty$.

Given any compact set $\mathcal{S} \subset \mathbb{R}^n$, we define the family of open neighborhoods

$$\mathcal{N}_\varepsilon \equiv \left\{ p \in \mathbb{R}^n \;\middle|\; |p|_{\mathcal{S}} < \varepsilon \right\} \quad ,$$

for all $0 < \varepsilon \le \varepsilon_0$, where $\varepsilon_0 > 0$ is some given constant.

The autonomous system given by some constant disturbance $w(\cdot) = c$ has an **stable set** $\boldsymbol{\mathcal{S}}$ if for each $0 < \varepsilon \le \varepsilon_0$ there is a $\delta > 0$ such that $x_0 \in \mathcal{N}_\delta$ implies that $x(t) \in \mathcal{N}_\varepsilon$ for all $t \ge 0$. The set $\mathcal{S}$ is called **asymptotically stable** if in addition $x(t) \to \mathcal{S}$ as $t \to \infty$ holds for all $x_0 \in \mathcal{N}_{\varepsilon_0}$.

The **basin of attraction** $\boldsymbol{\mathcal{A}^+}$ of some asymptotically stable set $\mathcal{S}$ is the largest set of initial points such that $\mathcal{S}$ is approached. It can be found by backwards integration of state trajectories from some suitable neighborhood of $\mathcal{S}$, and will always be an open set.

The autonomous system given by some constant disturbance $w(\cdot) = c$ has a **completely unstable set** $\boldsymbol{\mathcal{S}}$ if $\mathcal{S}$ is an asymptotically stable set for the time-inverted system $\dot{x} = -X(x, c)$.

The **domain of repulsion** $\boldsymbol{\mathcal{R}^+}$ of some completely unstable set $\mathcal{S}$ is the largest set of initial points such that $\mathcal{S}$ is repelled, that is, the basin of attraction of the time inverted system. It can be found by forward integration of state trajectories from some suitable neighborhood of $\mathcal{S}$, and will always be an open set.

The **positive limit set** $\boldsymbol{\Gamma^+}$ of some trajectory $x(\cdot)$ is - intuitively spoken - the set a state defined in the future tends to as $t \to \infty$. If the state approaches an equilibrium point or a limit circle, those are the positive limit sets. More formally, x_+ belongs to the positive limit set of a bounded state trajectory $x(\cdot)$ if there exists a sequence of time $\{t_n\}$ with $t_n \to \infty$ such that $x(t_n) \to x_+$ as $n \to \infty$.

A set $\mathcal{S} \subset \mathbb{R}^n$ is called **invariant with respect to a constant** $\boldsymbol{w(\cdot) = c}$, if for any $p \in \mathcal{S}$ the trajectory through p is defined in past and future, and is entirely lying in $\mathcal{S}$. This implies that the boundary $\partial\mathcal{S}$ consists of state trajectories. For example, every equilibrium point, every closed and bounded periodic orbit, and every collection of trajectories defined both in future and past are invariant sets. Note that invariant sets are not defined for time varying systems.

It is known that the limit sets Γ^+ of bounded trajectories generated by time varying systems of the form $\dot{x} = X(x, t)$ are nonempty and compact, and $x(\cdot) \to \Gamma^+$ as $t \to \infty$ [BN69, chap. 5]. In case that the system of concern is autonomous, that is of form $\dot{x} = X(x)$, the positive limit set will be invariant [Yos66, chap. 3] [Kha96, chap. 3].

3 Asymptotically autonomous systems

Let us in the following investigate the qualitative behavior of disturbed systems of the form $\dot{x} = X(x, w)$ subject to $\mathcal{L}_2$ disturbances which are bounded, piecewise continuous in time, and converging to zero, that is $|w(t)| \to 0$ as $t \to \infty$. We are inclined to believe that all bounded trajectories $x(\cdot)$ generated in this way have nonempty and compact positive limit sets Γ^+ which are matching the positive limit sets Γ^+_∞ belonging to the bounded trajectories $x_\infty(\cdot)$ of the autonomous system $\dot{x} = X(x, 0)$. Unfortunately, this assert is not quite accurate, see [Thi94a] for illustrative counterexamples. A more sophisticated approach than intuition is needed to understand the qualitative behavior of time varying systems which approach autonomous systems as $t \to \infty$. We seek inspiration in the work of L. Markus [Mar56] and Yoshizawa [Yos66, chap. 3] to generalize their results to a broader class of positive limit sets than stable equilibria. Furthermore, we allow for piecewise continuity in the time variable to accommodate the previous mentioned class of systems with piecewise continuous disturbance signals.

3.1 Definition (Asymptotically autonomous system) *Let $\Sigma : \ \dot{x} = X(x, t)$ and $\Sigma_\infty : \ \dot{x} = X_\infty(x)$ be continuous and locally Lipschitz in x for all fixed $t \in \mathbb{R}^+$, and piecewise continuous in t for all fixed $x \in \mathbb{R}^n$. We say that **Σ is asymptotic to Σ_∞**, denoted **$X(x, t) \to X_\infty(x)$**, in case for each compact $\mathcal{K} \subset \mathbb{R}^n$ and each $\varepsilon > 0$ there is a $T(\mathcal{K}, \varepsilon) \in \mathbb{R}^+$ such that*

$$|X(x, t) - X_\infty(x)| < \varepsilon$$

for all $x \in \mathcal{K}$ and all $t \geq T(\mathcal{K}, \varepsilon)$.

It is easy to tell when disturbed systems of the form (1) are asymptotically autonomous systems.

3.2 Proposition *Given a disturbed system of the form (1), assume that the disturbance $w(\cdot) \in \mathcal{L}_2^{loc}$ is bounded, piecewise continuous, and decreasing to zero, that is satisfying $|w(t)| \to 0$ as $t \to \infty$.*

Then the system $\dot{x} = X(x, w(t))$ is asymptotic to $\dot{x} = X(x, 0)$.

Proof: Clearly $X_\infty(x) = X(x, 0)$ is continuous and locally Lipschitz by the same properties of $X(x, w)$. On any compact set $\mathcal{K} \in \mathbb{R}^n$ there is a $\delta(\mathcal{K}, \varepsilon)$ such that $|X(x, w) - X(x, 0)| \leq \varepsilon$ for $|w| \leq \delta$ (again by smoothness of $X(x, w)$). Finally, by convergence of the disturbance there is a suitable $T(\mathcal{K}, \varepsilon) = T(\mathcal{K}, \delta(\mathcal{K}, \varepsilon))$ satisfying $|X(x, w(t)) - X(x, 0)| \leq \varepsilon$ for $t \geq T$. □

It seems reasonably to assume that the trajectories of the time variant and asymptotic autonomous system behave similar to the trajectories of the autonomous system as time goes to infinity. L. Markus [Mar56] states that the positive limit set Γ^+ of a trajectory of the asymptotic autonomous system Σ (also called the perturbed system) consists of a union of autonomous trajectories, and this result is repeated by Yoshizawa [Yos66, chap. 3]. More formally we have the following theorem:

3.3 Theorem *[Mar56] Let the system $\Sigma: \dot{x} = X(x,t)$ be asymptotic to $\Sigma_\infty: \dot{x} = X_\infty(x)$, and denote their trajectories $x(\cdot)$ and $x_\infty(\cdot)$ respectively. Then all $x(\cdot)$ which are bounded in future have a non-empty and compact positive limit set Γ^+, and $x(t) \to \Gamma^+$. Moreover, Γ^+ consists of a union of autonomous trajectories $x_\infty(\cdot)$, that is, Γ^+ is an invariant set of the autonomous system Σ_∞.*

3.4 Example: Duffing's equation
The dynamical system

$$\ddot{v} + v - \epsilon v^3 = 0$$

is conservative with potential energy $V(x) = \frac{1}{2}v^2 - \frac{1}{4}\epsilon x^4$. It can be written in the standard form

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_1 - \epsilon x_1^3 \quad . \end{aligned} \tag{2}$$

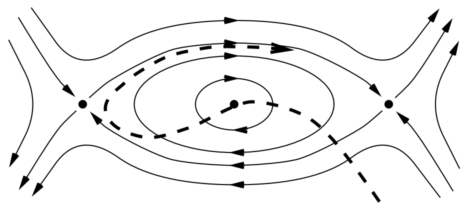

Figure 1: System which is asymptotically to Duffing's equation

The orbits of (2) are given by the level curves $V(x) = E$. For $\epsilon = \frac{1}{4} > 0$ the potential has a minimum at $x_1 = x_2 = 0$ and two maxima at $x_1 = \pm\sqrt{\frac{1}{\epsilon}} = \pm 2$, $x_2 = 0$. These are the location of one center and two saddle points, see figure 1.

Assume that some system $\dot{x} = X(x,t)$ is asymptotically autonomous to (2), then the positive limit set Γ^+ of some disturbed trajectory $x(\cdot)$ (shown as a dashed arrow in figure 1) might be the saddle connection given by $V(x) = 1$, that is the union of the two saddle points and the two connecting trajectories. In fact, it suffices to apply a time dependent vector field in a compact set containing a part of one of the connecting trajectories, which is oriented orthogonal to the autonomous trajectories and forces the perturbed trajectory towards the connecting trajectory. To prevent "overshoot" this orthogonal vector field must decay to zero in an appropriate way.

Note that all trajectories in Γ^+ are defined in future and past. The positive limit set Γ^+ consists according to theorem 3.3 of a union of several different autonomous trajectories defined in future and past, but a few of them being positive limit sets of the autonomous system. ⋆

The original proof given in [Mar56] on the part concerning the invariantness of Γ^+ is as follows:

From the uniform convergence of Σ to Σ_∞ on compact subsets of $\mathbb{R}^n$, it is clear that $x_\infty(\cdot) \subset \Gamma^+$, where $x_\infty(\cdot)$ is the solution of Σ_∞ through $p \in \Gamma^+$. Thus, Γ^+ is the union of solutions of Σ_∞.

Since the above proof is *very* short, the principal structure of the problem at hand is not easily seen. The curious reader might ask the following questions:

I: How comes that the existence of a perturbed trajectory bounded in future of the system Σ, that is, the existence of a compact positive limit set Γ^+, implies the existence of autonomous trajectories belonging to Σ_∞ which are bounded in *past and future*?

II: Is the positive limit set Γ^+ of $x(\cdot)$ belonging to Σ a union of positive limit sets $\cup\Gamma^+_\infty$ of trajectories belonging to Σ_∞ ?

III: Does the equality $\Gamma^+ = \Gamma^+_\infty$ hold?

The last properties have been the source of misunderstandings and misquotings, these assertions are not fulfilled in general. Illustrative counterexamples can be found in [Thi94a]. On the other hand, L. Markus states a result where the equality asked for in question III holds:

3.5 Theorem *[Mar56] Let x_* be a asymptotically stable equilibrium of Σ_∞. Then there exists a (sufficient small) neighborhood $\mathcal{N}$ of x_* and a time T such that $\Gamma^+ = \{x_*\}$ for all solutions $x(\cdot)$ of Σ intersecting $\mathcal{N}$ at a time later than T.*

Also here we are inclined to ask curious questions:

IV: Does theorem 3.5 hold for other positive limit sets of the autonomous system Σ_∞ than equilibria?

V: How large can the neighborhood $\mathcal{N}$ be?

VI: Does theorem 3.5 hold for completely unstable equilibrium points?

We prove in this paper that these questions have positive answers. Indeed, the following theorem holds:

3.6 Theorem *Let the system $\dot{x} = X(x,t)$ be asymptotic to $\dot{x} = X_\infty(x)$, and denote their trajectories $x(\cdot)$ and $x_\infty(\cdot)$ respectively. Let $\Omega \subset \mathbb{R}^n$ be a compact set. Assume that Γ^+_∞ is some positive limit set of the autonomous system Σ_∞ which is either asymptotically stable or completely unstable, and let $\mathcal{A}^+_\infty$ denote the basin of attraction, or $\mathcal{R}^+_\infty$ the domain of repulsion, under the dynamics of Σ_∞. Assume that the proper inclusions $\Gamma^+ \subset \Omega \subset \mathcal{A}^+_\infty$ or $\Gamma^+ \subset \Omega \subset \mathcal{R}^+_\infty$ hold.*

Then any trajectory $x(\cdot)$ which enters Ω in some finite time T, and stays in Ω for all times $t \geq T$, approaches Γ^+_∞ as $t \to \infty$. Hence $\Gamma^+ \subset \Gamma^+_\infty$.

Moreover, if Γ^+_∞ equals an equilibrium point or a periodic orbit, the equality $\Gamma^+ = \Gamma^+_\infty$ holds.

3.7 Example: Van der Pol equation
The system

$$\ddot{v} + \epsilon(v^2 - 1)\dot{v} + v = 0 \quad , \quad \epsilon > 0$$

is transformed by the Lienard transformation

$$x_1 = v \quad , \quad x_2 = \dot{v} - \epsilon(v - \frac{1}{3}v^3)$$

into the standard form

$$\begin{aligned} \dot{x_1} &= x_2 + \epsilon(x_1 - \frac{1}{3}x_1^3) \\ \dot{x_2} &= -x_1 \quad . \end{aligned} \tag{3}$$

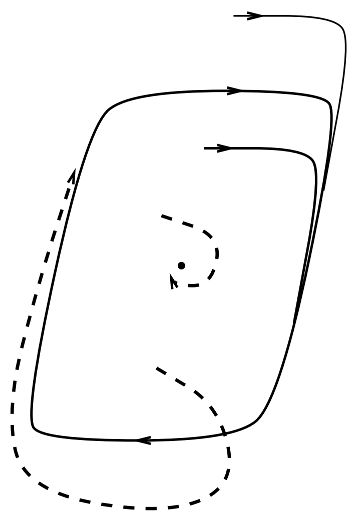

Figure 2: System which is asymptotically to Van der Pol equation

It is known [Gri90] that (3) has a unique limit cycle which is almost global asymptotically orbital stable, that is, the limit cycle C^+ is an asymptotically stable limit set Γ^+_∞ with basin of attraction $\mathcal{A}^+_\infty = \mathbb{R}^2 \setminus \{(0,0)\}$. Moreover, the completely unstable equilibrium point $(0,0)$ has a domain of repulsion $\mathcal{R}^+_\infty$ which equals the interior of C^+.

Assume now that there is some system $\dot{x} = X(x,t)$ which is asymptotically autonomous to (3). By almost global asymptotically orbital stability of C^+ it follows that any perturbed trajectory $x(\cdot)$ which is defined in the future also is bounded in the future, all unbounded perturbed trajectories have finite escape time. We apply now theorem 3.6.

Hence any such bounded perturbed trajectory $x(\cdot)$ which stays away from the equilibrium point at the origin has the asymptotically stable set C^+ as positive limit set. In this case we have the identity $\Gamma^+ = C^+$.

However, there might be some perturbed trajectory $x(\cdot)$ bounded for all times in a compact set laying inside the periodic cycle C^+, and then Γ^+ equals the completely unstable positive limit set $\{(0,0)\}$. $\star$

4 The proofs

We proceed here to display two essential observations in the following lemmas before reproving the main theorem of [Mar56]:

Given a perturbed trajectory $x(\cdot)$, assume we start an autonomous trajectory $x_\infty(\cdot)$ at time T in the point $x_T = x(T)$, and that we let both progress a finite time, ΔT, say. If both trajectories are defined on the closed time interval $[T, T + \Delta T]$, it is clear that they are bounded, hence the distance $|x(T + \Delta T) - x_\infty(T + \Delta T)|$ is finite. The essential observation is now that this distance can be made arbitrarily small for both positive and negative finite time increments ΔT considering a sufficient large T.

4.1 Lemma *Let the perturbed system $\dot{x} = X(x,t)$ be asymptotic to $\dot{x} = X_\infty(x)$, and assume that some specific perturbed trajectory $x(\cdot)$ is bounded in the future. Denote the family of autonomous trajectories starting at time T in $x(T)$ by $x_\infty^T(\cdot)$.*

Then for any fixed $\Delta T \in \mathbb{R}$ such that the members of $x_\infty^T(\cdot)$ are defined on $[T, T + \Delta T]$ (or on $[T + \Delta T, T]$ for negative increments) we have

$$|x(T + \Delta T) - x_\infty^T(T + \Delta T)| \to 0 \quad \text{as} \quad T \to \infty \quad .$$

Proof: We see from definition 3.1 that the vector field $X(x,t)$ is closely approximated by the autonomous vector field $X_\infty(x)$ at all $x \in \mathcal{K}$, as $t \to \infty$. Assume now that a given time increment ΔT is positive, then there is a finite time $T(\varepsilon) > 0$ such that

$$|X(x,t) - X_\infty(x)| < \varepsilon$$

on any compact set. It follows that the distance satisfies

$$|x(T + \Delta T) - x_\infty^T(T + \Delta T)| \le \int_T^{T+\Delta T} |X(x(x_*, s), s) - X_\infty(x(x_*, s))|\, ds < \delta \qquad (4)$$

along the perturbed trajectory $x(x_*, t)$ with initial point $x(T) = x_*$, and this holds for all $\delta = \varepsilon \Delta T > 0$. Now, by definition there exists a $T(\varepsilon) > 0$ for each $\varepsilon > 0$, hence it follows that there exists a $T(\frac{\delta}{\Delta T}) = T(\varepsilon) > 0$ for all $\delta > 0$, and clearly δ can be chosen to satisfy $\delta \to 0$ for all fixed ΔT and $T \to \infty$.

The case of a negative time increment $\Delta T \le 0$ is shown analogously considering the integral $\int_{T+\Delta T}^{T} ds$. □

In the course of proving theorem 3.3 we want to use lemma 4.1 to construct sequences of points $x_\infty^i \equiv x_\infty^{T_i}(T_i + \Delta T)$ which hopefully converge to a point in the compact limit set Γ^+. There is still one observation needed to follow this path: The positive limit set is compact, hence the union of autonomous trajectories considered in theorem 3.3 is bounded. It follows that each member of this union is defined on $\mathbb{R}$. On the other hand, the family of autonomous trajectories used in lemma 4.1 is defined on $[T, T + \Delta T]$ (or on $[T + \Delta T, T]$ for negative increments), therefore we have yet no indication that the maximal interval of definition is large enough to make the above mentioned approach work.

In general we can not show that all members of the family of autonomous trajectories are defined in past and future, but less can do the job: We are able to show that the members of the family $x_\infty^T(\cdot)$ have a maximal interval of definition which approaches $\mathbb{R}$ as $T \to \infty$.

4.2 Lemma *Assume that the conditions of lemma 4.1 hold. Then the members of* $x_\infty^T(\cdot)$ *are defined on* $(a(T), b(T))$, *where*

$$(a(T), b(T)) \to \mathbb{R} \ \text{ as } \ T \to \infty \quad .$$

Proof: Define a tube of diameter $\delta > 0$ around the perturbed trajectory $x(\cdot)$ by

$$\mathcal{T}_\delta \equiv \left\{ x \in \mathbb{R}^n \;\middle|\; \min_{t \in \mathbb{R}^+} |x(t) - x| \leq \delta \right\} \quad .$$

The tube $\mathcal{T}_\delta$ is a compact set because $x(\cdot)$ is bounded in future. Since the the members of the family $x_\infty^T(\cdot)$ are trajectories of an autonomous system, we can without loss of generality redefine their time axis such that the initial point x_T is met at time $t = 0$, that is $x_\infty^T(0) = x_T$. Hence we know that the members have a maximal interval of definition, $(a(T), b(T))$ say, which includes 0.

Following the proof of lemma 4.1 it is evident that the members of $x_\infty^T(\cdot)$ are bounded inside $\mathcal{T}_\delta$ on $[A(T), B(T)]$ say. Clearly, by boundedness in $\mathcal{T}_\delta$, it follows that the maximal interval of definition of the family members satisfies $[A(T), B(T)] \subset (a(T), b(T))$.

Finally, we show that $A(T) \to -\infty$, $B(T) \to \infty$ for $T \to \infty$, hence $(a(T), b(T)) \to \mathbb{R}$ as $T \to \infty$: By inequality (4) there is a sequence of times $T_0 \leq T_1 \leq \cdots \leq T_i$ such that

$$|x(T_i + \Delta T) - x_\infty^{T_i}(\Delta T)| \leq \frac{\delta}{2^i}$$

for all finite $\Delta T \in [A(T_0), B(T_0)]$. Since $x_\infty^{T_i}(\Delta T)$ are bounded in $\mathcal{T}_{\frac{\delta}{2^i}}$, it follows that $x_\infty^{T_i}(\cdot)$ are bounded in $\mathcal{T}_\delta$ for a larger time interval than $[A(T_0), B(T_0)]$. More precisely, the sequence $T_0 \leq T_1 \leq \cdots \leq T_i$ can be chosen such that

$$|X(x, t) - X_\infty(x)| < \frac{\varepsilon}{2^i}$$

for all $t \geq T_i$. Hence $x_\infty^{T_i}$ is bounded inside $\mathcal{T}_\delta$ at least for all $t \in [2^i A(T_0), 2^i B(T_0)]$. □

Proof of Theorem 3.3: We remember that all $x(\cdot)$ which are bounded have a non-empty and compact limit set Γ^+ [Yos66, chap. 3].

Take any $x_0 \in \Gamma^+$, and progress this point any time $\Delta T \in \mathcal{I}$ along the autonomous trajectory with initial point $x_\infty(0) = x_0$. By lemma 4.2 $\mathbb{R}$ is the maximal interval of definition of this particular autonomous trajectory. We show that $x_{\Delta T} \equiv x_\infty(\Delta T) \in \Gamma^+$:

By definition of positive limit sets there is a sequence $T_i \to \infty$ such that $x(T_i) \to x_0$ as $i \to \infty$, where $x(\cdot)$ is the perturbed trajectory of concern. We are now investigating the sequence $x(T_i + \Delta T)$ as $i \to \infty$, hopefully it has the limit $x_\infty(\Delta T)$. The collection

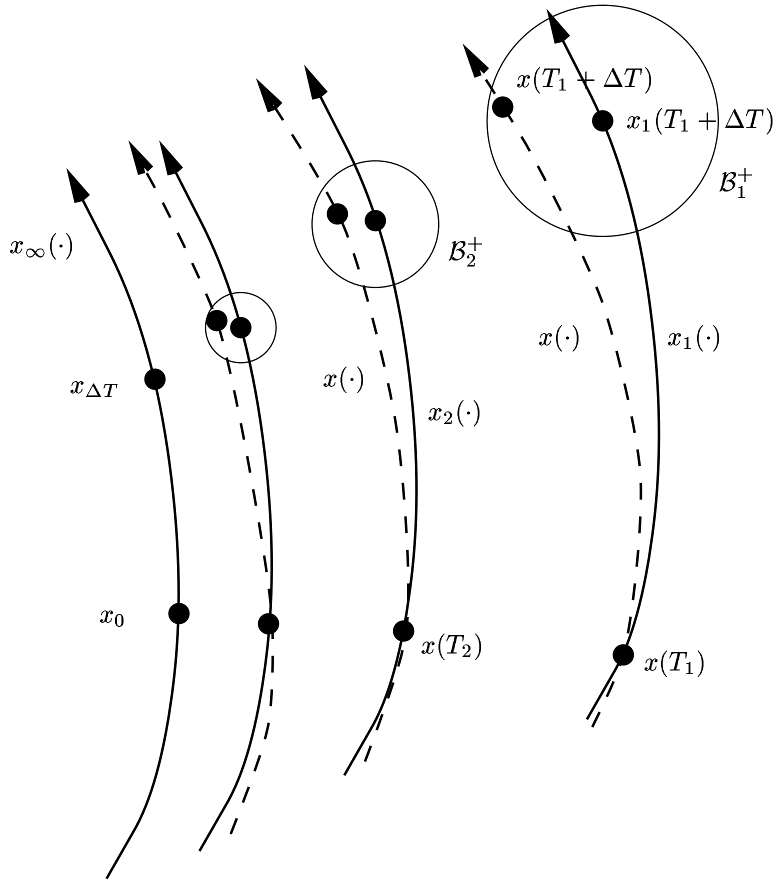

Figure 3: Convergence of $x(T_i + \Delta T) \to x_{\Delta T}$

of dashed arrows in figure 3 symbolizes different pieces of the perturbed trajectory $x(\cdot)$, namely the pieces given by the time intervals $(T_i, T_i + \Delta T)$, or $(T_i + \Delta T, T_i)$ in case that $\Delta T < 0$. Denote the family of autonomous trajectories with initial point $x(T_i)$ at $t = T_i$ by $x_\infty^i(\cdot)$.

By lemma 4.2 we can assume without loss of generality that the autonomous family $x_\infty^i(\cdot)$ is defined on $(T_i, T_i + \Delta T)$ (or $(T_i + \Delta T, T_i)$ in case that $\Delta T < 0$) for all finite $\Delta T \in \mathbb{R}$.

From lemma 4.1 we know that the point $x(T_i + \Delta T)$ satisfies

$$|x(T_i + \Delta T) - x_\infty^i(T_i + \Delta T)| \to 0$$

as $T_i \to \infty$, hence as $i \to \infty$. Moreover, it follows from continuity of autonomous trajectories on their initial conditions that

$$|x_\infty^i(T_i + \Delta T) - x_\infty(\Delta T)| \to 0$$

as $T_i \to \infty$.

Consequently, $x(T_i + \Delta T) \to x_\infty(\Delta T)$ as $i \to \infty$. By definition $\lim_{i\to\infty} x(T_i + \Delta T)$ belongs to Γ^+. We conclude that all points of the autonomous trajectory with initial point $x_0 \in \Gamma^+$ belong to Γ^+.

We have showed that $x_\infty(t) \in \Gamma^+$ for all $\Delta T \in \mathbb{R}$. Hence we have the final conclusion: Γ^+ is a union of autonomous trajectories $x_\infty(\cdot)$ defined in future and past. $\square$

Even if the components of the positive limit set Γ^+ are not necessarily positive limit sets of the autonomous system, there are some connections between them. The following theorem is a generalization of Theorem 2 obtained by L. Markus [Mar56], considering the stability properties of $x(\cdot)$ near an asymptotically stable invariant set, or a completely unstable invariant set, of Σ_∞.

4.3 Theorem *Let the system $\dot{x} = X(x,t)$ be asymptotic to $\dot{x} = X_\infty(x)$, and denote their trajectories $x(\cdot)$ and $x_\infty(\cdot)$ respectively. Let $\Omega \subset \mathbb{R}^n$ be a compact set. Assume that $\mathcal{S}$ is some compact invariant set of the autonomous system Σ_∞ which is either asymptotically stable or completely unstable, and let $\mathcal{A}^+_\infty$ denote the basin of attraction, or $\mathcal{R}^+_\infty$ the domain of repulsion, under the dynamics of Σ_∞. Assume that the proper inclusions $\mathcal{S} \subset \Omega \subset \mathcal{A}^+_\infty$ or $\mathcal{S} \subset \Omega \subset \mathcal{R}^+_\infty$ hold.*

Then any trajectory $x(\cdot)$ which enters Ω in some finite time T, and stays in Ω for all times $t \geq T$, approaches $\mathcal{S}$ as $t \to \infty$. Hence $\Gamma^+ \subset \mathcal{S}$.

Proof: By definition $\mathcal{A}^+_\infty$ and $\mathcal{R}^+_\infty$ are open sets while Ω is closed. It follows that there is a minimal distance $\varepsilon > 0$ between the boundary of $\mathcal{A}^+_\infty$ or $\mathcal{R}^+_\infty$ and Ω. By definition of the basin of attraction, or the domain of repulsion, it follows that all trajectories going from $\partial\mathcal{A}^+_\infty$ to $\partial\mathcal{S}$, or conversely, from $\partial\mathcal{S}$ to $\partial\mathcal{R}^+_\infty$, are not inside Ω for all $t \in \mathbb{R}$. Also, there are no trajectories in $\mathcal{A}^+_\infty$ which do not reach $\partial\mathcal{S}$, or conversely, there are no trajectories in $\mathcal{R}^+_\infty$ which do not leave $\partial\mathcal{S}$. Moreover, by stability, or complete unstability, of $\mathcal{S}$, there are no trajectories inside Ω which are defined on $\mathbb{R}$, leaving $\partial\mathcal{S}$, and reaching $\partial\mathcal{S}$ again.

Consequently, the only trajectories of Σ_∞ defined in past and future, and entirely laying inside Ω, are those inside the invariant set $\mathcal{S}$. It follows from theorem 3.3 that Γ^+ is a union of trajectories of Σ_∞ defined in past and future, hence $\Gamma^+ \subset \mathcal{S}$ follows. $\square$

4.4 Example: pathological system
Assume that $\dot{x} = X(x,t)$ is asymptotic to a system which is described in spherical coordinates by

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= 1-r \quad . \end{aligned}$$

It is easily seen that the union of equilibria other than the origin is forming a unit circle $\mathcal{S}$, and $\mathcal{S} = \cup\Gamma^+_\infty \backslash \{(0,0)\}$ is a asymptotically stable set with basin of attraction $\mathcal{A}^+_\infty = \mathbb{R}\backslash\{(0,0)\}$. We have a situation which topologically resembles the Van der Pol equation (3.7). It follows from theorem 4.3 that all perturbed trajectories defined in the future which are not approaching the origin are approaching $\mathcal{S}$, and then $\Gamma^+ \subset \mathcal{S}$ holds. Note though that we are not able to conclude $\Gamma^+ = \mathcal{S}$ since theorem 3.6 does not apply, Γ^+ might as well consist of only some of the equilibrium points. $\star$

Proof of Theorem 3.6: The only thing that remains to prove in relation to theorem 4.3 is that if Γ^+_∞ equals an equilibrium point or a periodic orbit, the equality $\Gamma^+ = \Gamma^+_\infty$ holds.

In these cases Γ^+_∞ consists of one single isolated orbit of Σ^+ defined both in past and future, and $\Gamma^+ = \Gamma^+_\infty$ is obvious in the case of an equilibrium point. Assume therefore that Γ^+_∞ consists of one single periodic orbit, then clearly by theorem 4.3 it follows that $\Gamma^+ \subset \Gamma^+_\infty$. But by theorem 3.3 the entire orbit is in Γ^+, hence $\Gamma^+ = \Gamma^+_\infty$. $\square$

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Comments and References

The notion of asymptotically autonomous systems has been introduced in 1956 by L. Markus in the paper [Mar56], and has been studied among few others by T. Yoshizawa [Yos63], and by Aaron Strauss and James A. Yorke [SY67]. The later research paper generalizes Markus results to systems of differential equations where state trajectories are not uniquely given by the initial data. They operate with sets called generalized positive limit sets. This is then applied to prove global attraction of equilibrium points under the asymptotical autonomous dynamics. Attraction of general invariant sets has not been studied there.

In the context of asymptotically autonomous systems we have certainly to remember the work of Horst R. Thieme from the nineties. Convergence properties of asymptotically autonomous differential equations are studied in the research paper [Thi92], and these results are refined in the planar case in [Thi94a] and [Thi94b], where Poincaré or Bendixon type results are proved to hold in two dimensions.

Finally, the fine-structure of positive limit sets of asymptotically autonomous differential equations are the concern of Konstatin Mischaikow, Hal Smith, and Horst R. Thieme in the paper [MST95] "Asymptotically Autonomous Semiflows: Chain Recurrence and Lyapunov Functions". There it has been shown that positive limit sets Γ^+ of asymptotically autonomous systems have to be chain-recurrent. Given an asymptotically autonomous system $\Sigma : \dot{x} = X(x,t)$ satisfying $X(x,t) \to X_\infty(x)$ for some autonomous limit system $\Sigma_\infty : \dot{x} = X_\infty(x)$, we know that the positive limit sets Γ^+ of Σ are given as an invariant set of Σ_∞. Chain-recurrence of Γ^+ means, loosely explained, that from each $x_0 \in \Gamma^+$ there is a chain of connected autonomous trajectories of the system Σ_∞ which lays entirely in Γ^+, and which is such that the chain is leading back to x_0 such that the autonomous flow has only to jump over countable many stagnation points of the flow of Σ_∞.

For example, the results of [MST95] show then that the invariant set depicted in the left

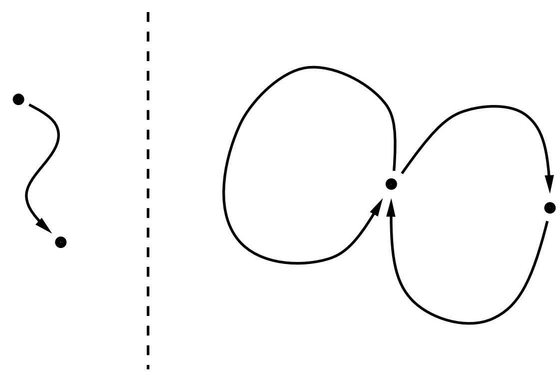

Figure 4: Left: invariant set Right: invariant and chain recurrent set

side of figure 2 can never be a positive limit set of the perturbes system Σ, because it is not chain-recurrent, whereas the invariant sets depicted in the right side of figure 4 can.

We can see that the condition of asymptotic stability or complete unstability of $\mathcal{S}$ in theorem 4.3 is linked to chain-recurrence of positive limit sets of perturbed trajectories $x(\cdot)$: the basin of attraction, or the domain of repulsion also can be defined for merely attractive sets, or sets which are attractive under the time-reversed dynamics in a similar way. But the condition of stability, or complete unstability, is needed to ensure that no trajectories leaving $\partial\mathcal{S}$, and reaching $\partial\mathcal{S}$, are existing. It follows that all existing chain-recurrent invariant sets are entirely inside the compact invariant set $\mathcal{S}$.

Robust set-stability in $\mathcal{H}_\infty$ control

Let us short explain how the results of the previous chapter merge with the here derived theory of asymptotically autonomous systems: We consider the smooth, continuous time system

$$\begin{aligned} \dot{x} &= X(x,u,w) \\ z &= Z(x,u) \quad , \end{aligned} \tag{5}$$

and we assume that the system is $\mathcal{S}$-detectable with respect to some compact and invariant set $\mathcal{S} \subset \mathsf{IR}^n$ of the autonomous dynamics

$$\dot{x} = X(x,0,0) \quad . \tag{6}$$

We know from the preceding chapter that a stabilizing control which renders the $\mathcal{L}_2$ gain less than or equal $\gamma > 0$ can be found whenever there is a positive definite C^1 solution to the Hamilton-Jacobi inequality

$$\begin{aligned} &\mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) \\ &= \tfrac{\partial V}{\partial x} X(x, u_{\min}(x, \tfrac{\partial V}{\partial x}), w_{\max}(x, \tfrac{\partial V}{\partial x})) - \gamma^2 \left| w_{\max}(x, \tfrac{\partial V}{\partial x}) \right|^2 + \left| Z(x, u_{\min}(x, \tfrac{\partial V}{\partial x}) \right|^2 \\ &\qquad \leq \; 0 \quad \text{for all } x \in \Omega \quad . \end{aligned} \tag{7}$$

It follows then that all bounded state trajectories of the undisturbed closed loop system

$$\dot{x} = X(x, u_{\min}(x), 0) \tag{8}$$

are satisfying $x(t) \to \mathcal{S}$ for $t \to \infty$, that is, $\mathcal{S}$ is attractive for all bounded $x(\cdot)$.

We have to impose another condition on the storage function V to ensure stability, and hence asymptotical stability of $\mathcal{S}$ with respect to the the dynamics (8). We say that $\alpha : \mathsf{IR}^+ \mapsto \mathsf{IR}^+$ is a function of class $\mathcal{K}_\infty$ if α is continuous, strictly increasing, and satisfies $\alpha(0) = 0$ and $\alpha(r) \to \infty$ for $r \to \infty$.

4.5 Theorem (Robust $\mathcal{H}_\infty$ control) *Assume that some C^1 solution $V : \Omega \mapsto \mathsf{IR}$ of the Hamilton-Jacobi inequality (7) satisfies*

$$\underline{\alpha}_V(|x|_\mathcal{S}) \leq V(x) \leq \overline{\alpha}_V(|x|_\mathcal{S}) \quad ,$$

where $\underline{\alpha}_V$, $\overline{\alpha}_V : \mathsf{IR}^+ \mapsto \mathsf{IR}^+$ are functions of class $\mathcal{K}_\infty$, and $\mathcal{S} \subset \mathsf{IR}^n$ is a compact invariant set of the open loop dynamics (6). Assume furthermore that the saddlepoint property

$$\mathsf{H}(u_{min}, w) \leq \mathsf{H}^{**} = \mathsf{H}(u_{min}, w_{max}) \leq \mathsf{H}(u, w_{max})$$

holds for all $x \in \mathrm{IR}^n$, and that the control system (5) is $\mathcal{S}$-detectable.

Then the system has L_2 gain less than or equal to γ, and $\mathcal{S}$ is an asymptotically stable set of the undisturbed, closed loop system (8).

Moreover, all $x(\cdot)$ generated by bounded, piecewise continuous $w(\cdot) \in \mathcal{L}_2$ satisfying $w(t) \to 0$ for $t \to \infty$ approach $\mathcal{S}$ as $t \to \infty$.

Proof: Since the saddlepoint property is assumed to hold globally on IR^n, it follows that any positive definite solution to the HJI (7) is a storage function satisfying

$$V(x_T) - V(x_0) \leq \int_0^T \gamma^2 |w(t)|^2 - |z(t)| \, dt \quad . \tag{9}$$

Hence, it follows that $V(x_T) \leq V(x_0) + \|w(\cdot)\|_T^2$ holds for all $x_0 \in \mathrm{IR}^n$, all $w(\cdot) \in \mathcal{L}_2^{\mathrm{loc}}$ and all finite $T > 0$.

Since the condition $\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}})$ implies that $V : \mathrm{IR}^n \mapsto \mathrm{IR}^+$ is radially unbounded (that is, $V(x) \to \infty$ for all $x \to \infty$, or equivalently, $V^{-1}([0,c])$ is compact for all $c > 0$), the trajectory $x(\cdot) : \mathrm{IR}^+ \mapsto \mathrm{IR}^n$ is bounded on any bounded subinterval $[0,T] \subset \mathrm{IR}^+$. If in addition $w(\cdot) \in \mathcal{L}_2$, we see immediately that any state $x(\cdot)$ with initial point x_0 is bounded inside the compact set

$$\Omega(x_0, \|w(\cdot)\|^2) = V^{-1}([0, V(x_0) + \gamma^2 \|w(\cdot)\|^2]) \quad .$$

Clearly, by boundedness of the state, the $\mathcal{L}_2$ gain γ is obtained.

Consider now all trajectories $x(\cdot)$ of the undisturbed, closed loop system (8). Since $w(\cdot) = 0$ for all such trajectories, it follows that $V(x_T) \leq V(x_0)$ holds for all $T \geq 0$. Therefore, for all x_0 satisfying $|x_0|_{\mathcal{S}} \leq \overline{\alpha}_V(c)$, and all $T \geq 0$, we have $x_T \in V^{-1}([0,c])$, and $|x_T|_{\mathcal{S}} \leq \underline{\alpha}_V(c)$ holds. Global stability of $\mathcal{S}$ with respect to the dynamics (8) follows immediately: if we wish to bound $x(\cdot)$ in $\mathcal{N}_\varepsilon$, it suffices to pick $\delta = ({\overline{\alpha}_V}^{-1} \circ \underline{\alpha}_V)(\varepsilon)$ and to set $x_0 \in \mathcal{N}_\delta$.

At this point we have showed that all trajectories generated by $w(\cdot)$ with finite $\mathcal{L}_2$ signal norm are bounded inside the compact set $\Omega(x_0, \|w(\cdot)\|^2)$, and furthermore, that $\mathcal{S}$ is globally stable with respect to the undisturbed and controlled dynamics (8). Moreover, from theorem 4.3 of chapter 3 it follows that the set $\mathcal{S}$ is globally attractive, and global attraction implies that the basin of attraction of $\mathcal{S}$ is given by $\mathcal{A}_\infty^+ = \mathrm{IR}^n$.

Clearly, the proper inclusions $\mathcal{S} \subset \Omega(x_0, c) \subset \mathcal{A}_\infty^+$ hold for all $w(\cdot) \in \mathcal{L}_2$ with $\|w(\cdot)\|^2 \leq c$. Finally, the stated result is given by a combination of proposition 3.2 and theorem 4.3 of the preceding paper. □

We see that a certain degree of robustness under $\mathcal{L}_2$ disturbances can be achieved. At this point, however, it is not at all clear under which circumstances there exists a C^1 solution to the HJI of concern. Even more annoying, it is not evident that a solution satisfying $\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}})$ for two functions of class $\mathcal{K}_\infty$ exist. We return to this problem in the next chapter, and give there some sufficient conditions which imply the above inequality.

Chapter 5

Analysis of Dissipative Systems

In this chapter the author's main contribution to nonlinear dissipation and it's relations to stability issues of general invariant sets is presented. We are combining the functionality of the following different tools mentioned in the introductiory chapter 1:

General dissipative analysis theorem 2.3,

La Salle's invariance principle theorem 2.1,

Performance envelopes theorem 2.1,

Set-stability theorem 2.2,

Robustness subject to non-zero disturbances theorem 2.2,

Smoothness of Lyapunov-like functions theorem 2.2,

Control Lyapunov functions theorem 2.4, and

Game theory theorem 2.5.

Although many aspects of these tools work nicely together, there are still open questions and riddles to be solved.

The overall idea is, that - because it is hard to solve nonlinear Hamilton-Jacobi inequalities - as much as possible information on the behavior of a dissipative system should be extracted from the data of the system and the structure of the storage function found.

One of the cornerstones of this chapter is that the required regularity properties of storage functions are relaxed: In the case that we are interested in the dissipation inequality alone, lower semicontinuity suffices. Without loss of generality we are considering systems defined on an open, n-dimensional subset of $\mathbb{R}^n$, called the reachable set. This approach takes account for the fact that not all systems of interest are reachable on the entire Euclidean space $\mathbb{R}^n$, and provides a platform for the analysis of regional stability issues.

It is then shown that continuity of all existing storage functions is a simple consequence of a fundamental property of the system of concern, which we here rename locally bounded excitation. Although not being a new concept, nor being satisfied in general, it is a useful result on regularity of storage functions.

To make a thorough investigation of the positive limit sets of state trajectories treatable, we have to consider continuous and locally Lipschitz storage functions. We can then use the power of the theory of generalized gradients, which is commonly used in non-smooth analysis and convex analysis, but has - as far as the author knows - never been applied before to investigate stability issues of dissipative systems.

The second cornerstone presented in this paper is the definition of four different subsets of $\mathbb{R}^n$ in terms of generalized gradients, which are shown to be closely related to the dissipative stability analysis of positive limit sets. We can extract valuable information on the (asymptotic) stability properties of undisturbed trajectories from the relations among these sets . Similar to the $\mathcal{H}_\infty$ problem presented earlier, a suitable generalization of the La Salle's invariance principle provides insight into the structure of the problem.

The third cornerstone is the use of a so-called strict Hamilton-Jacobi inequality, that is, a HJI which is satisfied strictly negative (not with equality) on $\mathbb{R}^n \backslash \mathcal{S}$. Storage functions which satisfy a strict HJI are then shown to posses beneficial properties like properness, or positive definiteness. Also, asymptotic stability of undisturbed trajectories are a consequence of the existence of solutions to strict HJI's.

The fourth cornerstone is the combination of Input-to-State-Stability (ISS) Lyapunov functions in the sense of Sontag and Lin with proper solutions to HJI's. This merge of different point of views gives immediately the existence of smooth storage functions under some additional conditions. It is also very useful to show robustness of stability properties with respect to time-persistent, not decaying, but $\mathcal{L}_\infty$-bounded disturbances.

The here reprinted paper

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considers only the analysis of stability properties of dissipative systems; the actual use of dissipation techniques in set-stabilizing state feedback control is shortly presented in the following commentary section.

The commentary section starts with some reflections on the existence of continuous and locally Lipschitz storage functions, which are the regularity conditions imposed on storage functions in the second half of the paper.

Thereafter, the use of differential games in dissipative state feedback control is shortly described. We show that the findings of the analysis of dissipative systems hold in general in state feedback systems, provided that the so-called state feedback Hamiltonian possesses a saddlepoint condition. In the context of differential games this saddlepoint property ensures the existence of a value function, which then equals the available storage of the dissipative problem under suitable conditions.

In particular, we show how former results on the existence of viscosity solutions to the Hamilton-Jacobi inequality related to system analysis are transformed to equivalent results on existence of viscosity solutions to the state feedback Hamilton-Jacobi inequality. Moreover, the asymptotic properties of controlled, but undisturbed state trajectories are a simple consequence of the analogous properties of the analysis case described in the paper.

One interesting generalization of the state feedback $\mathcal{H}_\infty$ control case is then shown to hold: the minimizing state feedback control $u_{\min}$ vanishes on the positive limit sets of controlled, but undisturbed trajectories. While this result is immediately seen in the $\mathcal{H}_\infty$ case by the structure of the saddlepoint equations (see the commentary section in chapter 3), it is, in the general dissipative case, a consequence of the structure of the dissipation inequality. The importance of this observation is the fact that positive limit sets of the autonomous undisturbed, but controlled system are also compact invariant sets of the autonomous undisturbed and uncontrolled system.

Finally, it is briefly shown that the former analysis results on smoothness of storage functions, and on robust set-stability, are still valid in the context of state feedback control.

This chapter provides a set of tools which the modern control engineer may use to solve robust set-stability problems under the constrain of dissipation. Moreover, for some subclasses of control problems, the existence of smooth storage functions is proven. This implies that fast converging numerical approximation schemes - like higher order FEM or spectral methods - can be applied.

It is - unfortunately - still an open question whether the existence of smooth storage functions can be proved for a broader class of dissipative problems than used in the following paper.

On dissipative systems and set-stability

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Abstract

The theory of state feedback $\mathcal{H}_\infty$ control of nonlinear systems is inherited from the now well understood linear theory, and so are the questions mostly asked and answered in the nonlinear context. Unfortunately, this approach does not touch interesting generalizations which are crucial for a deeper understanding of nonlinear dissipative control. This paper investigates the relation between dissipative systems and asymptotic stability properties of more general invariant sets than equilibrium points.

Among other new results, the regional stability of invariant sets of general dissipative systems is studied here. Also the robustness of dissipative control systems with respect to time-persistent disturbances, and the regularity of storage functions are investigated. Non-standard modes of operation such as periodic orbits of nonlinear oscillators, multiple equilibria, or stability of specified state trajectories are but some areas of application.

The main tools used are a combination of La Salle's invariance principle and the input-to-state stability property (ISS) with fundamental properties of dissipative storage functions. Simple sufficient conditions for the existence of smooth storage functions are shown.

1 Introduction

Nonlinear systems exhibit behavior essentially different from linear systems. For example, nonlinear continuous time systems may have multiple equilibrium points, periodic orbits, or bifurcation phenomena - not to mention chaotic behavior. This vast possibility of complex motion makes nonlinear systems difficult to treat in the context of regulation and control. On the other hand, linear control theory is now solid founded and essentially well understood. As a consequence, many questions asked and investigated in the nonlinear universe are inherited from problem formulations belonging to linear control theory, the modern theory of local nonlinear $\mathcal{H}_\infty$ control being no exception (see for example the papers [Isi92, IA92b, IA92a, vdS92a, BHW93, IK95]).

This paper tries to answer questions not commonly asked in the community of nonlinear $\mathcal{H}_\infty$ control. More precisely, it generalizes $\mathcal{H}_\infty$ analysis - that is, the $\mathcal{L}_2$ gain analysis of the bounded real lemma together with the stability analysis of the equilibrium point zero - in four main directions to be discussed here:

I The very basic ideas of stabilization by $\mathcal{H}_\infty$ theory can without problems be transferred to the case of **general dissipation** in the sense of Willems [Wil72a]. To achieve this goal, it must be assumed that the supply rate of concern has a certain regularity, which - of course - is satisfied by the commonly used $\mathcal{H}_\infty$ supply rate.

II The dogma of to-be-stabilized-equilibrium-at-the-origin is discarded. There is no reason to carry on this line of approach clearly inherited from linear theory. Here the stabilization of **invariant sets** is considered, we are particular interested in stabilizing positive limit sets of the undisturbed system. Included in this framework is the stabilization of single or multiple equilibria, periodic orbits, specific trajectories, and any union of such non-standard modes of operation.

III In a linear context local stability implies global stability. This property does not hold for nonlinear systems, and most results in nonlinear control are local only. From an engineering point of view this is a very unsatisfactory state of art, since designers of real world control systems have to be sure that the designed regulator works as intended on some given compact region. It is therefore mandatory to consider dissipation on compact sets here called **performance envelopes**, and to confine the state trajectories to these compact sets, a requirement which we denote **regional stability**.

IV Fourth, in a linear $\mathcal{H}_\infty$ context asymptotic stability of the origin for zero disturbances implies asymptotic stability of the origin for non-zero, but vanishing-in-time-disturbances of bounded $\mathcal{L}_2$ norm. Moreover, practical stability for non-vanishing, but bounded in $\mathcal{L}_\infty$ norm disturbances is implicitly given because linear asymptotically stable systems are by force exponentially stable. This is not at all true for nonlinear systems, and therefore the community of nonlinear $\mathcal{H}_\infty$ control considers stability under zero disturbances mostly. The practical oriented control engineer is always faced with the existence of non-zero disturbances, and therefore the impact of **time-vanishing disturbances** and **time-persistent disturbances** on the asymptotic behavior of state trajectories is taken into account here.

The paper deals with the analysis of dissipative nonlinear systems and regularity of storage functions. Section 2 recalls the basic definitions and properties of disturbed dynamic systems, that is, nonlinear systems of ODE which are affected by $\mathcal{L}_2^{\mathrm{loc}}$ signals called disturbances.

Section 3 explains the concept of dissipative systems, mostly following the approach of Jan C. Willems [Wil72a] but for one refinement: for the use in later sections the concept of almost regular, regular and strictly regular supply rates is introduced. Then the notion of viscosity solutions of Hamilton-Jacoby inequalities (HJI) is presented in subsection 3.1 according to James [Jam93a], and the special cases of the nonlinear generalizations of the positive real lemma and the bounded real lemma are displayed. In subsection 3.2 the HJI is investigated on the reachable set, that is the open subset in state space which can be reached in finite time from any point of minimal storage. In order to introduce the possibility of multiple points of minimal storage in a natural manner, we have to make a new assumption on the structure of the dynamical system which will be trivially satisfied in many applications. The existence of continuous storage functions is also investigated in subsection 3.2, it will be a consequence of the property of locally bounded excitation. Finally, subsection 3.3 deals with the new concept of regional stability, that is the property that even disturbed state trajectories can be bounded in some compact and safe regions in state space, here called performance envelopes. In contrary to many other expositions [Isi92, IA92b, IA92a, vdS92a, BHW93, IK95] we do not restrict ourselves to the case of local dissipation.

The new contribution to the theory of set-stability of dissipative systems is found in Section 4. It consists mainly of a novel combination of some dissipative control techniques with the La Salle's invariance principle. Preliminary work in this direction has been published in [CMPP97] and [CS97]. See also the related work of David J. Hill [Hil92] and David J. Hill and Peter J. Moylan [HM80a]. To be more specific, the present approach accesses asymptotic stability properties of some invariant set in the framework of general dissipation in the sense of Willems [Wil72a], by invoking arguments similar to those used in the proof of La Salle's invariance principle [SL61]. To do so, the new notions of Hamiltonian null sets, storage null sets, storage kernels and performance kernels must be introduced. It is for the first time shown that the maximal disturbance vanishes on the union of positive limit sets of all bounded and undisturbed trajectories; this is a natural generalization of a similar, well known property of systems with one equilibrium point only. The convergence of undisturbed motions in state space in relation to general invariant sets is investigated in subsection 4.1 (see [Cro96] for possible applications of periodic orbits in control of Hopf bifurcations in compressor systems). New convergence results with respect to invariant sets are displayed, grouped in four cases: first, set-detectability which is a generalization of the usual zero-detectability assumption in $\mathcal{H}_\infty$ control, second strict negative definiteness of the HJI, third, positive definiteness of the performance function, and finally, a combination of these three cases which results in a remarkable identity of the former mentioned null sets and kernels.

The standard $\mathcal{H}_\infty$ theory is useful for robustness properties with respect to unstructured modelling errors, but stability of motions is often confined to the case of zero disturbances. This drawback from an applied point of view is now resolved in Section 5. The qualitative behavior of disturbed systems subject to disturbances which are bounded, piecewise continuous in time, and converging to zero as $t \to \infty$ are investigated in subsection 5.1. This approach is based on the concept of asymptotic autonomous systems introduced by L. Markus [Mar56] and reworked in [Cro98a]. The attraction properties of compact sets with respect to bounded trajectories generated by such decaying disturbances is then proved for the first time. Differentiability and smoothness of storage functions is investigated in subsection 5.2. The notion of input-to-state stability (ISS) introduced by Sontag [Son89a, Son89b], and reformulated by Sontag, Lin, and Wang [Lin92, Son95c, SW95b, SW96] is recalled and used to show surprisingly simple new sufficient conditions for the existence of smooth storage functions. It is also shown that dissipative systems of the above mentioned cases two and three are ISS and admit a smooth ISS-Lyapunov function under some additional conditions. This knowledge is then used in subsection 5.3 to derive useful performance envelopes and practical stability results for dissipative systems which are under influence of bounded disturbances. This consists a new and important application of the ISS property in the context of nonlinear dissipative systems.

Section 6 summarizes the new results of this paper. Classes of dissipative problems are listed together with their asymptotic properties, regularity results of storage functions, and robustness results subject to non-zero disturbances. A short discussion on the numerical aspects of HJI solvers in relation to the presented classes follows. The actual process of numerical integration of the HJI by polynomial expansion [CMPP97], finite difference methods [Jam93b], higher order FEM or smooth spectral methods is not described here.

A later publication is devoted state feedback control of disturbed and controlled dynamical systems. The new results of this paper are used to formulate set-stabilizing regional feedback laws which are robust with respect to non-zero disturbances. The cornerstone of this part is a generalization of the saddlepoint property fulfilled by nonlinear $\mathcal{H}_\infty$ control. Various applications in dissipative set-stabilizing control and in particular, in robust $\mathcal{H}_\infty$ control will be given.

2 Basics

We need some definitions: The non-negative reals are denoted $\mathbb{R}^+$. A real valued function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ belongs to **class $\mathcal{K}$** if it is smooth, strictly increasing, and satisfies $\alpha(0) = 0$. It belongs to **class $\mathcal{K}_\infty$** if in addition $\alpha(r) \to \infty$ for $r \to \infty$.

The symbol $\mathcal{R}$ denotes always an n-dimensional open set, more precisely the reachable set defined later on. A real valued function $f : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$ is **locally bounded** if it is bounded on any compact subset of its domain $\mathcal{R}$. A real valued function $f : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$ is **radially unbounded**, or **proper**, if $f(x) \to \infty$ for all $x \to \partial\mathcal{R}$, where $\partial\mathcal{R}$ is the boundary of the domain (In the case that $\mathcal{R}$ is unbounded we understand by $x \to \partial\mathcal{R}$ that $x \to \infty$ whenever appropriate).

A function $f : \mathcal{R} \mapsto \mathbb{R}$ is called **locally Lipschitz** if for each $x \in \mathcal{R}$ there exists a closed ball $\mathcal{B}_\varepsilon$ of radius $\varepsilon > 0$, centered at x, and a constant $k \geq 0$ such that for each pair $x_1, x_2 \in \mathcal{B}_\varepsilon$ there holds

$$|f(x_1) - f(x_2)| \leq k|x_1 - x_2| \quad .$$

Let $|\cdot|$ denote the usual Euclidean vector norm on $\mathbb{R}^n$. Given any closed subset $\mathcal{A} \subset \mathbb{R}^n$, we define the distance between $\mathcal{A}$ and some point $p \in \mathbb{R}^n$ by

$$|p|_{\mathcal{A}} \equiv \min_{q\in\mathcal{A}} |p-q| \quad . \tag{1}$$

Notice that the usual Euclidean vector norm then is given by $|\cdot| = |\cdot|_{\{0\}}$. Let $\mathbb{R}^+$ denote the future, that is the nonnegative real axis $[0,\infty)$, and $\mathbb{R}^- \equiv (-\infty, 0]$ the past.

A **perturbed**, or **disturbed system** is a system of the form

$$\dot{x} = X(x,w) \quad , \tag{2}$$

where $x(\cdot) : \mathbb{R}^+ \mapsto \mathcal{R} \subset \mathbb{R}^n$ is called the **state**, and $w(\cdot) : \mathbb{R}^+ \mapsto \mathcal{W} \subset \mathbb{R}^l$ the exogenous input, also called **disturbance**. The symbol $X(x,w)$ denotes a continuous vector field $X : \mathcal{R} \times \mathcal{W} \subset \mathbb{R}^l \mapsto \mathbb{R}^n$, which is locally Lipschitz in x, uniformly in w. More precisely, for each compact set $\mathcal{K} \subset \mathcal{R}$ there is a constant $k > 0$ such that

$$|X(x_1,w) - X(x_2,w)| \leq k|x_1 - x_2|$$

for all $x_1,\ x_2 \in \mathcal{K}$ and all $w \in \mathcal{W} \subset \mathbb{R}^l$.

Applying the theory of ordinary differential equations [BN69, chap. 3] [CL55, chap. 1] on systems of the form (2), it is easily seen that these conditions ensure the existence and uniqueness of state solutions whenever the disturbance is a piecewise continuous signal. We use the notation $x(\cdot)$ for the unique signal $x(\cdot, t_0, x_0, w(\cdot))$ generated by the input $w(\cdot)$, where the initial condition at time t_0 is x_0. We denote the value of a signal $x(\cdot) : \mathcal{I} \mapsto \mathbb{R}^n$ at time t by $x(t) \in \mathbb{R}^n$. Here $\mathcal{I} \subset \mathbb{R}$ is the maximal time interval where the signal $x(\cdot)$ is defined. Some point in $\mathbb{R}^n$ is denoted x, and in particular, the initial point of some given signal $x(\cdot)$ at initial time t_0 is defined by $x_0 \equiv x(t_0)$.

It is assumed that all signals are $\mathcal{L}_2^{\mathrm{loc}}$, that is, the integral $\int_a^b |y(t)|^2\, dt$ is finite for all $a, b \in \mathcal{I} \subset \mathbb{R}$. Moreover, in case that the disturbance is not a continuous signal, we assume further that the state exist uniquely for all disturbances of concern, and is a C^1 signal but on a set of measure zero. In the following it will be important that the state signal is a continuous function of the triplet (t, t_0, x_0), and the signal space of disturbances must be chosen accordingly to this requirement.

One important class of disturbed systems satisfying the later assumptions is given by the requirements that the disturbance is bounded and piecewise continuous, and $\frac{\partial X}{\partial x} : \mathcal{R}\times\mathcal{W} \mapsto \mathbb{R}^n \times \mathbb{R}^n$ is continuous. Then we can show using the techniques of [CL55, chap. 1 & 2] that the state is a continuous function of the triplet (t, t_0, x_0) which is continuously differentiable on some neighborhood of (t_0, x_0) for each fixed t, and piecewise continuously differentiable in t for all fixed (t_0, x_0).

Other interesting classes of problems meeting the above assumptions may be found using Caratheodory theory [CL55, chap. 2], but for simplicity we will not follow this path.

Given some unique disturbance signal $w(\cdot)$, the disturbed system (2) can be regarded as a time variant differential system, and, in case that the disturbance is constant in time, as an autonomous system. In both cases, we say that the state is **defined in the future** if a solution $x(\cdot, t_0, x_0, w(\cdot))$ exists for all $t \geq t_0$, it is **defined in the past** if it exists for all $t \leq t_0$. A system where all state trajectories are defined in the future for all initial conditions and all disturbances is called **forward complete**, and if the equivalent property also holds in the past, the system is called **complete**.

We say that the state is **bounded in the future (bounded in the past, bounded)** if it is defined in the future (past, future and past) and satisfies $|x(t)| \leq k$, $k \geq 0$ in the future (past, future and past). In case that the state is not defined in the future, it has a finite escape time $T \in \mathbb{R}$, and $|x(t)| \to \infty$ as $t \to T$.

A compact set $\mathcal{S} \subset \mathcal{R}$ is called **stable** under the dynamics of the **undisturbed system**

$$\dot{x} = X(x, 0) \tag{3}$$

if for all $\varepsilon > 0$ sufficiently small there exists a $\delta > 0$ such that all trajectories of (3) with initial point $|x_0|_{\mathcal{S}} \leq \delta$ are defined on $\mathbb{R}^+$ and satisfy $|x(t)|_{\mathcal{S}} \leq \varepsilon$ for all $t \geq 0$. The compact set $\mathcal{S}$ is called **attractive** if there exists a neighborhood $\mathcal{N} \supset \mathcal{S}$ such that all trajectories of (3) with initial point $x_0 \in \mathcal{N}$ are defined on $\mathbb{R}^+$ and satisfy $|x(t)|_{\mathcal{S}} \to 0$ for $t \to \infty$. Then we say that **$x(\cdot)$ converges to $\mathcal{S}$ as $t \to \infty$**, and we write shortly **$x(\cdot) \to \mathcal{S}$**. The compact set $\mathcal{S}$ is called **asymptotically stable** if it is stable and attractive.

The **basin of attraction $\mathcal{A}^+$** of some attractive set $\mathcal{S}$ is the largest set of initial points such that $\mathcal{S}$ is approached. It can be found by backwards integration of state trajectories from some suitable neighborhood of $\mathcal{S}$, and will always be an open set.

The **positive limit set Γ^+** of some trajectory $x(\cdot)$ is - intuitively spoken - the set a state defined in the future tends to as $t \to \infty$. If the state approaches an equilibrium point or a limit circle, those are the positive limit sets. More formally, x_+ belongs to the positive limit set of a state trajectory $x(\cdot)$ bounded in future if there exists a sequence of time $\{t_n\}$ with $t_n \to \infty$ such that $x(t_n) \to x_+$ as $n \to \infty$.

A set $\mathcal{S} \subset \mathbb{R}^n$ is called **positive (negative) invariant with respect to $w(\cdot)$** if all state trajectories generated by some unique disturbance $w(\cdot)$, and starting in $\mathcal{S}$ are defined in the future (past) and never leave $\mathcal{S}$ as $t \to \infty$ $(t \to -\infty)$. In case that $\mathcal{S}$ is positive (negative) invariant with respect to all $w(\cdot) : \mathbb{R} \mapsto \mathcal{W}$, where $\mathcal{W}$ is some pre-described signal value set, we say simply that $\mathcal{S}$ is **positive (negative) $\mathcal{W}$-invariant**.

A set $\mathcal{S} \subset \mathbb{R}^n$ is called **invariant with respect to a constant $w(\cdot) = c$**, if it is both positive and negative invariant, and this implies that the boundary $\partial\mathcal{S}$ consists of state trajectories. For example, every equilibrium point, every closed and bounded periodic orbit, and every collection of trajectories defined both in future and past are invariant sets. Note that invariant sets are not defined for time varying systems.

It is known that the limit sets Γ^+ of bounded trajectories generated by time varying systems of the form $\dot{x} = X(x, t)$ are nonempty and compact, and $x(\cdot) \to \Gamma^+$ as $t \to \infty$ [BN69, chap. 5]. In case that the system of concern is autonomous, that is of form $\dot{x} = X(x)$, the positive limit set will be invariant [Yos66, chap. 3] [Kha96, chap. 3]. Therefore, all states generated by $w(\cdot) = c$, which are entering a compact positive invariant set $\mathcal{M}$ with respect to the same $w(\cdot) = c$, are approaching the biggest invariant set $\mathcal{S} \subset \mathcal{M}$.

3 Dissipative systems

The theory of dissipative dynamical systems has been developed by Jan C. Willems [Wil72a] (see also [HM80b]) and has been useful in applications like $\mathcal{H}_\infty$ control, and passive systems. It was originally formulated in terms of abstract input, state and output spaces, the relations among these are described by the state transition function (which is assumed to be consistent, deterministic, has the semigroup property and the stationary property), and the readout function. Here we will restrict ourselves to a subclass of dynamical systems which have these properties.

An **dynamic**, or **uncontrolled system** is a system of the form

$$\begin{aligned} \dot{x} &= X(x, w) \\ z &= Z(x) \quad , \end{aligned} \tag{4}$$

where $x(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R} \subset \mathbb{R}^n$ is called the **state**, $w(\cdot) : \mathbb{R}^+ \mapsto \mathcal{W} \subset \mathbb{R}^l$ the exogenous input, also called **disturbance**, and $z(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$ the **performance**, or to-be-controlled signal. The symbol $X(x, w)$ denotes a locally Lipschitz vector field on $\mathcal{R} \subset \mathbb{R}^n$ uniformly in w, depending continuously on the input w. Finally, the vector valued function $Z(x)$ specifies continuously the performance measure.

Assume that a dynamical system is given together with a real valued function $s : \mathbb{R}^l \times \mathbb{R}^p \mapsto \mathbb{R}$, called the supply rate. The supply rate is a measure of some abstract energy flow fed into the system. More precisely, we define:

3.1 Definition *A* ***supply rate*** *of a system is a real valued, continuous function $s : \mathbb{R}^l \times \mathbb{R}^p \mapsto \mathbb{R}$ which is* ***almost regular****, that is, satisfies $s(w, 0) \geq 0$ for all w, and $s(0, z) \leq 0$ for all z .*

All supply rates satisfying $s(0, 0) = 0$, $s(w, 0) > 0$ for $w \neq 0$, and $s(0, z) < 0$ for $z \neq 0$ are called ***regular****.*

All supply rates of the form $s(w, z) = \alpha_w(|w|) - \alpha_z(|z|)$, where $\alpha_w, \alpha_z : \mathbb{R}^+ \mapsto \mathbb{R}^+$ are functions of class $\mathcal{K}_\infty$, are called ***strictly regular****.*

The regularity assumptions specify, loosely speaking, that no energy is fed into a system without exogenous nonzero input, and that no energy is extracted from the system by a zero performance output.

3.2 Definition *A dynamical system (4) with supply rate $s : \mathbb{R}^l \times \mathbb{R}^p \mapsto \mathbb{R}$ is called* ***dissipative*** *if there exists a nonnegative locally bounded function $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}^+$, called the* ***storage function****, such that, along the states of (4),*

$$V(x_T) - V(x_0) \leq \int_0^T s(w(t), z(t))\, dt \tag{5}$$

for all initial points $x_0 \in \mathcal{R}$, exogenous inputs $w(\cdot)$, and times $T \geq 0$, where the final point is $x_T = x(T, 0, x_0, w(\cdot))$. The above inequality is called ***dissipation inequality****.*

Clearly, we must insist that $\mathcal{R} \subset \mathrm{I\!R}^n$ is chosen such that $x(t) \in \mathcal{R}$ for all $t > 0$ and all possible $w(\cdot)$. The original definition in [Wil72a] requires not local boundedness of the storage function, but we are following the definition used in [Jam93a] to achieve viscosity results in a later section.

3.3 Example: Passive systems
In case that the uncontrolled system satisfies the dissipation inequality with the almost regular passivity supply rate $s(w,z) \equiv w^T z$, the system can be realized with passive nonlinear electrical components, and we call the system **passive**. ⋆

Willems definition is cast in a very general setting, and regularity requirements such as continuity or differentiability are not imposed on storage functions. However, James [Jam93a] showed that any dissipative system possesses a lower semicontinuous storage function.

3.4 Proposition *[Jam93a] If a locally bounded function V satisfies the dissipation inequality (5), so does its **lower semicontinuous envelope V_*** defined by*

$$V_*(x) \equiv \lim_{z \to x} \inf V(z) \quad ,$$

and hence V_* is a lower semicontinuous storage function.

Proof: Fix $T \geq 0$, $w(\cdot)$, and $x \in \mathcal{R} \subset \mathrm{I\!R}^n$. Select any sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_0 = \lim_{i\to\infty} x_i$ and $V_*(x_0) = \lim_{i\to\infty} V(x_i)$. Since $V \geq V_*$, the dissipation inequality (5) implies

$$V_*(x(T,0,x_i,w(\cdot))) - V(x_i) \leq \int_0^T s(w(t), Z(x(t,0,x_i,w(\cdot))))\, dt \quad .$$

Send $i \to \infty$ to obtain

$$\lim_{i\to\infty} \inf V_*(x(T,0,x_i,w(\cdot))) - V_*(x_0) \leq \int_0^T s(w(t), Z(x(t,0,x_0,w(\cdot))))\, dt \quad ,$$

since $x(\cdot,0,x_0,w(\cdot))$ is continuous in x_0. By definition of lower semicontinuity we have $\liminf_{i\to\infty} V_*(x(T,0,x_i,w(\cdot))) \geq V_*(x(T,0,x_0,w(\cdot)))$, hence V_* satisfies (5). □

Clearly, the dissipation inequality bounds the difference of the internal energy of the system in some abstract sense. We are interested to know how much internal energy can be be extracted from the system.

3.5 Definition *The **available storage** $V_A : \mathcal{R} \subset \mathrm{I\!R}^n \mapsto \mathrm{I\!R}^+ \cup \{\infty\}$ is the function*

$$V_A(x) \equiv \sup_{w(\cdot),T} \int_0^T -s(w(t), z(t))\, dt \quad , \tag{6}$$

where the supremum is taken over all $w(\cdot)$, all $T \geq 0$, and all signals with initial point $x_0 = x \in \mathcal{R}$ satisfying $x(t) \in \mathcal{R}$ for all $t \geq 0$.

The available storage is nonnegative (consider $T = 0$ for a proof), and it can be used to decide whether a given system is dissipative or not.

3.6 Theorem *[Wil72a, Jam93a] The available storage is a locally bounded function if and only if the system of concern is dissipative, and then the available storage is itself a lower semicontinuous storage function satisfying*

$$0 \leq V_A(x) \leq V(x) \quad \text{for all} \quad x \in \mathcal{R} \subset \mathbb{R}^n$$

for all possible storage functions $V(x)$.

3.7 Example: $\mathcal{L}_2$ gain
We say that a system has an $\mathcal{L}_2$ gain less then or equal to $\gamma > 0$ if there exists a locally bounded function $K : \mathbb{R}^n \mapsto \mathbb{R}^+$ such that the $\mathcal{L}_2$ gain inequality

$$\int_0^T |z(t)|^2 \, dt \leq \gamma^2 \int_0^T |w(t)|^2 \, dt + K(x_0) \tag{7}$$

holds for all initial points $x_0 \in \mathbb{R}^n$, all $T \geq 0$, and all $\mathcal{L}_2^{\text{loc}}$ disturbances $w(\cdot)$, and the trajectories $x(\cdot)$ such generated are defined in future. It is not hard to show that all systems which are dissipative with the strictly regular $\mathcal{H}_\infty$ supply rate

$$s(w,z) = \alpha_w(|w|) - \alpha_z(|z|) \equiv \gamma^2 |w|^2 - |z|^2 \quad , \tag{8}$$

are satisfying the $\mathcal{L}_2$ gain inequality (7) with the least possible initial point function $K(x) = V_A(x)$. Furthermore, if there exists a radially unbounded storage function, the state space trajectories are bounded, and the system has an $\mathcal{L}_2$ gain less then or equal to $\gamma > 0$. ⋆

Let us now investigate the amount of energy required to steer an trajectory from some initial point x_* to any other point $x \in \mathbb{R}^n$. Although one could choose any initial point, it is most logical to assume that the system starts in a state of minimum storage. We say that $\boldsymbol{x_*}$ is a **point of minimal storage** if $x_* \in \mathcal{X}_* \equiv \ker V_A$, the kernel of the available storage.

3.8 Assumption *There exists a point $x_* \in \mathcal{R} \subset \mathbb{R}^n$ of minimal storage, and all storage functions have been normalized such that*

$$V(x_*) = \min_{x \in \mathcal{R}} V(x) = 0 \quad .$$

3.9 Definition *The* ***required supply*** *$V_R : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}^+ \cup \{\infty\}$ of a dissipative system with supply rate $s : \mathbb{R}^l \times \mathbb{R}^p \mapsto \mathbb{R}$ is the function defined by*

$$V_R(x_T) \equiv \inf_{w(\cdot), x_*, T} \int_0^T s(w(t), z(t)) \, dt \quad , \tag{9}$$

where the infimum is taken over all $w(\cdot)$, all points of minimal storage x_*, all $T \geq 0$, and all trajectories with initial point $x_0 = x_*$ and final point $x_T = x(T, 0, x_*, w(\cdot))$.

Clearly, for a finite amount of required energy on $\mathbb{R}^n$, the system must be **reachable** from x_*, that is, for all $x \in \mathbb{R}^n$ there must exist at least one exogenous input $w(\cdot)$ such that $x = x(T, 0, x_*, w(\cdot))$ for some finite $T \geq 0$. In case that x is not reachable from some point of minimal storage, we define $V_R(x) = \infty$, and then V_R can impossibly be a storage function on $\mathbb{R}^n$.

Unfortunately, the condition that $\mathbb{R}^n$ is reachable from x_* will often be violated in practice. On the other hand, we may without loss of generality consider the dissipation inequality (5) only on an open subset of $\mathbb{R}^n$.

3.10 Definition *The* ***reachable set*** *is the open subset $\mathcal{R} \subset \mathbb{R}^n$ defined by*

$$\mathcal{R} \equiv \left\{ x_T \in \mathbb{R}^n \;\middle|\; \text{there are } T < \infty \;,\; w(\cdot) \text{ such that } x_T = x(T, 0, x_*, w(\cdot)) \right\} \;,$$

where the initial point $x_0 = x_*$ is any point of minimal storage.

Often there exists more than one point of minimal storage, and then $\mathcal{R}$ may be split into disjoint components. We will consider this as a collection of several systems, each of them defined on a connected component of $\mathcal{R}$. We want to avoid unnecessary complications and assume therefore the following for the rest of this text:

3.11 Assumption (Reachable set) *The reachable set $\mathcal{R}$ is an open, n-dimensional and connected subset, and has a sufficient smooth boundary. All other points of minimal storage are contained in $\mathcal{R}$, and $\mathcal{R}$ is invariant under the choice of initial point of minimal storage. Moreover, the set of all points of minimal storage, denoted $\mathcal{X}_*$, is compact.*

The set $\mathcal{R}$ may contain several disjoint points of minimal storage: we see from the dissipation inequality that for each initial point $x_0 = x_* \in \mathcal{X}_*$, an for each path satisfying

$$\int_0^T s(w(t), z(t))\, dt = 0 \;\;,$$

the final point x_T is a point of minimal storage.

In particular it follows by almost regularity of the supply rate that every point on a trajectory starting at some $x_0 = x_*$, and generated by the trivial disturbance $w(\cdot) = 0$, is a point of minimal storage. We conclude that the set of all minimal points $\mathcal{X}_*$ is an **invariant** set of the undisturbed dynamics (3) that is, $\mathcal{X}_*$ consists of a union of bounded trajectories of (3), defined on $\mathbb{R}$.

Without loss of generality, we consider from now on the dynamics of the system (4) restricted to the reachable set $\mathcal{R} \subset \mathbb{R}^n$.

3.12 Theorem *[Wil72a] Let the system of concern be dissipative with supply rate $s : \mathbb{R}^l \times \mathbb{R}^p \mapsto \mathbb{R}$, and assume that $V : \mathcal{R} \mapsto \mathbb{R}$ is defined on the reachable set $\mathcal{R}$, and that there exists a point x_* of minimal storage satisfying $V_A(x_*) = 0$. Then $V_R : \mathcal{R} \mapsto \mathbb{R}$ satisfies $V_R(x_*) = 0$ and*

$$0 \leq V_A(x) \leq V(x) \leq V_R(x) \;\; \text{for all} \;\; x \in \mathcal{R} \;\;.$$

Moreover, V_R is locally bounded, and the required supply is itself a l.s.c. storage function.

It follows directly from theorem 3.12 that the set of points of minimal available storage satisfies $\mathcal{X}_* = \ker V_A = \ker V = \ker V_R$, hence in the following $\mathcal{X}_*$ is shortly called the set of minimal storage.

In general, dissipative systems allow for an infinite number of storage functions. Given two storage functions V_1 and V_2 with respect to the same supply rate $s(w,z)$, it is easily seen from the dissipation inequality (5) that $cV_1 + (1-c)V_2$ for all $0 \leq c \leq 1$ also is a storage function. This observation is formalized:

3.13 Theorem *[Wil72a] The set of all possible storage functions of a dissipative system is convex. In particular, $cV_A + (1-c)V_R$ for all $0 \leq c \leq 1$ is a storage function for a dissipative system whose state space is confined to the reachable set $\mathcal{R}$.*

Assume for a moment that the system (4) is dissipative, and that there exists a storage function $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}^+$ which is continuously differentiable along any possible trajectory $x(\cdot)$. Then we are able to study the **differential dissipation inequality** (where we define $V(t) = V(x(t))$ for all $x(\cdot)$)

$$\tfrac{d}{dt} V(t) \leq s(w(t), z(t)) \quad , \tag{10}$$

which then is equivalent to the dissipation inequality (5). We note that the function $\frac{d}{dt}V - s(w,z)$ is non-positive for all $t \in \mathbb{R}^+$ and all $w(\cdot)$ by dissipation of the system (4), and vice versa, the existence of a non-negative function V satisfying $\frac{d}{dt}V - s(w,z) \leq 0$ for all $w(\cdot)$ implies that the system is dissipative with respect to the supply rate s.

More formally, we define the **pre-Hamiltonian** $\mathsf{H} : \mathcal{R} \times \mathbb{R}^n \times \mathcal{W} \mapsto \mathbb{R}$ by the equation

$$\mathsf{H}(x,p,w) \equiv pX(x,w) - s(w, Z(x)) \quad . \tag{11}$$

Notice that the cotangent vector p is denoted by a row vector. The pre-Hamiltonian is a continuous and locally bounded function, affine in p. It follows that $\mathsf{H}(x,p,w)$ is convex in p. Moreover, in case that $s(w,\cdot) : \mathbb{R}^p \mapsto \mathbb{R}$ and $Z : \mathcal{R} \mapsto \mathbb{R}^p$ are locally Lipschitz, it follows that $(x,p) \mapsto \mathsf{H}(x,p,w)$ is locally Lipschitz for all w.

It is known [Jam93a] that the **Hamiltonian** $\mathsf{H}^* : \mathcal{R} \times \mathbb{R}^n \mapsto \mathbb{R} \cup \{\infty\}$ defined by

$$\mathsf{H}^*(x,p) \equiv \sup_{w \in \mathcal{W}} \{pX(x,w) - s(w, Z(x))\} \tag{12}$$

is an extended real valued lower semicontinuous function, locally bounded from below. In case that H^* is finite at some point (x,p), then $\mathsf{H}^* : \mathcal{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ is finite, continuous and locally Lipschitz on a neighborhood around (x,p). Moreover, if only disturbance signals $w(\cdot) : \mathbb{R}^+ \mapsto \mathcal{W}$ taking values in a compact subset $\mathcal{W} \subset \mathbb{R}^l$ are considered, the Hamiltonian is continuous on $\mathcal{R} \times \mathbb{R}^n$.

It follows from the preceding discussion and from (12), that the existence of a C^1 function V which satisfies the **Hamilton-Jacobi inequality (HJI)**

$$\mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \leq 0 \quad \text{for all} \quad x \in \mathcal{R} \tag{13}$$

implies the dissipation inequality (5) to hold for all disturbances $w(\cdot)$.

In case that there exists a maximizing exogenous input for the uncontrolled system (4)

$$w_{\max}(x,p) \equiv \arg \max_{w \in \mathcal{W}} \mathsf{H}(x,p,w) \tag{14}$$

the Hamiltonian function is given by

$$\mathsf{H}^*(x,p) \equiv \mathsf{H}(x,p,w_{\max}) \quad . \tag{15}$$

It must be remembered that $w_{\max}(x,p)$ is the worst possible disturbance concerning the negativity of the Hamiltonian. We will see in the next section that $w_{\max}$ has little effect on the stability properties of the state, whereas other disturbances may destroy asymptotic stability completely.

It is important to notice that the dissipation inequality (5) is a variational formulation of an abstract energy concept, where we consider the variation of a state path from x_0 to x_T due to different disturbances $w(\cdot)$. On the other hand, the HJI (13) is a partial differential inequality in $x \in \mathbb{R}^n$, which is valid on a dense subspace, and which is independent of the actual disturbance, and independent of the path of motion.

3.1 A weak formulation of the HJI

From optimal control theory it is well known that value functions associated with variational problems can not assumed to be globally smooth. To handle this lack of regularity we use the recently developed theory of weak (or viscosity) solutions for nonlinear first order PDE [BP87, CEL84, CIL92]. This theory with its convergence and comparison theorems has proved to be a powerful tool.

3.14 Definition (Viscosity solution) *A locally bounded function $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$ is a* ***weak*** *or* ***viscosity solution*** *to the HJI (13) if for every C^1 function $\Psi : \mathcal{R} \mapsto \mathbb{R}$ and every local minimum $x_0 \in \mathbb{R}^n$ of $V_* - \Psi$ (where V_* is the l.s.c. envelope of V) one has*

$$\mathsf{H}^*(x_0, \tfrac{\partial}{\partial x}\Psi(x_0)) \leq 0 \quad .$$

The following theorem characterizes dissipation in terms of the HJI (13) in the weak sense, and does not require V to be a C^1 function as in the discussion of the preceding section.

Notice that the supremum in the definition of the Hamiltonian (12) need not to be attained, hence the weak formulation of the HJI reads

$$\mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) = \sup_{w \in \mathcal{W}} \{ \tfrac{\partial V}{\partial x}^T X(x,w) - s(w, Z(x)) \} \leq 0 \quad . \tag{16}$$

3.15 Theorem *[Jam93a] If the uncontrolled system (4) is dissipative with (locally bounded) storage function $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$, then V satisfies the HJI (16) in the weak sense.*

Conversely, if a locally bounded function $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$ is a viscosity solution to the HJI (16), then the uncontrolled system (4) is dissipative, and its l.s.c. envelope V_* is a l.s.c. storage function.

Proof: James [Jam93a] considers only systems which are affine in $w(\cdot)$ and have an equilibrium point at zero. However, since our more general system has an Hamiltonian which is an extended real valued lower semicontinuous function, locally bounded from below, the original proof applies as mentioned in [Jam93b]. $\square$

From now on we will assume without loss of generality that each viscosity solution to the HJI (13), - that is, each locally bounded storage function - is lower semicontinuous and takes values in the non-negative reals $\mathbb{R}^+$.

The next theorem says that the property of dissipation is not destroyed under perturbations of the data defining the system.

3.16 Theorem *[Jam93a] Let $\Sigma^\varepsilon = (X^\varepsilon, Z^\varepsilon)$, $\varepsilon > 0$ denote a family of systems of the form (4), and let $X^\varepsilon \to X$ and $Z^\varepsilon \to Z$ locally uniformly as $\varepsilon \to 0$, for some limit system $\Sigma = (X, Z)$.*

Assume that each system is dissipative with respect to a supply rate $s^\varepsilon(w,z) \to s(w,z)$ locally uniformly, and assume that the (locally bounded) l.s.c. storage functions $V^\varepsilon : \mathbb{R}^n \mapsto \mathbb{R}$ satisfy

$$\sup_{\varepsilon>0} \|V^\varepsilon\|_{\mathcal{L}_\infty^{loc}} < \infty \quad .$$

Assume furthermore that the corresponding Hamiltonians satisfy

$$\liminf_{\varepsilon\to 0, z\to x, q\to p} \mathsf{H}_\varepsilon^*(z,q) \geq \mathsf{H}^*(x,p) \quad .$$

Then the limit system $\Sigma = (X, Z)$ is dissipative, and

$$V(x) \equiv \liminf_{\varepsilon\to 0, z\to x} V^\varepsilon(z)$$

is a storage function.

Proof: Note first that V is well defined, locally bounded and lower semicontinuous. Let $\Psi : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$ be C^1, and assume without loss of generality that $V - \Psi$ attains a local minimum at $x_0 \in \mathbb{R}^n$. There is a subsequence $\varepsilon_i \to 0$ as $i \to \infty$ such that

$$V^{\varepsilon_i}(x^{\varepsilon_i}) \to V(x_0) \quad , \quad x^{\varepsilon_i} \to x_0 \quad \text{as} \quad \varepsilon_i \to 0 \quad ,$$

and $V^{\varepsilon_i} - \Psi$ attains a local minimum at x^{ε_i} [BP87]. Since each Σ^ε is dissipative, by theorem 3.15 we have

$$\mathsf{H}_{\varepsilon_i}^*(x^{\varepsilon_i}, \tfrac{\partial}{\partial x}\Psi) \leq 0 \quad , \quad i = 1, 2, \cdots \quad .$$

Send $i \to \infty$ to obtain $\mathsf{H}^*(x, \frac{\partial}{\partial x}\Psi) \leq 0$. By theorem 3.15 Σ is dissipative, and V is a storage function. $\square$

We remember the fundamental lack of uniqueness of storage functions for dissipative systems. Consequently, we can not expect uniqueness of weak solutions of the HJI (13). However, the available storage V_A is the minimum solution, and the required supply V_R the maximum solution of the HJI (13). It is possible to show that V_A and V_R solve the Hamilton-Jacobi **equality** $\mathsf{H}^*(x, \frac{\partial V}{\partial x}) = 0$ in a weak sense under some additional assumptions [BH96].

3.1.1 Passive systems and the positive real lemma

In this subsection we assume for simplicity of the exposition that the uncontrolled system of concern is affine in the exogenous input $w(\cdot)$, and has an equilibrium point at $x = 0$, that is, the system is of the form

$$\begin{array}{ll} \dot{x} = A(x) + B(x)w & , \quad A(0) = 0 \\ z = C(x) & , \quad C(0) = 0 \quad . \end{array} \tag{17}$$

Moreover, $\mathbb{R}^l = \mathbb{R}^p$ is assumed. In this context we define that the affine system (17) is **passive** if it is dissipative with respect to the almost regular supply rate $s(w, z) = w^T z$, and the l.s.c. storage function satisfies $V(0) = 0$.

The next result is a weak sense version of a nonlinear generalization of the positive real lemma (also called Kalman-Yacubovitch-Popov lemma) due to Moylan [Moy74], and it is a corollary of theorem 3.15.

3.17 Theorem *[Jam93a] The affine system (17) is passive if and only if there exists a locally bounded nonnegative function $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ such that $V(0) = 0$ and such that the partial differential inequality*

$$\tfrac{\partial V}{\partial x} A(x) + \sup_{w \in \mathcal{W}} \{ \tfrac{\partial V}{\partial x} B(x) w - w^T z \} \leq 0 \tag{18}$$

is satisfied weakly on $\mathbb{R}^n$.

In case that the disturbance signal is not confined to have values in a compact set, the above inequality reads

$$\begin{array}{ll} \tfrac{\partial V}{\partial x} A(x) \leq 0 & \text{and} \\ \tfrac{\partial V}{\partial x} B(x) = C(x) & \text{on } \mathbb{R}^n \end{array} \tag{19}$$

with the following meaning: for all C^1 functions Ψ and all $x_0 \in \mathbb{R}^n$ where the difference $V_* - \Psi$ attains a local minimum, there holds

$$\begin{array}{ll} \tfrac{\partial \Psi}{\partial x}(x_0) A(x_0) \leq 0 & \text{and} \\ \tfrac{\partial \Psi}{\partial x}(x_0) B(x_0) = C(x_0) & . \end{array}$$

Local stability results for passive, and more general, dissipative systems of the affine form (17) are well known [BIW91, HM76, HM80b, Wil72a], and the most common formulation is the following:

3.18 Theorem *[Jam93a] Let $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ be a locally bounded nonnegative function which is positive definite and continuous at $x = 0$, and which satisfies $V(0) = 0$ and the PDI (18) weakly on $\mathbb{R}^n$. Then $x = 0$ is a stable equilibrium for the disturbance free system*

$$\dot{x} = A(x) \quad .$$

Proof: The positive definiteness of V implies the existence of a function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ of class $\mathcal{K}$ satisfying $V(x) \geq \alpha(|x|)$. This implies $V_*(x) \geq \alpha(|x|)$. Let $\varepsilon > 0$. Since V is continuous at $x = 0$, there exists a $\delta > 0$ such that $|x| < \delta$ implies $V_*(x) < \alpha(\varepsilon)$. By theorems 3.15, 3.16 and 3.18, V_* is a storage function, and setting $w(\cdot) = 0$ the dissipation inequality implies $V_*(x(t) \leq V_*(x_0)$ for all $t \geq 0$, where $x(\cdot)$ is the state trajectory with initial condition x_0. Then

$$\alpha|x(t)| \leq V_*(x(t) \leq V_*(x_0) < \alpha(\varepsilon)$$

for all $t \geq 0$ whenever $|x_0| \leq \delta$. Therefore, $|x(t)| < \varepsilon$ for all $t \geq 0$ if $|x_0| \leq \delta$, and so $x = 0$ is a stable equilibrium. □

Notice that additional assumptions are made about the storage function. These have system dependent criteria: continuity follows from a strong form of local controllability [HM80b], or from a weaker form of local controllability presented in section 3.2, and there called locally bounded excitation. Positive definiteness is a consequence of a form of detectability [HM80b], which is called zero-detectability in section 4.1.

Notice also that theorem 3.18 does not investigate asymptotic stability of the equilibrium point, and says nothing about the size of the neighborhood of $x = 0$ where stability is achieved. Moreover, no other disturbance free invariant sets than equilibria are considered, and the boundedness of state trajectories subject to $w(\cdot) \neq 0$ is not investigated. We will touch these questions in the following sections.

3.1.2 Finite $\mathcal{L}_2$ gain systems and the bounded real lemma

It is commonly assumed that the plant of concern has an equilibrium point at the origin, that is, it is of the form

$$\begin{array}{lll} \dot{x} = X(x,w) & , & X(0,0) = 0 \ , \\ z = Z(x) & , & Z(0) \quad = 0 \ . \end{array} \tag{20}$$

We define that the system (20) has **$\mathcal{L}_2$ gain less than or equal to γ** if there exists a locally bounded function $K : \mathcal{R} \mapsto \mathbb{R}^+$ with $K(0) = 0$ such that

$$\int_0^T |z(t)|^2 \, dt \leq \gamma^2 \int_0^T |w(t)|^2 \, dt + K(x_0) \ \text{ for all } \ T \geq 0 \ ,$$

where x_0 is the initial point of the motion, or equivalently, if (20) is dissipative with respect to the strictly regular supply rate $s(w,z) = \gamma^2|w|^2 - |z|^2$, and the l.s.c. storage function satisfies $V(0) = 0$.

The next result is a weak sense version of a nonlinear generalization of the bounded real lemma (also called the Kalman-Yacubovitch-Popov lemma) due to van der Schaft [vdS92c, vdS92a], and it is a corollary of theorem 3.15.

3.19 Theorem *[Jam93b]* *Assume that the system (20) is reachable from the origin. Then (20) has $\mathcal{L}_2$ gain less than or equal to γ if and only if there exists a locally* *bounded nonnegative l.s.c. function $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ such that $V(0) = 0$ and such that the partial differential inequality*

$$\mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) = \sup_{w \in \mathcal{W}} \{ \tfrac{\partial V}{\partial x}^T X(x,w) - \gamma^2 |w|^2 + |Z(x)|^2 \} \leq 0 \tag{21}$$

is satisfied weakly on $\mathbb{R}^n$.

In case that the system of concern is affine in the exogenous input $w(\cdot)$ and has an equilibrium point at $x = 0$, that is, the system is of the form

$$\begin{array}{lll} \dot{x} = A(x) + B(x)w & , & A(0) = 0 \\ z = C(x) & , & C(0) = 0 \ , \end{array} \tag{22}$$

then the pre-Hamiltonian is given by

$$\mathsf{H} = \tfrac{\partial V}{\partial x}(A(x) + B(x)w) - \gamma^2 |w|^2 + C^T(x)C(x) \ .$$

A simple completion of the squares argument shows that the supremum in the definition of the Hamiltonian (12) is attained by the the maximizing disturbance

$$w_{\max}(x) = \tfrac{1}{2\gamma^2} B^T(x) \tfrac{\partial V}{\partial x}^T(x) \ ,$$

hence the Hamiltonian becomes

$$\mathsf{H}^* = \tfrac{1}{4\gamma^2} \tfrac{\partial V}{\partial x}(x) B(x) B^T(x) \tfrac{\partial V}{\partial x}^T(x) + \tfrac{\partial V}{\partial x}(x) A(x) + C^T(x) C(x) \ .$$

Also here, Hill and Moylan [HM76, HM80a] have a locally stability result which reads:

3.20 Proposition *[vdS92a] Suppose there exists a nonnegative C^1 solution V to the quadratic HJI*

$$\mathsf{H}^* = \tfrac{1}{4\gamma^2} \tfrac{\partial V}{\partial x} B B^T \tfrac{\partial V}{\partial x}^T + \tfrac{\partial V}{\partial x} A + C^T C \leq 0 \ , \tag{23}$$

and assume that the system (22) is zero-observable, that is, $z(\cdot) = 0$ and $w(\cdot) = 0$ implies $x(0) = 0$. Then V is positive definite, and the origin is a locally asymptotically stable equilibrium for the autonomous system

$$\dot{x} = A(x) \ . \tag{24}$$

Moreover, if V is proper, that is, for each $c > 0$ the preimage $V^{-1}([0,c])$ is compact, then the origin is a globally asymptotically stable equilibrium.

Conversely, if the origin is a globally asymptotically stable equilibrium of (24), then every C^1 solution of (23) is nonnegative.

Proposition 3.20 investigates asymptotic stability of the equilibrium point, and tells about the size of the neighborhood of $x = 0$ where stability is achieved. But still, no other disturbance free invariant sets than equilibria are considered, and the boundedness of state trajectories subject to $w(\cdot) \neq 0$ is not investigated. The research article [vdS92c] mentions the existence of a locally defined smooth storage function provided that the linearized $\mathcal{H}_\infty$ problem is solvable. Moreover, it has been showed in [IK95] that also the general system (20) attains the maximizing disturbance $w_{\max}$ locally near zero.

3.21 Example: Nonlinear oscillator
We investigate the uncontrolled planar system

$$\begin{aligned} \dot{x_1} &= x_1((r^2-1)(r^2-4) + r(r^2-4)w) - x_2 \\ \dot{x_2} &= x_2((r^2-1)(r^2-4) + r(r^2-4)w) + x_1 \\ z &= r^2 - 1 \quad , \end{aligned} \tag{25}$$

where $r = \sqrt{x_1^2 + x_2^2}$. In polar coordinates $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$, where $r \geq 0$ and $0 \leq \theta < 2\pi$, the nonlinear oscillator (25) is given by the equations

$$\begin{aligned} \dot{r} &= r(r^2-1)(r^2-4) + r(r^2-4)w \\ \dot{\theta} &= 1 \\ z &= r^2 - 1 \quad . \end{aligned} \tag{26}$$

It is easily seen that the set $\mathcal{S}$ defined by

$$\mathcal{S} \equiv \left\{ x \in \mathbb{R}^2 \;\middle|\; r = 1 \right\}$$

is a compact invariant set of the undisturbed system, and is asymptotically stable for all $x(\cdot)$ subject to $w(\cdot) = 0$. The open set

$$\mathcal{R} \equiv \left\{ x \in \mathbb{R}^2 \;\middle|\; 0 < r < 2 \right\}$$

is the largest possible basin of attraction of $\mathcal{S}$ subject to $w(\cdot) = 0$. Moreover, all points in $\mathcal{R}$ are reached from $\mathcal{S}$ in finite time.

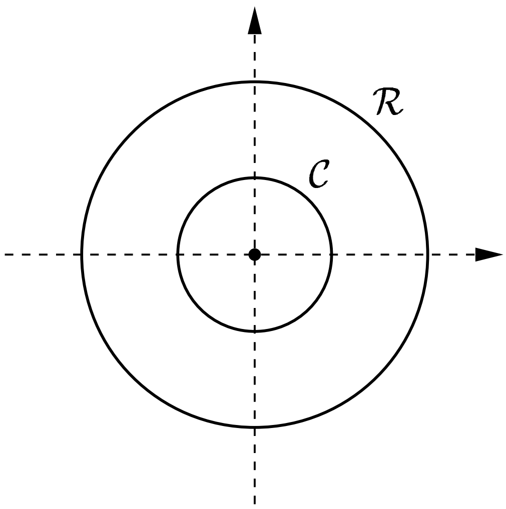

Figure 1: Nonlinear oscillator

We show in the following that system (25) is dissipative with respect to the strictly regular $\mathcal{H}_\infty$ supply rate $s(w,z) = \gamma^2|w|^2 - |z|^2$ for all $\gamma \geq 1$. To do so, consider the available storage V_A given by

$$V_A(x) \equiv \sup_{w(\cdot),T} \int_0^T -\gamma^2|w(t)|^2 + |z(t)|^2 \, dt \quad .$$

We see that $V_A(x_0) \geq \int_0^T |z(t)|^2 \, dt > 0$ for all initial points $x_0 \in \mathcal{R}\backslash\mathcal{S}$ implies that all points inside $\mathcal{S}$ are the only possible points of minimal storage. Since any point on $\mathcal{S}$ can be reached in finite time from any initial point $x_0 \in \mathcal{S}$ by some drifting trajectory $x(\cdot)$ subject to $w(\cdot) = 0$, it follows that the set $\mathcal{R}$ is invariant under the choice of initial point x_0. Hence, the set $\mathcal{S}$ is a candidate for the set of points of minimal available storage X_*. Moreover, if $\mathcal{S} = X_*$ is proven, the fundamental assumption 3.11 is satisfied, and $\mathcal{R}$ as defined before is the reachable set.

Notice that we have not yet showed that system (25) is dissipative. We can show dissipation by two means: either we show that $V_A(x) \leq V_R(x)$, and that $V_R(x)$ is locally bounded on $\mathcal{R}$, hence dissipation follows by theorem 3.6. Or we find some other l.s.c storage function V satisfying the weak HJI (16), and dissipation follows by theorem 3.15.

To show that the possible storage functions are locally bounded on $\mathcal{R}$ if they exist, we investigate the required supply

$$V_R(x) \equiv \inf_{w,T} \int_{-T}^0 \gamma^2|w(t)|^2 - |z(t)|^2 \, dt \quad .$$

Since $V_R(x) \leq \int_0^T |z(t)|^2 \, dt < \infty$ for all final points in $\mathcal{R}$ it follows that the required supply is locally bounded on $\mathcal{R}$ if existing. Notice that showing $V_A(x) \leq V_R(x)$ requires knowledge of all trajectories satisfying the **inf** or **sup** conditions, hence is a variational problem hardly solved.

Fortunately, the HJI (13) can be solved analytically: assume that there is a storage function of the form $V = V(r^2)$, then

$$\tfrac{\partial V}{\partial x} = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \tfrac{dV}{d(r^2)} \quad .$$

The pre-Hamiltonian and the maximizing disturbance become

$$\begin{aligned} \mathsf{H} &= 2\tfrac{dV}{d(r^2)} r^2((r^2-1)(r^2-4) + r(r^2-4)w) - \gamma^2|w|^2 + (r^2-1)^2 \quad , \\ w_{\max} &= \tfrac{1}{\gamma^2} r^2(r^2-4)^2 \tfrac{dV}{d(r^2)} \quad . \end{aligned}$$

It follows that the HJI (13) reads

$$\mathsf{H}^* = \left(r^2(r^2-4)\tfrac{dV}{d(r^2)} + (r^2-1)\right)^2 - (1-\tfrac{1}{\gamma^2})r^4(r^2-4)^2\left(\tfrac{dV}{d(r^2)}\right)^2 \;\leq\; 0 \quad , \tag{27}$$

on $0 < r < 2$, and clearly (27) has a solution if and only if $\gamma \geq 1$. Moreover, any solution to the ODE

$$r^2(r^2-4)\tfrac{dV}{d(r^2)} + (r^2-1) = 0$$

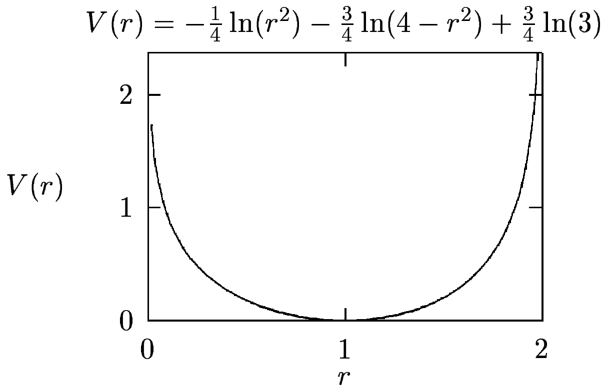

Figure 2: Smooth Storage Function

with minimal set condition $V(1) = 0$ solves the HJI (27), hence the smooth storage function (see the graph in figure 2)

$$V(r) = -\frac{1}{4}\ln(r^2) - \frac{3}{4}\ln(4-r^2) + \frac{3}{4}\ln(3) \tag{28}$$

solves the HJI (27) for all $\gamma \geq 1$.

We see that V is locally bounded on $\mathcal{R}$, $V(x) \geq 0$, $V(x) \to \infty$ for all $x \to \partial\mathcal{R}$, and $V(x) = 0$ on $\mathcal{S}$. We have showed that system (25) is dissipative with respect to the strictly regular $\mathcal{H}_\infty$ supply rate $s(w,z) = \gamma^2|w|^2 - |z|^2$ for all $\gamma \geq 1$. Notice that a smooth storage function exists for all $\gamma \geq 1$. $\star$

3.2 On continuity of storage functions

We have seen that any non-negative and continuously differentiable solution $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}^+$ to the HJI (13) is a storage function of the dissipation inequality (5). The converse is only true under some more restrictive conditions: in case that there exists a point of minimal storage x_* satisfying $V(x_*) = 0$ for all possible storage functions, and in case that the system is reachable from x_*, theorem 3.12 shows that the required supply V_R is a bounded storage function. But in general the required supply is not a C^1 function, an issue which will be discussed in a later section.

In a recent paper [BH96] J. A. Ball and J. W. Helton investigate the continuity of storage functions belonging to $\mathcal{H}_\infty$ control problems. These results are here generalized to include other supply rates. We have to confine the continuity of storage functions to the reachable set. Moreover, the system of concern has to have an additional property, implying that the amount of dissipated or released energy can be related to the distance between initial and final point of a trajectory. We remember that a real valued function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ belongs to **class $\mathcal{K}$** if it is smooth, strictly increasing, and satisfies $\alpha(0) = 0$.

3.22 Definition (Excitation) *The uncontrolled system (4) has* ***locally bounded excitation*** *if for each point $x \in \mathcal{R}$ there exists a neighborhood $\mathcal{P} \in \mathcal{R}$ and a class $\mathcal{K}$ function $\alpha_{\mathcal{P}} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that for each pair of points $x_1, x_2 \in \mathcal{P}$ there exists a disturbance $w(\cdot)$ and a trajectory $x(\cdot)$ which reaches x_2 in final time from x_1, and such that*

$$\left| \int_{t_1}^{t_2} s(w(t), z(t))\, dt \right| \leq \alpha_{\mathcal{P}}(|x_1 - x_2|)$$

holds along this trajectory.

We want to stress that the fulfillment of definition 3.22 has two equally important interpretations: first, a system with locally bounded excitation is locally controllable if we consider the disturbance as exciting control. Second, there is a path which bounds the amount of energy exchanged with the surrounding environment, and this energy is uniform bounded by the distance between initial and final point, and not by the length of the path. We notice that the minimizing trajectory connecting x_1 and x_2 is not confined to the compact set $\mathcal{P}$ around x_0, but clearly it must be entirely inside $\mathcal{R}$. Notice also that the bound $\alpha_{\mathcal{P}}$ is uniform on the above mentioned open neighborhood, but not necessarily uniform on $\mathcal{R}$. The property of locally bounded excitation has been investigated before in [HM80b], there it has been called "locally controllable". We prefer though to use the term controllable in systems where a control signal $u(\cdot)$ is applied, based on some control law like state feedback or dynamic feedback. Since the exogenous disturbance is unknown and unpredictable, the term excitation seems more logic to use. Notice also that the state universe of all dynamical systems considered in this paper is the reachable set according to definition 3.10 and assumption 3.11. Therefore the following is a slight variation of lemma 6 in [HM80b].

3.23 Proposition (Continuous storage function) *[HM80b] Given a dissipative uncontrolled system (4) which mets assumption 3.11 and definition 3.22, there exists at least one storage function $V : \mathcal{R} \mapsto \mathbb{R}^+$, and all existing storage functions are continuous on $\mathcal{R}$.*

Proof: Existence of at least one storage function is easily deduced considering the required supply V_R: by theorem 3.12 there holds

$$0 \leq V_A(x) \leq V(x) \leq V_R(x) \quad \text{for all} \quad x \in \mathbb{R}^n \quad ,$$

and by definition 3.9 we have $V_R(x) < \infty$ for all $x \in \mathcal{R}$. Hence dissipation implies that all candidate storage functions are locally bounded on $\mathcal{R}$. On the other hand, by assumption 3.11 the restriction of the system dynamics onto the reachable set does not alter the dynamics as such, hence $V_R : \mathcal{R} \mapsto \mathbb{R}^+$, the restriction of the required supply onto the reachable set, is a storage function.

Continuity of any existing storage function $V : \mathcal{R} \mapsto \mathbb{R}^+$ is shown by contradiction: Assume that there is a storage function which is not continuous at some point x_0. Then there is a sequence of points $x_n \to x_0$ for $n \to \infty$ such that $|V(x_n) - V(x_0)| \geq \varepsilon > 0$ for all $n \in \mathbb{N}$, and such that $x_0, x_n \in \mathcal{P}$. By definition 3.22 we have a sequence of disturbances $w_n(\cdot)$ and a sequence of minimal trajectories $x_n(\cdot)$ which connect the initial point x_0 and the final points x_n in time t_n.

First we assume that there exists a subsequence $x_n \to x_0$ such that $V(x_n) - V(x_0) \geq \varepsilon > 0$ for all $n \in \mathbb{N}$. Then the dissipation inequality (5) and definition 3.22 show that

$$0 < \varepsilon \leq V(x_n) - V(x_0) \leq \left| \int_0^{t_n} s(w_n(t), z(t)) \, dt \right| \leq \alpha_{\mathcal{P}}(|x_n - x_0|) \ ,$$

and a contradiction occurs since $\alpha_{\mathcal{P}}(|x_n - x_0|) \to 0$ as $n \to \infty$.

In case that there does not exist a subsequence as assumed above, there does certainly exist a subsequence satisfying $V(x_n) - V(x_0) \leq \varepsilon < 0$ for all $n \in \mathbb{N}$. Then we consider sequence of disturbances $w_n(\cdot)$ and a sequence of minimal trajectories $x_n(\cdot)$ which connect the initial points x_n and the final point x_0 in time t_n (note that the reversion of initial and final point is possible due to definition 3.22). A similar contradiction occurs if we consider the difference $V(x_0) - V(x_n)$. □

3.24 Example: Nonlinear oscillator, continued
We show that, for $\gamma = 1$, the system (25) has the property of locally bounded excitation. We see easily from the system dynamics in polar co-ordinates (26) that there is a trajectory connecting each pair x_1, x_2 contained in any compact set $\mathcal{P} \subset \mathcal{R}$: in fact, the disturbance $w(\cdot)$ can steer the trajectory in the radial direction, and the drift vectorfield $X(x, 0)$ can be used to reach any tangential direction - if necessary, we drift almost one circle around to reach x_2 if the initial point x_1 is in the counter-clockwise direction. It is also easily seen that the integral

$$\left| \int_{t_1}^{t_2} \gamma^2 |w(t)|^2 - |z(t)|^2 \, dt \right| \leq \alpha_{\mathcal{P}}(|x_1 - x_2|)$$

holds along any trajectory in the radial or the counter-clockwise direction. But if we want to reach a point in the clockwise direction, it is not at all clear that the above integral will be bounded by the distance between initial and finial point, since then we must drift almost one entire rotation. We see also from (26) that the constant disturbance $w(\cdot) = -(r^2 - 1)$ implies that $\dot{r} = 0$, hence we can drift along a circle of any radius in the counter-clockwise direction. Now, in case that $\gamma = 1$, it follows from $z = (r^2 - 1)$ that the integrand is zero. We conclude that $\alpha_{\mathcal{P}}$ can be chosen to satisfy the integral condition in the radial directions, and the system has the property of locally bounded excitation if $\gamma = 1$ is chosen. It follows by proposition 3.23 that the case $\gamma = 1$ possesses a continuous storage function. ⋆

3.3 On the performance envelope of dissipative systems

We remember from the previous section that the reachable set $\mathcal{R}$ may be unbounded in general. But real world systems do only sustain finite stress, and therefore we would like to be assured that the state trajectories are constrained in some safely bounded and closed subset $\Omega \subset \mathbb{R}^n$, here called **performance envelope**.

In case that the reachable set $\mathcal{R} \subset \mathbb{R}^n$ is a bounded set we can always take the closure $\Omega = \overline{\mathcal{R}}$ as performance envelope if the system stress level is not violated on $\overline{\mathcal{R}}$.

If $\mathcal{R}$ is unbounded, we have in principle the possibility to investigate the vector field $X(x,w)$ directly to prove that some given compact set $\Omega \subset \mathbb{R}^n$ is positive invariant for all disturbances belonging to some pre-described set $w(\cdot) \in \mathcal{W}$, but in practice this investigation will often be prohibitive complex.

On the other hand, to show dissipation of a system, we have to solve the HJI (13), which is a rather complicated process. Therefore we are inclined to extract as much information from any viscosity solution at hand as possible. The boundedness of state trajectories can be deduced, since the dissipation inequality (5) shows that all trajectories with $s(w,z) \leq 0$ satisfy $V(x_T) \leq V(x_0)$ for all $T \geq 0$. This observation bounds all such trajectories inside a performance envelope if the storage function has an additional property:

3.25 Lemma *Assume that some l.s.c. storage function $V : \mathcal{R} \mapsto \mathbb{R}^+$ of a dissipative system is given, and assume that the set $\Omega \subset \mathbb{R}^n$, $\Omega \equiv V^{-1}([0,c]) \cap \overline{\mathcal{R}}$ for some $c > 0$ is bounded, hence compact.*

Then all trajectories $x(\cdot)$ with initial condition $x_0 \in \Omega$ do not leave Ω if driven by some $w(\cdot)$ such that $s(w,z) \leq 0$.

Proof: The conclusion of this lemma is a simple consequence of the dissipation inequality (5) and l.s.c. of the storage function: By definition (see [Ped89]) V is l.s.c. if and only if all preimages $V^{-1}(]c,\infty[)$, $c \in \mathbb{R}$, are open sets. Since $V \geq 0$, it follows that $V^{-1}([0,c]) = V^{-1}(]-\infty,c])$ is a closed set, hence Ω is compact. Finally, all trajectories with $s(w,z) \leq 0$ satisfy $V(x_T) \leq V(x_0)$ for all $T \geq 0$, hence $x_T \in V^{-1}([0,c])$. But by definition 3.10 $x_T \in \overline{\mathcal{R}}$ for all $T \in \mathbb{R}$, hence $x_T \in \Omega$. □

Notice that the given storage function may be such that the set $V^{-1}([0,c]) \cap \overline{\mathcal{R}}$ never has a bounded component, in which case the approach proposed here is not applicable. We see also that the preimage of the required supply according to definition 3.9 exists for all $c > 0$ and satisfies $V_R^{-1}([0,c]) \subset \mathcal{R}$. However, the preimages of other $V(x) \leq V_R(x)$ need not to be subsets of $\mathcal{R}$, hence we consider the intersection $\Omega \equiv V^{-1}([0,c]) \cap \overline{\mathcal{R}}$ only.

In praxis we have still the problem to decide which pair of initial points and disturbances are such that $s(w,z) \leq 0$ is satisfied. This problem is partially resolved by the following corollary.

3.26 Corollary *Assume that some l.s.c. storage function $V : \mathcal{R} \mapsto \mathbb{R}^+$ of a dissipative system satisfies lemma 3.25, and that the associated supply rate is almost regular. Then Ω is positive invariant with respect to all trajectories subject to $w(\cdot) = 0$.*

Assume furthermore that the supply rate is strictly regular, and consider all trajectories with initial condition x_0 inside a proper, compact subset $\Omega_0 \subset \Omega$ of the form $\Omega_0 \equiv V^{-1}([0,c_0]) \cap \overline{\mathcal{R}}$, $0 < c_0 < c$. Then all such $x(\cdot)$ generated by disturbances which satisfy

$$\int_0^T \alpha_w(|w(t)|)\, dt \leq c - c_0$$

are bounded inside the performance envelope Ω.

Proof: The first conclusion of this corollary is a simple consequence of definition 3.1: by regularity of the supply rate we have $s(0,z) \leq 0$. Then apply lemma 3.25.

Assume that the supply rate is strictly regular, then the dissipation inequality (5) shows that all such trajectories satisfy

$$V(x_T) \leq V(x_0) + \int_0^T \alpha_w(|w(t)|)\,dt \leq c - c_0 + c_0 \quad ,$$

hence $x_T \in \Omega$ for all $T \geq 0$. □

Corollary 3.26 considers only boundedness of state trajectories subject to $w(\cdot) = 0$, or subject to an integral constraint of the form $\int_0^T \alpha_w(|w(t)|)\,dt \leq c - c_0$, no information on the asymptotic properties of such state trajectories is given.

The investigation of asymptotic properties of state trajectories subject to $w(\cdot) = 0$ are delayed to the following sections 4 and 4.1, whereas the asymptotic properties of state trajectories subject to $|w(t)| \to 0$ as $t \to \infty$ are treated in section 5.1.

In practice it may be difficult to assure that the disturbances are of one of the previous classes, whereas it often will be easy, due to physical properties of the plant, to specify some constant $c \in \mathbb{R}^+$ such that $|w(t)| \leq c$ for all $t \in \mathbb{R}$. The analysis of performance envelopes of systems and the asymptotic properties of state trajectories generated by bounded disturbances are investigated in section 5.3.

4 La Salle's invariance principle in a dissipative setting

This section is devoted to the stability analysis of dissipative systems with respect to invariant sets of the undisturbed dynamics subject to $w(\cdot) = 0$. The overall idea is to use the storage function of a dissipative system as a Lyapunov function to investigate the stability properties of some state trajectories of the system. The notion of almost regular, regular, and strictly regular supply rates will be useful to prove stability of trajectories generated by exogenous inputs $w(\cdot) = 0$. Related work has earlier been published by David J. Hill [Hil92] in the framework of behavioral systems, where asymptotic stability of some preferred set of internal states is considered for systems represented by families of input-output operators, with quadratic supply rates. See also the research paper by David J. Hill and Peter J. Moylan [HM80a]. We postpone asymptotic stability issues of more general trajectories not satisfying $w(\cdot) = 0$ to later sections.

Clearly, boundedness of state trajectories is an important stability requirement in all real world applications. On the other hand, asymptotic stability of state trajectories may be mandatory in certain control problems. For this purpose it is beneficial to recall boundedness and invariance properties of autonomous systems of the form

$$\dot{x} = X(x) \quad . \tag{29}$$

We assume that the integral curves of (29) are uniquely given on some suitable set, and we denote them $x(\cdot) = x(\cdot, t_0, x_0)$. We recall that a set $\mathcal{A} \subset \mathbb{R}^m$ is called **invariant** if all trajectories of the autonomous system (29) starting in $\mathcal{A}$ are defined in the future and in the past, and evolve entirely inside $\mathcal{A}$. The set is called **positive invariant** if all trajectories starting in $\mathcal{A}$ are defined in the future and never leave $\mathcal{A}$ as time increases.

It is our purpose to use a solution to the Hamilton-Jacobi inequality as a Lyapunov function in order to establish asymptotic stability properties of dissipative systems. We recall a theorem knows as the La Salle's invariance principle, which has been published in the early sixties by La Salle and Lefschetz [SL61].

4.1 Theorem (La Salle and Lefschetz) *Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^1 function and let Ω denote a connected component of the pre-image $V^{-1}(]-\infty, c])$, $c \in \mathbb{R}$. Assume that Ω is bounded, and that*

$$\tfrac{d}{dt}V \leq 0 \tag{30}$$

within Ω along any trajectory of the autonomous system (29). Let $\mathcal{V} \subset \Omega$ be the largest set where $\frac{\partial}{\partial t}V = 0$, and let $\mathcal{A}$ be the largest invariant set contained in $\mathcal{V}$.

Then Ω is positive invariant and every solution in Ω tends to $\mathcal{A}$ as $t \to \infty$.

In other words: Ω is a basin of attraction for the stable invariant set $\mathcal{A}$. The original proof of Theorem 4.1 shows that any such C^1 function V satisfying $\frac{d}{dt}V \leq 0$ is not assumed to be positive definite.

In the following we are interested in storage functions which are not necessarily continuously differentiable. On the other hand we want not to burden this exposition with technical details, and therefore we assume for the rest of this section that continuous and locally Lipschitz storage functions exists. We follow the standard approach in nonsmooth analysis [Cla83]. In special, all convex functions and all C^1 functions are locally Lipschitz on the interior of their domain [HUL93]. Due to Rademachers theorem any locally Lipschitz function is differentiable almost everywhere (in Lebesgue measure), so that the following definition makes sense. We denote by **con $\{p_i\}$ the convex hull** of the cotangent vectors p_i, that is,

$$\mathrm{con}\,\{p_i\} \equiv \left\{ \textstyle\sum_i c_i p_i \;\middle|\; c_i \geq 0 \;\;,\;\; \textstyle\sum_i c_i = 1 \right\} \quad .$$

4.2 Definition *The **generalized gradient ∂V** of a continuous, locally bounded, and locally Lipschitz function $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$ is the set-valued function which maps to each $x \in \mathcal{R}$ a nonempty, closed and convex set of cotangent vectors $p \in \mathbb{R}^n$ in the following way:*

$$\partial V(x) \equiv con \left\{ \lim_{i\to\infty} \tfrac{\partial V}{\partial x}(x_i) \;\middle|\; x_i \to x \;,\; x_i \notin \mathcal{O} \right\} \quad , \tag{31}$$

where $\mathcal{O} \subset \mathcal{R}$ is any set of Lebesgue measure zero which includes all the points where V fails to be differentiable.

In other words, we are considering the sequence of all $\{x_i\}$ converging to x while avoiding $\mathcal{O}$ such that the classical gradient $\frac{\partial V}{\partial x}(x_i)$ converges, then the convex hull of all such limit gradients is the generalized gradient ∂V. From now on, we denote by the symbol $\frac{\partial V}{\partial x}$ any function $\frac{\partial V}{\partial x} : \mathcal{R} \mapsto \mathrm{IR}^n$ such that $\frac{\partial V}{\partial x}(x) \in \partial V(x)$ for all $x \in \mathcal{R}$. We see immediately that the generalized gradient for any continuously differentiable function collapses at each x to a singleton, and then $\frac{\partial V}{\partial x}$ denotes the classical gradient.

The **generalized directional gradient** of a locally bounded function $V : \mathcal{R} \subset \mathrm{IR}^n \mapsto \mathrm{IR}$ is the function $V^0 : \mathcal{R} \times \mathrm{IR}^n \mapsto \mathrm{IR}$ defined by

$$\begin{aligned} V^0(x,v) &\equiv \limsup_{y \to x} \left\{ \tfrac{\partial V}{\partial x}(y)v \;\middle|\; y \notin \mathcal{O} \right\} \\ &= \max_{p \in \partial V(x)} pv \quad . \end{aligned} \tag{32}$$

In the following we want to investigate the asymptotic properties of state trajectories subject to $w(\cdot) = 0$. It turns out that certain subsets of the reachable set, related to the HJI (13), the storage function and the performance function, play an important part in this investigation.

4.3 Definition *Let $V : \mathcal{R} \subset \mathrm{IR}^n \mapsto \mathrm{IR}$ denote any continuous, locally bounded, locally Lipschitz, and non-negative function. We define the following subsets of $\mathcal{R} \subset \mathrm{IR}^n$:*
the ***Hamiltonian null set***

$$\mathcal{N} \equiv \left\{ x \in \mathcal{R} \;\middle|\; \textit{there exists}\, p(x) \in \partial V(x) \;\textit{with}\; \mathsf{H}^*(x, p(x)) = 0 \right\} \quad ,$$

the ***storage null set***

$$\mathcal{V} \equiv \left\{ x \in \mathcal{R} \;\middle|\; \textit{there exists}\, p(x) \in \partial V(x) \;\textit{with}\; p(x)X(x,0) = 0 \right\} \quad ,$$

the ***storage kernel***

$$\ker V \equiv \left\{ x \in \mathcal{R} \;\middle|\; V(x) = 0 \right\} \quad ,$$

and the ***performance kernel***

$$\ker Z \equiv \left\{ x \in \mathcal{R} \;\middle|\; Z(x) = 0 \right\} \quad .$$

We see that the sets $\ker V$ and $\ker Z$ are closed sets: they are the preimages of a compact set under a continuous map. Notice also that the storage null set is defined by specifying zero disturbances.

There is a relation between the set of points in $\mathcal{R}$ where the function $V(t) \equiv V(x(t))$ is constant along the trajectories $x(\cdot)$ of the undisturbed system $\dot{x} = X(x,0)$, and the storage null set $\mathcal{V}$: in case that V is differentiable, constantness of $V(t)$ clearly is equivalent to that the classical gradient satisfies $\frac{\partial V}{\partial x}X(x,0) = 0$, and in case that V is not differentiable, constantness of $V(t)$ implies that the equation $\frac{\partial V}{\partial x}X(x,0) = 0$ holds for some $\frac{\partial V}{\partial x} \in \partial V$ (and if the equation $\frac{\partial V}{\partial x}X(x,0) = 0$ holds for all $\frac{\partial V}{\partial x} \in \partial V$, the reverse conclusion is valid). Hence, the set of points where $V(t)$ is constant along the trajectories $x(\cdot)$ of the undisturbed system $\dot{x} = X(x,0)$, is a subset of $\mathcal{V}$.

A similar interpretation holds for the Hamiltonian null set $\mathcal{N}$: assume that a maximizing disturbance $w_{\max} : \mathrm{IR}^n \mapsto \mathrm{IR}^l$ exists, then the set of points where the function

$h(t) \equiv V(x(t)) - s(w_{\max}(x), Z(x))$ is constant along the trajectories $x(\cdot)$ of the (maximally) disturbed system $\dot{x} = X(x, w_{\max}(x))$ is a subset of the Hamiltonian null set.

We denote in the following the storage null set of the available storage V_A by the symbol $\mathcal{V}_A$, and the storage null set of the required supply V_R by $\mathcal{V}_R$. We are able to state the following (quite obvious, but useful) facts on kernel and null sets.

4.4 Proposition (storage kernel and null sets) *For any continuous and locally Lipschitz storage function $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ the storage null set is a closed set, and $\mathcal{V} \supset \ker V$ holds.*

Assume that the maximal disturbance (14) is a continuous functions on the reachable set $\mathcal{R}$. Then, for any continuous and locally Lipschitz storage function, the Hamiltonian null set is a closed set.

Furthermore, $\mathcal{X}_* = \ker V_A = \ker V = \ker V_R$ holds for any possible storage function V.

Proof: Assume that there is a sequence of points $\{x_i\}$ with $\frac{\partial V}{\partial x}(x_i)X(x_i, 0) = 0$ for some $\frac{\partial V}{\partial x} \in \partial V$, which is converging to x. We have to show that there exists a $p \in \partial V(x)$ satisfying $pX(x, 0) = 0$.

Now, the sequence $\{\frac{\partial V}{\partial x}(x_i)\}$ is bounded (since V is locally Lipschitz, and we can without loss of generality avoid $\mathcal{O}$), hence there exists a subsequence converging to some $p \in \partial V$. Clearly, for this subsequence,

$$\lim_{i \to \infty} \frac{\partial V}{\partial x}(x_i)X(x_i, 0) = pX(x, 0) = 0 \quad ,$$

hence we conclude that $\mathcal{V}$ is closed.

It is a simple consequence of definition 4.3 that $\mathcal{V} \supset \ker V$ holds whenever V is a storage function: Since both $\mathcal{V}$ and $\ker V$ are closed sets, and $p = 0$ is included in $\partial V(x_*)$ for each x_* satisfying $V(x_*) = 0$, the result follows trivially.

A similar argumentation shows that the Hamiltonian null set under the stated conditions is a closed set.

Theorem 3.12 implies that V_R is a storage function due to the standing assumption 3.11 that there exists a point of minimal storage x_* from which $\mathcal{R}$ can be reached. Hence $V_A(x) \leq V(x) \leq V_R(x)$, and it follows directly that $\mathcal{X}_* = \ker V_A \supset \ker V \supset \ker V_R$. Now, by definition 3.9 we have $V_R(x_*) = 0$ for all $x_* \in \mathcal{X}_*$, and $\mathcal{X}_* = \ker V_A = \ker V = \ker V_R$ follows. □

4.5 Theorem (Invariance principle) *Assume that the HJI (13) has a continuous and locally Lipschitz viscosity solution $V : \mathcal{R} \mapsto \mathbb{R}$. Let $\Omega \subset \mathbb{R}^n$ be any compact set. Assume that all $x(\cdot)$ with $x_0 \in \Omega$ generated by $w(\cdot) = 0$ are bounded in future inside Ω.*

Then all such $x(\cdot)$ approach the largest invariant set $\mathcal{A}$ contained in the intersection

$$\mathcal{A} \;\subset\; \mathcal{V} \cap \Omega \cap \mathcal{R} \;.$$

Proof: By assumption 3.11 we have at least one point of minimal storage $x_* \in \mathcal{R}$. Then $V_*(x) \equiv V(x) - V(x_*)$ also solves the HJI (13) on $\mathcal{R}$, hence we can assume without loss of generality that $V(x)$ is normed such that $V(x_*) = \min_{x \in \mathcal{R}} V(x) = 0$, and therefore V is a l.s.c. storage function on $\mathcal{R}$.

We remember that all bounded $x(\cdot)$ have a non-empty, compact and invariant positive limit set Γ^+, and $x(\cdot) \to \Gamma^+$ as $t \to \infty$.

Now, observe that $V(x)$ is (lower semi-) continuous and defined on the compact set $\Omega \cap \overline{\mathcal{R}}$. Hence $V(x)$ is bounded from below, and there exists a local point of minimal storage $x_\Omega \in \Omega \cap \overline{\mathcal{R}}$ satisfying $V(x_\Omega) \geq V(x_*) = 0$ [Ped89].

We show now that the positive limit set of any trajectory $x(\cdot)$ subject to $w(\cdot) = 0$ which is bounded in future in Ω satisfies $\Gamma^+ \subset \mathcal{V}$: given any such $x(\cdot)$, the storage function $V(x(\cdot))$ is decreasing, since the dissipation inequality (5) together with almost regularity of the supply rate implies that

$$V(x(t)) - V(x_0) \leq 0 \quad \text{for all} \quad t \geq 0 \quad .$$

Also, $V(\cdot)$ is bounded from below, hence approaches some minimal value, say $c_\Gamma \geq 0$, as $t \to \infty$. By continuity we conclude that $V(x) = c_\Gamma$ on the positive limit set Γ^+. It follows that $\Gamma^+ \subset \mathcal{V}$, and in case that V is C^1, $\frac{d}{dt}V = 0$ on Γ^+.

Finally, Γ^+ is an invariant set, and contained in the maximal invariant set $\mathcal{A} \subset \mathcal{V} \cap \Omega \cap \mathcal{R}$. Therefore any trajectory $x(\cdot)$ subject to $w(\cdot) = 0$, and which is bounded in Ω, is approaching $\mathcal{A}$ as $t \to \infty$. □

It is in general not easy to find a weak solution of the HJI (13), and it is not at all clear nor proved that the sets $\mathcal{V}$ and $\mathcal{N}$ are invariant under the non-uniqueness of all possible storage functions. However, the control engineer has in practice the possibility to specify his favorite performance kernel and Hamiltonian null set, hopefully not excluding the existence of a storage function by this approach. Then the following theorem may help to identify the storage null set of a particular storage function.

4.6 Theorem (Subsets) *Assume that the HJI (13) has a continuous and locally Lipschitz viscosity solution $V : \mathcal{R} \mapsto \mathbb{R}$. Moreover, assume that all $x(\cdot)$ subject to $w(\cdot) = 0$ are bounded in future inside some compact $\Omega \subset \mathbb{R}^n$. Then $\mathcal{V} \subset \mathcal{N}$ holds. If in addition the supply rate is regular, then $\mathcal{V} \subset \ker Z$ follows.*

Proof: According to definition 4.3 we have only to consider trajectories subject to $w(\cdot) = 0$ to investigate the properties of the storage null set $\mathcal{V}$. Therefore theorem 4.5 applies. Hence, the almost regularity of the supply rate together with (12) and (13) shows that

$$\frac{\partial V}{\partial x} X(x,0) \leq \frac{\partial V}{\partial x} X(x,0) - s(0,z) \leq \mathsf{H}^* \leq 0 \tag{33}$$

holds for all $\frac{\partial V}{\partial x} \in \partial V$.

We show first that $\mathcal{V} \subset \mathcal{N}$: by inequality (33) and the previous discussion it follows that for all $\frac{\partial V}{\partial x}(x) \in \partial V(x)$ such that $\frac{\partial V}{\partial x}(x) X(x,0) = 0$ there holds

$$0 \leq -s(0,z) \leq \mathsf{H}^* \leq 0 \quad \text{on} \quad \mathcal{V} \quad , \tag{34}$$

hence we have $\mathcal{V} \subset \mathcal{N}$. Notice that the same $\frac{\partial V}{\partial x}(x) \in \partial V(x)$ which defines the storage null set $\mathcal{V}$, defines also the Hamiltonian null set $\mathcal{N}$.

We see easily that regularity of the supply rate implies $\mathcal{V} \subset \ker Z$: any such trajectory evolving entirely on $\mathcal{V}$ satisfies $z(\cdot) = 0$ by equation (34) and regularity of s. It follows that $\mathcal{V} \subset \ker Z$. □

Notice also that we have $\frac{\partial V}{\partial x} X(x, 0) < 0$ for all motions evolving on $\Omega \backslash \mathcal{N}$. Trajectories on $\mathcal{N}$ are satisfying $\frac{\partial V}{\partial x} X(x, 0) = 0$ only if $s(0, z) = 0$, which implies $z(\cdot) = 0$ by regularity of the supply rate.

In the standard nonlinear $\mathcal{H}_\infty$ control theory where the stability of the equilibrium point zero is investigated, it is known [IA92b, IA92a, vdS92a, IK95, BHW93] that the maximal disturbance $w_{\max}$ vanishes at the origin. A similar property holds for the generalized problem treated here. We denote the **union of all positive limit sets** of bounded $x(\cdot, t_0, x_0, w = 0)$ by the symbol $\cup\boldsymbol{\Gamma}^+_{\boldsymbol{w=0}}$.

4.7 Corollary *Assume that the supply rate is regular, and the supremum in the definition of the Hamiltonian (12) is attained at each $(x, p) \in \mathcal{R} \times \mathrm{IR}^n$. Assume furthermore that the HJI (13) has a continuous and locally Lipschitz viscosity solution $V : \mathcal{R} \mapsto \mathrm{IR}$, and define the set-valued function*

$$w_{max}(x) \equiv w_{max}(x, \tfrac{\partial V}{\partial x}(x)) \quad , \quad \tfrac{\partial V}{\partial x} \in \partial V \quad , \tag{35}$$

where the maximizing disturbance $w_{\max}(x, p)$ is given by (15). Moreover, assume the existence of some compact set $\Omega \subset \mathrm{IR}^n$ such that all $x(\cdot)$ with $x_0 \in \Omega$, and generated by $w(\cdot) = 0$, are bounded in future inside Ω, and assume that the intersection between $\cup\Gamma^+_{w=0}$ and $\mathcal{X}_*$ is non-empty. Then there holds

$$w_{max}(x) = 0 \quad on \quad \Gamma^+_{w=0} \subset \mathcal{X}_* \quad .$$

Proof: Using the Hamiltonian (12), the maximal disturbance (15) and the HJI (13) we have the inequalities

$$\tfrac{\partial V}{\partial x} X(x, 0) - s(0, z) \leq \tfrac{\partial V}{\partial x} X(x, w_{\max}(x)) - s(w_{\max}(x), z) = \mathsf{H}^* \leq 0 \quad \text{for all} \quad \tfrac{\partial V}{\partial x} \in \partial V \quad .$$

The proof of theorem 4.5 and theorem 4.6 shows that $\Gamma^+_{w=0} \subset \mathcal{V} \subset \mathcal{N}$ and $\Gamma^+_{w=0} \subset \ker Z$. Hence, by regularity of the supply rate there follows

$$0 \leq \tfrac{\partial V}{\partial x} X(x, w_{\max}(x)) - s(w_{\max}(x), 0) \leq 0 \quad \text{on} \quad \Gamma^+_{w=0} \quad .$$

Since $\Gamma^+_{w=0} \subset \mathcal{X}_*$ is assumed, it follows that for all $x \in \Gamma^+_{w=0}$ the generalized gradient satisfies $0 \in \partial V(x)$. Hence,

$$-s(w_{\max}, 0) = 0 \quad \text{on} \quad \Gamma^+_{w=0} \subset \mathcal{X}_*$$

follows. Finally, $w_{\max} = 0$ on $\Gamma^+_{w=0} \subset \mathcal{X}_*$ is a consequence of the regularity of the supply rate s. □

In case that the union of positive limit sets $\cup\Gamma^+_{w=0}$ is not connected, the proof of theorem 4.5 indicates that each component of $\cup\Gamma^+_{w=0}$ is a local minimum of the function $V(x)$, but the constant value $V(x) = c_\Gamma$ will in general be different from component to component. Nevertheless, we can assume that $V(x)$ is positive semidefinite.

Theorem 4.5 shows that $\cup\Gamma^+_{w=0}$ is a subset of the intersection of the Hamiltonian null set $\mathcal{N}$, the storage null set $\mathcal{V}$, and the performance kernel $\ker Z$ if the supply rate is regular. In practice, the sets $\mathcal{N}$ and $\mathcal{V}$ can not be computed before a solution to the HJI (13) is known, and - even more inconvenient - the sets $\mathcal{N}$ and $\mathcal{V}$ may be depending on the special storage function found.

4.1 Attraction of compact sets

We may be concerned with the question whether some given compact set $\mathcal{S} \subset \Omega$ is attractive for all bounded trajectories subject to $w(\cdot) = 0$. By theorem 4.5 and theorem 4.6 we see that the intersection $\mathcal{S} \cap \ker Z$ must be non-empty if the supply rate is regular - otherwise there will be no positive limit set satisfying $\Gamma^+ \subset \mathcal{S}$.

In order to conclude attraction of the set $\mathcal{S}$ subject to $w(\cdot) = 0$, we can impose some conditions on the performance function $Z(x)$. Observability of the state trajectory on $\mathcal{S}$, that is $Z(x)|_{x\in\Omega\setminus\mathcal{S}} \neq 0$, may be too severe an assumption. Instead we will impose a weaker detectability assumption on the system:

4.8 Definition *Given some compact set $\mathcal{S} \subset \mathcal{R}$, the system (4) is called **$\mathcal{S}$-detectable** if all bounded trajectories $x(\cdot) = x(\cdot, t_0, x_0, 0)$ (subject to $w(\cdot) = 0$) generating the zero-output $z(\cdot) = 0$ are approaching $\mathcal{S}$ as $t \to \infty$.*

*In case that $\mathcal{S}$ is the origin, we say the control system is **zero-detectable**.*

Given $\mathcal{S}$-detectability, we can prove attraction of the set $\mathcal{S}$ subject to $w(\cdot) = 0$.

4.9 Corollary ($\mathcal{S}$-detectability) *Given a regular supply rate, assume that theorem 4.5 applies, and that the uncontrolled system (4) is $\mathcal{S}$-detectable. Then all $x(\cdot)$ subject to $w(\cdot) = 0$ which are bounded in future converge to $\mathcal{S}$ as $t \to \infty$. Hence $\Gamma^+ \subset \mathcal{A} \subset \ker Z$ and $\Gamma^+ \subset \mathcal{S}$ hold.*

A similar property has been published as theorem 6 in [HM80a] in the context of a family of input-output operators of a behavioral system with states.

Proof: All such trajectories have due to theorem 4.5 and theorem 4.6 positive limit sets satisfying $\Gamma^+ \subset \ker Z$, hence by $\mathcal{S}$-detectability the compact set $\mathcal{S}$ is approached. It follows that $\Gamma^+ \subset \mathcal{S}$. $\square$

In practice the notion of $\mathcal{S}$-detectability is very awkward: it requires essentially knowledge of all trajectories satisfying $z(\cdot) = 0$. This problem can be circumvented if we study solutions of the **strict Hamilton-Jacobi inequality**

$$\mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \leq -\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \quad , \tag{36}$$

where $\alpha_{\mathsf{H}} : \mathrm{IR}^+ \mapsto \mathrm{IR}^+$ is any function of class $\mathcal{K}$ (see appendix A for the notion of $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions). Clearly, the existence of a l.s.c. viscosity solution of the strict HJI implies that the dissipation inequality (5) holds for all disturbances $w(\cdot)$.

4.10 Corollary (strict convergence) *Assume that all requirements of theorem 4.5 are satisfied using the strict HJI (36), where $\mathcal{S}$ is any compact subset of $\mathcal{R}$. Then theorem 4.5 applies using the HJI (13), and all such $x(\cdot)$ generated by $w(\cdot) = 0$ which are bounded in future converge to $\mathcal{S}$ as $t \to \infty$. Hence $\Gamma^+ \subset \mathcal{A} \subset \mathcal{V} \subset \mathcal{N} \subset \mathcal{S}$ holds.*

Proof: Clearly the strict HJI (36) implies the HJI (13), hence theorem 4.5 holds. We need only to prove the attraction of $\mathcal{S}$. Notice that the strict HJI (36) implies $\mathcal{N} \subset \mathcal{S}$, and by theorem 4.6 $\Gamma^+ \subset \mathcal{A} \subset \mathcal{V} \subset \mathcal{N} \subset \mathcal{S}$ follows. □

The rate of convergence of the motion is dependent on the shape of V and the magnitude of $\frac{\partial V}{\partial x} X(x, 0)$. The later is by the strict HJI (36) estimated by $\frac{\partial V}{\partial x} X(x, 0) \leq -\alpha_{\mathsf{H}}$. In many practical applications it will be therefore advantageous to solve the strict HJI using a $\mathcal{K}$ or even $\mathcal{K}_\infty$ function α_{H} as large as possible.

Finally we show that asymptotic stability of some compact set $\mathcal{S}$ also can be achieved using a condition on the performance measure $Z(x)$ instead of forcing the strict HJI (36) to be satisfied.

4.11 Corollary (performance kernel) *Assume that theorem 4.5 applies using a regular supply rate. Assume furthermore that the uncontrolled system (4) is such that there exists a compact set $\mathcal{S} \subset \mathcal{R}$ and a $\mathcal{K}$ function $\underline{\alpha}_Z : \mathrm{IR}^+ \mapsto \mathrm{IR}^+$ such that*

$$\underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)| \quad .$$

Then $\Gamma^+ \subset \mathcal{A} \subset \mathcal{V} \subset \ker Z \subset \mathcal{S}$, and all $x(\cdot)$ subject to $w(\cdot) = 0$ which are bounded in future converge to $\mathcal{S}$ as $t \to \infty$.

Proof: It is easy to see that $\underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)|$ implies $\ker Z \subset \mathcal{S}$. By theorem 4.5 and theorem 4.6 we have $\Gamma^+ \subset \mathcal{A} \subset \mathcal{V} \subset \ker Z$ for all such positive limit sets, hence $x(\cdot) \to \mathcal{S}$ as $t \to \infty$. □

Also here, the rate of convergence of the motion is dependent on the shape of V and the magnitude of $\frac{\partial V}{\partial x} X(x, 0)$. Now we have the estimate $\frac{\partial V}{\partial x} X(x, 0) \leq -s(0, \underline{\alpha}_Z)$, and in case that the supply rate is strictly regular $\frac{\partial V}{\partial x} X(x, 0) \leq -\alpha_z \circ \underline{\alpha}_Z$. It may be a good idea to use performance measures which are bounded from below by a $\mathcal{K}$ or even $\mathcal{K}_\infty$ function $\underline{\alpha}_Z$ as large as possible.

Let us for the moment assume that the uncontrolled system (4) has an invariant set $\mathcal{S}$ when regarded as an autonomous system with respect to the zero disturbance $w(\cdot) = 0$. If we want to prove that the system is dissipative with respect to some given supply rate, and that the set $\mathcal{S}$ is attractive in the sense that all undisturbed motions which are bounded in future converge to $\mathcal{S}$ as $t \to \infty$, then we have to find a candidate storage function by solving the HJI (13). This is in general a difficult task, and may be eased considerably if we know the kernel $\ker V$, and can use this knowledge to construct a suitable finite dimensional basis expansion of V. Then we have to compute the null sets $\mathcal{N}$ and $\mathcal{V}$, and the kernel $\ker Z$ in on order to evaluate the possibility of attraction. Moreover, we have to know whether the undisturbed motions are bounded in future or not.

These problems can be solved more efficiently if we try to find a storage function which takes advantage of lemma 3.25, theorem 4.5, the strict HJI (36), and corollary 4.11 simultaneously.

4.12 Proposition (identity of sets) *Assume that the supply rate is regular and that a continuous and locally Lipschitz viscosity solution to the strict HJI (36) exists, such that*

$$\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}}) \quad , \quad \mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \leq -\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \quad , \text{ and } \quad \underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)| \quad .$$

Here $\mathcal{S} \subset \mathcal{R}$ is a compact set, and $\underline{\alpha}_V$, $\overline{\alpha}_V$, α_{H}, and $\underline{\alpha}_Z$ are four functions of class $\mathcal{K}$.

Then the identity

$$\mathcal{N} = \mathcal{V} = \ker V = \ker Z = \mathcal{S}$$

holds, and any trajectory $x(\cdot)$ subject to $w(\cdot) = 0$ which is bounded in future satisfies $x(\cdot) \to \mathcal{S}$. Moreover, $\mathcal{S}$ is locally asymptotically stable, and in case that V is proper, globally asymptotically stable.

Assume in addition that $\mathcal{S}$ consists only of one isolated trajectory of the undisturbed system (3), then

$$\mathcal{N} = \mathcal{V} = \ker V = \ker Z = \mathcal{S} = \mathcal{A} = \Gamma^+ \quad .$$

Proof: The convergence of such trajectories to the set $\mathcal{S} \subset \mathcal{A}$ follows by corollaries 4.10, and 4.11.

Note that the assumption $\underline{\alpha}_V \leq V \leq \overline{\alpha}_V$ implies $\ker V = \mathcal{S}$. The strict HJI $\mathsf{H}^* \leq -\alpha_{\mathsf{H}}$ implies $\mathcal{S} \supset \mathcal{N}$, and $\mathcal{S} \supset \ker Z$ follows from condition $\underline{\alpha}_Z \leq |Z|$.

Now, by theorem 4.5 we have $\mathcal{V} \supset \mathcal{A}$, and theorem 4.6 implies $\mathcal{V} \subset \mathcal{N}$ for almost regular supply rates, and $\mathcal{V} \subset \ker Z$ for all regular supply rates.

Finally, proposition 4.4 implies $\mathcal{V} \supset \ker V$.

Combining these simple facts, we have

$$\Gamma^+ \subset \mathcal{A} \subset \mathcal{V} = \mathcal{N} = \mathcal{S} = \ker V \quad \text{for all } s(w,z) \text{ almost regular, and}$$
$$\Gamma^+ \subset \mathcal{A} \subset \mathcal{V} = \ker Z = \mathcal{S} = \ker V \quad \text{for all } s(w,z) \text{ regular.}$$

hence $x(\cdot) \to \mathcal{S}$ has been showed.

Now, $\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}})$ implies that there exists a $c_0 > 0$ such that the preimage $V^{-1}([0,c])$ is compact for all $0 \leq c \leq c_0$, hence all such undisturbed trajectories with initial point $V(x_0) \leq V(c_0)$ are contained in the preimages $V^{-1}([0, V(x_0)])$ for all $t \in \mathbb{R}^+$. Since clearly $\lim_{x_0 \to \mathcal{S}} V^{-1}([0, V(x_0)]) = \mathcal{S}$, stability of the set $\mathcal{S}$ is proven. In case that V is proper, the same argumentation shows global stability of $\mathcal{S}$. Together with attraction of $\mathcal{S}$ locally or globally asymptotic stability follows.

Assume furthermore that $\mathcal{S}$ consists only of one single trajectory of the undisturbed system (3). Then the largest invariant set contained in $\mathcal{S}$ equals $\mathcal{S}$ itself, and $\mathcal{A} = \mathcal{S}$ follows. Since the positive limit set of an autonomous system is invariant, we have $\Gamma^+ = \mathcal{A} = \mathcal{S}$, and the result follows as stated. □

4.13 Remark *The existence of two $\mathcal{K}$ functions $\underline{\alpha}_V$, $\overline{\alpha}_V$ satisfying $\underline{\alpha}_V(|x|_\mathcal{S}) \leq V(x) \leq \overline{\alpha}_V(|x|_\mathcal{S})$ implies that the preimage $V^{-1}([0, c])$ is compact for many $c \in \mathbb{R}$. It follows that lemma 3.25 or corollary 3.26 can be applied without difficulties to prove boundedness of state trajectories.*

In fact, the existence of two such $\mathcal{K}$ functions satisfying $\underline{\alpha}_V(|x|_\mathcal{S}) \leq V(x) \leq \overline{\alpha}_V(|x|_\mathcal{S})$ is often a simple consequence of the structure of the system at hand. We have the following generalization of a result on positive definiteness of C^1 storage functions due to Hill and Moylan [HM76]. See also Theorem 1 in the paper [Hil92].

4.14 Definition *Any lower semicontinuous and locally bounded function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is called **positive $\mathcal{S}$-definite** with respect to a compact set $\mathcal{S} \subset \mathcal{R}$ if there exist two $\mathcal{K}$ functions $\underline{\alpha}_V$, $\overline{\alpha}_V$ such that*

$$\underline{\alpha}_V(|x|_\mathcal{S}) \leq V(x) \leq \overline{\alpha}_V(|x|_\mathcal{S})$$

*for all $x \in \mathcal{R}$. It is called **proper** if $V(x) \to \infty$ for all $x \to \partial\mathcal{R}$.*

The usual definition of positive definiteness does not imply $V(x) = 0$ for all $x \in \mathcal{S}$. However, it is convenient to include this specification, since all storage functions are majorized by the required supply, which by definition satisfies $V_R(x) = 0$ for all $x \in \mathcal{S}$ in case that $\mathcal{S}$ is a suitable compact subset of points of minimal storage, namely in case that $\mathcal{S} = \ker V_R \subset \mathcal{X}_*$. Also, the existence of an upper bound $\overline{\alpha}_V \in \mathcal{K}$ follows from local boundedness of V_R on $\mathcal{R}$.

4.15 Lemma *Assume that $\mathcal{S} \subset \mathcal{R}$ is a compact set, and that the uncontrolled system (4) is dissipative.*

1. *If the uncontrolled system (4) is $\mathcal{S}$-detectable and the supply rate is regular, then all points of minimal storage $x_* \notin \mathcal{S}$ are connected to $\mathcal{S}$ by a trajectory $x(\cdot)$ subject to $w(\cdot) = 0$ which consists solely of points of minimal storage.*
2. *If the performance function satisfies $\underline{\alpha}_Z(|x|_\mathcal{S}) \leq |Z(x)|$, $\underline{\alpha}_Z \in \mathcal{K}$ and the supply rate is regular, all l.s.c storage functions satisfy $\underline{\alpha}_V(|x|_\mathcal{S}) \leq V(x)$ for some $\mathcal{K}$ function $\underline{\alpha}_V$. If in addition $Z : \mathcal{R} \mapsto \mathbb{R}^p$ is proper and $s(w, z)$ is strictly regular, all l.s.c storage functions are proper.*

3. *All l.s.c storage functions satisfying the strict HJI (36) satisfy $\underline{\alpha}_V(|x|_{\mathcal{S}}) \le V(x)$ for some $\mathcal{K}$ function $\underline{\alpha}_V$. If in addition $-\mathsf{H}^*(x, \frac{\partial V}{\partial x}(x))$ is proper, all such l.s.c storage functions are proper.*

4. *If in addition $\mathcal{S} = \mathcal{X}_*$, properties 2 and 3 imply that all such storage functions are positive $\mathcal{S}$-definite.*

Proof: To show property 1, take any point of minimal storage $x_* \in \mathcal{R}\backslash\mathcal{S}$. Then the trajectory $x(\cdot)$ subject to $w(\cdot) = 0$ with initial point $x_0 = x_*$ converges to $\mathcal{S}$ as $t \to \infty$ by $\mathcal{S}$-detectability, hence it suffices to show that $V(x(t)) = 0$ for all $t \in \mathrm{IR}^+$. By regularity of the supply rate, and by dissipation of the system we have $V(x(t)) - V(x_*) \le 0$, and property 1 follows immediately.

To show property 2, that is, positive definiteness with respect to $\mathcal{S}$ we use the available storage V_A. By theorem 3.6 we have

$$V(x_0) \ge V_A(x_0) = \sup_{w,T} \int_0^T -s(w(t), z(t))\, dt \ge \int_0^T -s(0, z(t))\, dt \quad ,$$

where the last inequality follows from the sup condition.

Now, to prove the result of property 2 assume that $x_0 \in \mathcal{R}\backslash\mathcal{S}$ and $z(t) = 0$ for all $t \in \mathrm{IR}^+$. This contradicts the assumption on the performance kernel. It follows that $z(t) \neq 0$ on a set of measure larger than zero for all initial conditions $x_0 \in \mathcal{R}\backslash\mathcal{S}$, hence with the regularity of the supply rate we have

$$V(x_0) \ge \int_0^T -s(0, z(t))\, dt > 0 \quad .$$

Property 2 implies that $x(\cdot) \to \mathcal{S}$, and moreover, the function $V(t) = V(x(t))$ is by regularity of the supply rate strictly decreasing on $\mathcal{R}\backslash\mathcal{S}$ (along the trajectories of the undisturbed system $\dot{x} = X(x, 0)$). It follows that the set of points of minimal storage x_* satisfies $X_* \subset \mathcal{S}$. Hence, the above inequality shows that there exists a $\mathcal{K}$ function $\underline{\alpha}_V$ satisfying $\underline{\alpha}_V(|x|_{\mathcal{S}}) \le V(x)$. In fact, the function defined by

$$\underline{\alpha}_V(r) \equiv \inf_{|x|_{\mathcal{S}}=r} V(x)$$

does the job.

We need to show properness of storage functions in property 2. Strict regularity of the supply rate together with positive definiteness and properness of the performance function implies that $-s(0, Z(x))$ is proper. Now $V(x_0) \ge \int_0^T -s(0, z(t))\, dt$ for all $T \in \mathrm{IR}^+$ implies that V is proper.

To prove property 3 , assume that $x_0 \in \mathcal{R}\backslash\mathcal{S}$. Then the strict HJI (36) implies that

$$V(x_T) - V(x_0) \le \int_0^T s(w(t), z(t)) - \alpha_{\mathsf{H}}(|x(t)|_{\mathcal{S}})\, dt \quad ,$$

and since V satisfies the common HJI (13), it follows that $V \geq 0$, hence

$$0 \leq V(x_T) \leq V(x_0) + \int_0^T s(w(t), z(t)) - \alpha_{\mathsf{H}}(|x(t)|_{\mathcal{S}})\, dt \quad .$$

Since this inequality is true for all motions starting in x_0, all $w(\cdot)$ and all $T > 0$ this implies

$$\begin{aligned} V(x_0) &\geq \sup_{w(\cdot),T} \int_0^T \alpha_{\mathsf{H}}(|x(t)|_{\mathcal{S}}) - s(w(t), z(t))\, dt \\ &\geq \int_0^T \alpha_{\mathsf{H}}(|x(t)|_{\mathcal{S}}) - s(0, z(t))\, dt \geq \int_0^T \alpha_{\mathsf{H}}(|x(t)|_{\mathcal{S}})\, dt \quad , \end{aligned}$$

where the last inequality follows by almost regularity of the supply rate. Again, property 3 implies that $x(\cdot) \to \mathcal{S}$ and that $\mathcal{X}_* \subset \mathcal{S}$. The existence of a $\mathcal{K}$ function $\underline{\alpha}_V$ satisfying $\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x)$ follows immediately. Clearly, properness of $-\mathsf{H}^*(x, \frac{\partial V}{\partial x})$ implies properness of V: We have for all $\frac{\partial V}{\partial x} \in \partial V$

$$\tfrac{\partial V}{\partial x} X(x,0) - s(0, Z(x)) \leq \mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \to -\infty \quad \text{for all} \quad x \to \partial\mathcal{R} \quad ,$$

and by almost regularity of the supply rate $-s(0, Z(x)) \geq 0$ is known, hence $\frac{\partial V}{\partial x} X(x,0) \to -\infty$ for all $x \to \partial\mathcal{R}$. The properness of all V satisfying the strict HJI (36) follows immediately.

The positive $\mathcal{S}$-definiteness in property 4 follows from the fact that (by theorems 3.12 and 3.6, and proposition 4.4) $\mathcal{X}_* \equiv \ker V_A = \ker V = \ker V_R = \mathcal{S}$, hence V_R is zero on $\mathcal{S}$, and therefore there exists by continuity and local boundedness of V with $0 \leq V_A \leq V \leq V_R \leq \infty$ on $\mathcal{R}$ a $\mathcal{K}$ function $\overline{\alpha}_V$ satisfying $V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}})$. Finally, if V_A satisfies properties 2 and/or 3, then $\ker V_A \subset \mathcal{S}$, and therefore there exists - again by continuity of V - a $\mathcal{K}$ function $\underline{\alpha}_V$ satisfying $\overline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x)$. □

Lemma 4.15 is very useful in connection with the corollaries 4.9, 4.10 and 4.11, since it implies the compactness of the preimage $V^{-1}([0, c])$ for a wide range of c, or even for all $c \in \mathbb{R}$. Hence, lemma 3.25 or corollary 3.26 can be applied to prove boundedness of state trajectories. In particular, the condition $\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}})$ of proposition 4.12 is not needed if $\mathcal{S} = \mathcal{X}_*$, since then it is a consequence of the two other imposed conditions.

5 Robustness with respect to non-zero disturbances

The standard $\mathcal{H}_\infty$ theory is useful for robustness properties with respect to unstructured modelling errors. We show that dissipative systems are often also robust with respect to disturbances. This is a major benefit from an applied point of view, and sufficient conditions for disturbance robust stability are given, as well as estimates of the maximal allowable envelope of performance and the maximal allowable $\mathcal{L}_\infty$ norm of the disturbance.

5.1 Set-stability with vanishing disturbances

Let us in the following investigate the qualitative behavior of disturbed systems of the form $\dot{x} = X(x, w)$ subject to $\mathcal{L}_2^{\text{loc}}$ disturbances which are bounded, piecewise continuous in time, and converging to zero, that is $|w(t)| \to 0$ as $t \to \infty$. We are inclined to believe that all bounded trajectories $x(\cdot)$ generated in this way have nonempty and compact positive limit sets Γ^+ which are matching the positive limit sets Γ_∞^+ belonging to the bounded trajectories $x_\infty(\cdot)$ of the autonomous system $\dot{x} = X(x, 0)$. Unfortunately, this assert is not quite accurate. A more sophisticated approach than intuition is needed to understand the qualitative behavior of time varying systems which approach autonomous systems as $t \to \infty$. We seek inspiration in the work of L. Markus [Mar56] and Yoshizawa [Yos66, chap. 3] to generalize their results to a broader class of positive limit sets than stable equilibria. Furthermore, we allow for piecewise continuity in the time variable to accommodate the previous mentioned class of systems with piecewise continuous disturbance signals.

5.1 Definition (Asymptotically autonomous system) *Let $\Sigma :\ \dot{x} = X(x, t)$ and Σ_∞ : $\dot{x} = X_\infty(x)$ be continuous and locally Lipschitz in x for all fixed $t \in \mathbb{R}^+$, and piecewise continuous in t for all fixed $x \in \mathbb{R}^n$. We say that **Σ is asymptotic to Σ_∞**, denoted **$X(x,t) \to X_\infty(x)$**, in case for each compact $\mathcal{P} \subset \mathbb{R}^n$ and each $\varepsilon > 0$ there is a $T(\mathcal{P}, \varepsilon) \in \mathbb{R}^+$ such that*

$$|X(x, t) - X_\infty(x)| < \varepsilon$$

for all $x \in \mathcal{P}$ and all $t \geq T(\mathcal{P}, \varepsilon)$.

It is easy to tell when disturbed systems of the form (2) are asymptotically autonomous systems.

5.2 Proposition *Given a disturbed system of the form (2), assume that the disturbance $w(\cdot) \in \mathcal{L}_2^{loc}$ is bounded, piecewise continuous, and decreasing to zero, that is satisfying $|w(t)| \to 0$ as $t \to \infty$. Then the system $\dot{x} = X(x, w)$ is asymptotic to $\dot{x} = X(x, 0)$.*

Proof: Clearly $X_\infty(x) = X(x, 0)$ is continuous and locally Lipschitz by the same properties of $X(x, w)$. On any compact set $\mathcal{P} \in \mathbb{R}^n$ there is a $\delta(\mathcal{P}, \varepsilon)$ such that $|X(x, w) - X(x, 0)| \leq \varepsilon$ for $|w| \leq \delta$ (again by continuity of $X(x, w)$). Finally, by convergence of the disturbance there is a suitable $T(\mathcal{P}, \varepsilon) = T(\mathcal{P}, \delta(\mathcal{P}, \varepsilon))$ satisfying $|X(x, w(t)) - X(x, 0)| \leq \varepsilon$ for $t \geq T$. $\square$

It seems reasonably to assume that the trajectories of the time variant and asymptotically autonomous system behave similar as the trajectories of the autonomous system as time goes to infinity. L. Markus [Mar56] states that the positive limit set Γ^+ of a trajectory of the asymptotically autonomous system Σ (also called the perturbed system) consists of a union of autonomous trajectories, and this result is repeated by Yoshizawa [Yos66, chap. 3]. More formally we have the following theorem:

5.3 Theorem *[Mar56, Cro98a] Let the system $\dot{x} = X(x, t)$ be asymptotic to $\dot{x} = X_\infty(x)$, and denote their trajectories $x(\cdot)$ and $x_\infty(\cdot)$ respectively. Then all $x(\cdot)$ which are bounded* *have a non-empty and compact limit set Γ^+ consisting of a union of bounded autonomous trajectories $x_\infty(\cdot)$, defined in past and future.*

Proof: See [Mar56] for a very short proof, and [Cro98a] for a detailed proof. □

Even if the components of the positive limit set Γ^+ are not necessarily positive limit sets of the autonomous system, there are some connections between them. The following theorem is a generalization of some results obtained by L. Markus [Mar56].

5.4 Theorem *[Cro98a] Let the system $\dot{x} = X(x,t)$ be asymptotic to $\dot{x} = X_\infty(x)$, and denote their trajectories $x(\cdot)$ and $x_\infty(\cdot)$ respectively. Let $\Omega \subset \mathbb{R}^n$ be a compact set. Let $\mathcal{A}_\infty^+$ denote the basin of attraction of some asymptotically stable, compact and invariant set $\mathcal{S}$ of the autonomous system Σ_∞, and assume that the proper inclusions $\mathcal{S} \subset \Omega \subset \mathcal{A}_\infty^+$ hold.*

Then any trajectory $x(\cdot)$ which enters Ω in some finite time T_0, and stays in Ω for all times $t \geq T_0$, approaches $\mathcal{S}$ as $t \to \infty$.

Proof: See [Cro98a] for a detailed proof. □

The proof of theorem 5.4 is based on asymptotic stability of some invariant set $\mathcal{S} \subset \mathbb{R}^n$, and it shows that the positive limit set of any perturbed trajectory bounded in Ω satisfies $\Gamma^+ \subset \mathcal{S}$. It is interesting to investigate the situation where $\mathcal{S}$ consists of one autonomous limit set Γ_∞^+ only, that is, $\mathcal{S}$ consists only of one single trajectory $x(\cdot)$ subject to $w(\cdot) = 0$, which is bounded in past and future. Clearly, we have $\Gamma^+ \subset \Gamma_\infty^+$.

5.5 Corollary *[Cro98a] Assume that the invariant set $\mathcal{S}$ in theorem 5.4 consists of one autonomous limit set Γ_∞^+ only, then the equality $\Gamma^+ = \Gamma_\infty^+$ holds.*

Proof: See [Cro98a] for a detailed proof. □

At this point we are able to investigate the attraction of invariant sets of dissipative systems also for state trajectories generated by decaying disturbances. Dissipation of the system is handled by theorem 4.5. Then, any simple combination of former results concerning the boundedness of state trajectories such as lemma 3.25 or corollary 3.26 with corollaries 4.9, 4.10 or 4.11 concerning the asymptotic behavior of undisturbed motions together with proposition 5.2 and any of theorem 5.4 or corollary 5.5 gives results on asymptotic properties of trajectories subject to decaying disturbances. In particular we want to stress the following combination:

5.6 Proposition (decaying disturbances) *Assume that theorem 4.5 is satisfied using the strict HJI (36) and a regular supply rate, and that there exists a compact set $\mathcal{S} \subset \mathcal{R}$,* *four functions of class $\mathcal{K}$, denoted $\underline{\alpha}_V$, $\overline{\alpha}_V$, α_{H}, and $\underline{\alpha}_Z$, and a continuous and locally Lipschitz viscosity solution V such that*

$$\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}}) \quad , \quad \mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \leq -\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \quad , \text{ and } \quad \underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)| \quad .$$

Then the set $\mathcal{N} = \mathcal{V} = \ker V = \ker Z = \mathcal{S}$ is asymptotically stable under the dynamics of the undisturbed system $\dot{x} = X(x, 0)$.

Let Ω be any compact set (performance envelope) such that the proper inclusions $\mathcal{S} \subset \Omega \subset \mathcal{A}^+_\infty$ hold, where $\mathcal{A}^+_\infty$ is the basin of attraction of $\mathcal{S}$. Consider all disturbance $w(\cdot) \in \mathcal{L}_2^{loc}$ which are bounded, piecewise continuous, and satisfying $|w(t)| \to 0$ as $t \to \infty$.

Then any trajectory generated by such decaying disturbances which is bounded in future in Ω satisfies $x(t) \to \mathcal{S}$ as $t \to \infty$.

Proof: Direct consequence of propositions 4.12 and 5.2, and theorem 5.4. □

According to remark 4.13 we can use lemma 3.25 or corollary 3.26 to prove boundedness of state trajectories in the compact preimage $\Omega = V^{-1}([0, c])$ for many $c \in \mathbb{R}$.

5.7 Example: Nonlinear oscillator, continued
The planar system (25) has the compact invariant set

$$\mathcal{S} \equiv \left\{ x \in \mathbb{R}^2 \;\middle|\; r = 1 \right\}$$

which is asymptotically stable for all trajectories with $x_0 \in \mathcal{R}$, subject to $w(\cdot) = 0$. The performance measure is satisfying $\alpha_Z(|x|_{\mathcal{S}}) = |Z(x)|$, where clearly $\alpha_Z(|x|_{\mathcal{S}}) = |x|_{\mathcal{S}} \in \mathcal{K}$. We showed that the system is dissipative with respect to the strictly regular $\mathcal{H}_\infty$ supply rate $s(w, z) = \gamma^2|w|^2 - |z|^2$ for all $\gamma \geq 1$.

It is not hard to prove that the PDI

$$\mathsf{H}^* = \left(r^2(r^2-4)\tfrac{dV}{d(r^2)} + (r^2-1)\right)^2 - (1 - \tfrac{1}{\gamma^2})r^4(r^2-4)^2\left(\tfrac{dV}{d(r^2)}\right)^2 \;\leq\; -(1 - \tfrac{1}{\gamma^2})(r^2-1)^2$$

can be satisfied on $0 < r < 2$. It follows that there is a solution to the strict HJI $\mathsf{H}^* \leq -\alpha_{\mathsf{H}}$ for all $\gamma > 1$, where $\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) = (1 - \frac{1}{\gamma^2})|x|^4_{\mathcal{S}} \in \mathcal{K}$. It follows from lemma 4.15 that there exists storage functions which are positive $\mathcal{S}$-definite, but not necessarily proper, continuous or smooth. Indeed, the smooth, positive $\mathcal{S}$-definite and proper storage function (28) satisfies the strict HJI $\mathsf{H}^* \leq -\alpha_{\mathsf{H}}$ for all $\gamma > 1$.

By properness of V, any trajectory with initial point $x_0 \in \mathcal{R}$, driven by any disturbance with finite $\mathcal{L}_2$ norm, is bounded in some compact set $\Omega \subset \mathcal{R}$. Since the basin of attraction of $\mathcal{S}$ for all undisturbed motions equals $\mathcal{A}^+_\infty = \mathcal{R}$, it follows by proposition 5.6 that all $x(\cdot)$ with initial point $x_0 \in \mathcal{R}$ approaches $\mathcal{S}$ as $t \to \infty$ if it is driven by piecewise continuous disturbances with finite $\mathcal{L}_2$ norm satisfying in addition $w(t) \to 0$ as $t \to \infty$. ⋆

5.2 On differentiability of storage functions

As we have seen in the previous sections, storage functions have certain regularity properties. More precisely, the existence of a viscosity solution to the HJI (13) is equivalent to the dissipation of the system of concern, and the associated storage functions are lower semicontinuous without loss of generality (see theorems 3.4 and 3.15). In case that definition 3.22 holds, that is, the system has the property of locally bounded excitation, proposition 3.23 assures the continuity of all existing storage functions.

On the other hand, there are many situations in applications and in system theoretic considerations where an higher degree of regularity is essential. For example, in $\mathcal{H}_\infty$ control it is desirable to use a continuous control feedback which implies that the associated storage function must be continuously differentiable on $\mathcal{R}$. In case that we want to combine dissipative control with other design approaches such as backstepping or state feedback linearization we must consider C^p, $p > 1$, storage functions. Also numerical methods used to approximate solutions of the HJI (13) may require higher regularity than viscosity or continuity: higher order FEM solvers need the existence of C^p, $p > 1$, solutions, and the use of efficient spectral methods with faster-than-polynomial convergence is not applicable if no C^∞ (smooth) solutions exist. We conclude that it is important to investigate the existence of solutions of higher regularity than continuity.

In the following we recall an important property of nonlinear systems known as input-to-state stability, first developed by E. D. Sontag [Son89a, Son89b] for systems with one equilibrium point (see also the survey [Son95c]), and later generalized by Yuandan Lin [Lin92] for stability with respect to closed, but not necessarily bounded, positive invariant sets. This property will in the next section be the cornerstone of the theory of dissipative control with semi-global set-stability. For more information on the ISS property see also [SW94, SW95b, SW95a, SW96].

To avoid unnecessary complications, we assume for the rest of this section that the reachable set $\mathcal{R}$ equals $\mathbb{R}^n$ (These complications arise in defining $\mathcal{K}_\infty$ functions on bounded $\mathcal{R} \subset \mathbb{R}^n$, see appendix A). Assume that the disturbed system

$$\dot{x} = X(x, w) \quad , \tag{37}$$

has a positive invariant and compact set $\mathcal{S}$ subject to $w(\cdot) = 0$, and suppose that $\mathcal{S}$ is a global attractor for all trajectories generated by zero disturbances. In the following we are considering disturbances which are locally essentially bounded, that is, $w(\cdot) \in \mathcal{L}_\infty^{\mathrm{loc}}$ is assumed. This implies trivially $w(\cdot) \in \mathcal{L}_2^{\mathrm{loc}}$. We are interested to know if trajectories generated by nonzero disturbances have the converging-input-converging-state property

$$w(t) \to 0 \quad \Longrightarrow \quad x(t) \to \mathcal{S} \;\text{ for }\; t \to \infty \quad ,$$

or the bounded-input-bounded-state property

$$w(t) \text{ bounded} \quad \Longrightarrow \quad x(t) \text{ near } \mathcal{S} \;\text{ for all }\; t \in \mathbb{R}^+ \quad .$$

Both properties are satisfied for linear systems which are asymptotically stable subject to $w(\cdot) = 0$, where $\mathcal{S} = \{0\}$, but they are not necessarily satisfied for nonlinear asymptotically stable systems. We tie these properties together in the following definition, see appendix A for the notion of $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions.

5.8 Definition *The system (37) is* ***input-to-state-stable****, shortly* ***ISS****, with respect to a compact set $\mathcal{S}$ if there exists a function $\beta \in \mathcal{KL}$ and a function $\sigma \in \mathcal{K}$ such that*

$$|x(t, x_0, w(\cdot))|_{\mathcal{S}} \leq \beta(|x_0|_{\mathcal{S}}, t) + \sigma(\|w|_{[0,t]}(\cdot)\|_\infty)$$

for all $t \in \mathbb{R}^+$, where $w|_{[0,T]}(\cdot)$ is the restriction of $w(\cdot)$ on $[0, T]$ satisfying $w(t) = 0$ for all $t > T$.

It is immediately seen by causality that $\sigma(\|w(\cdot)\|_\infty)$ can be used in case that $w(\cdot) \in \mathcal{L}_\infty$.

The ISS property implies several nice features: first of all, the set $\mathcal{S}$ must be positive invariant for all trajectories generated by zero disturbances, because for any initial condition $x_0 \in \mathcal{S}$ and $w(\cdot) = 0$ we have $|x(t, x_0, w)|_{\mathcal{S}} \leq 0$ for all $t \in \mathbb{R}^+$.

Second, the bounded-input-bounded-state property follows for all initial points within a neighborhood of $\mathcal{S}$: For every initial point satisfying $|x_0|_{\mathcal{S}} \leq k$ it follows $|x(t, x_0, w)|_{\mathcal{S}} \leq \beta(k, t) + \sigma(\|w_{[0,t]}(\cdot)\|_\infty) \leq \beta(k, 0) + \sigma(\|w(\cdot)\|_\infty)$ for all $t \in \mathbb{R}^+$, independently of x_0.

Third, a system which is ISS is also asymptotically stable with respect to $\mathcal{S}$, subject to $w(\cdot) = 0$. To show this, take any initial condition and set $w(\cdot) = 0$, then $|x(t, x_0, 0)|_{\mathcal{S}} \leq \beta(|x_0|_{\mathcal{S}}, 0)$ shows that the state is bounded near $\mathcal{S}$, and therefore is defined in the future, and moreover, $|x(t, x_0, 0)|_{\mathcal{S}} \leq \beta(|x_o|_{\mathcal{S}}, t) \to 0$ as $t \to \infty$, which accounts for the asymptotic stability of $\mathcal{S}$.

Fourth, the converging-input-converging-state property is easily deducted by contradiction: Given any time sequence $\{t_i\}$ with $t_i \to \infty$ and any decaying disturbance with $w_i \equiv \|w|_{[t_i,\infty]}(\cdot)\|_\infty \to 0$, we assume that there exists an $\epsilon > 0$ such that $|x(\cdot)|_{\mathcal{S}} > \epsilon$ for all t_i. But then there exists a time $t_i \in \mathbb{R}^+$ such that $\sigma(w_i) \leq \frac{\epsilon}{2}$, and by the ISS property the set given by $|x(\cdot)|_{\mathcal{S}} \leq \frac{\epsilon}{2}$ is approached for $t \to \infty$, thus contradicting the existence of such an $\epsilon > 0$.

The definition of ISS is as such not easily handled in the context of dissipative systems. However, it has been shown that the ISS property has an equivalent formulation in terms of a certain Lyapunov like inequality which can be used to deduce the set-stability of disturbance free trajectories. We need the following definition.

5.9 Definition *Any continuous function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is called* ***ISS-Lyapunov function*** *with respect to a compact set $\mathcal{S}$ if it is* ***positive $\mathcal{S}$-definite and proper****, that is, if there exist two $\mathcal{K}_\infty$ functions $\underline{\alpha}_V$, $\overline{\alpha}_V$ such that*

$$\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}}) \quad , \tag{38}$$

and if there exists some functions $\alpha_x \in \mathcal{K}_\infty$ and $\alpha_w \in \mathcal{K}$ such that V satisfies the ***ISS partial differential inequality****, shortly named* ***ISS-PDI****,*

$$\tfrac{\partial V}{\partial x}(x) X(x, w) \leq \alpha_w(|w|) - \alpha_x(|x|_{\mathcal{S}}) \tag{39}$$

in the weak sense on $\mathbb{R}^n$, for all $w \in \mathbb{R}^l$.

The ingenious work of Lin, Sontag and Wang has shown many equivalent formulation of the ISS property in terms of ISS-Lyapunov functions and various decay estimates. They define that a continuous ISS-Lyapunov function is any continuous positive $\mathcal{S}$-definite and proper function $V : \mathbb{R}^n \mapsto \mathbb{R}$ which satisfies the **ISS-integral inequality**

$$V(x_T) - V(x_0) \leq \int_0^T \alpha_w(|w(t)|) - \alpha_x(|x(t)|_{\mathcal{S}})\, dt \tag{40}$$

for all $w(\cdot) \in \mathcal{L}_\infty^{\text{loc}}$ and all $T \geq 0$, and some $\alpha_w \in \mathcal{K}$ and $\alpha_x \in \mathcal{K}_\infty$. However, since by theorem 3.15 any viscosity solution V to (39) also satisfies (40), and conversely, we use (39) to define continuous ISS-Lyapunov functions. We want to emphasize the connections between continuous and smooth - that is C^∞ - ISS-Lyapunov functions.

5.10 Proposition *[Lin92, Son95c, SW94, SW95b, SW96] Any of the three following statements are equivalent:*

1. *The system (37) is ISS with respect to a compact set $\mathcal{S}$.*
2. *The system (37) admits a continuous ISS-Lyapunov function.*
3. *The system (37) admits a smooth (C^∞) ISS-Lyapunov function.*

The existence of a smooth ISS-Lyapunov function is hard to prove, it is based on an partition of the unity argument first proved by Lin, Sontag and Wang [LSW96]. We want at this point stress that the existence of a continuous ISS-Lyapunov function V satisfying the ISS-PDI (39) does *not* imply, that there is a smooth ISS-Lyapunov function $\tilde{V}$ satisfying the ISS-PDI (39) with the *same* set of $\mathcal{K}_\infty$ comparison functions as the continuous V. It becomes clear implicitly in the proofs of the papers [SW94, SW95b] that there only exists a smooth ISS-Lyapunov $\widetilde{V}$ satisfying

$$\begin{gathered}\tfrac{d}{dt}\widetilde{V} = \tfrac{\partial \widetilde{V}}{\partial x}(x)X(x,w) \leq \alpha_w(|w|) - \widetilde{\alpha_x}(|x|_{\mathcal{S}}) \quad \text{and} \\ \widetilde{\underline{\alpha}_V}(|x|_{\mathcal{S}}) \leq \widetilde{V}(x) \leq \widetilde{\overline{\alpha}_V}(|x|_{\mathcal{S}})\end{gathered}$$

on $\mathbb{R}^n$, where the $\mathcal{K}_\infty$ functions $\widetilde{\underline{\alpha}_V}$, $\widetilde{\overline{\alpha}_V}$ and $\widetilde{\alpha_x}$ are such that

$$\widetilde{\underline{\alpha}_V} \leq \underline{\alpha}_V \quad , \quad \widetilde{\overline{\alpha}_V} \geq \overline{\alpha}_V \quad \text{and} \quad \widetilde{\alpha_x} \leq \alpha_x \quad .$$

The ISS property is interesting not only from a system theoretic point of view; it will help to construct useful performance envelopes and stability properties in the following sections. It is easily seen that dissipative systems which satisfy the conditions of corollary 4.10 or corollary 4.11 have the ISS property.

5.11 Lemma *Given a system with a supply rate satisfying $s(w,z) \leq \alpha_w(|w|)$ for all z, where $\alpha_w \in \mathcal{K}$, assume that there exists a positive $\mathcal{S}$-definite, proper and continuous viscosity solution to the strict HJI (36), where α_z and α_{H} are functions of class $\mathcal{K}_\infty$. Then there exists a smooth ISS-Lyapunov function satisfying the ISS-PDI*

$$\tfrac{d}{dt}V = \tfrac{\partial V}{\partial x}(x)X(x,w) \leq \alpha_w(|w|) - \tilde{\alpha_{\mathsf{H}}}(|x|_{\mathcal{S}})$$

in the classical sense, with $\alpha_{\mathsf{H}} \geq \tilde{\alpha_{\mathsf{H}}} \in \mathcal{K}_\infty$.

We see that the condition on the supply rate is satisfied for all strictly regular supply rates.

Proof: By the strict HJI (36) and the definition of the Hamiltonian (12) it follows that

$$\tfrac{d}{dt}V = \tfrac{\partial V}{\partial x}(x)X(x,w) \leq s(w,z) - \alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \ ,$$

hence $\frac{\partial V}{\partial x}X \leq \alpha_w - \alpha_{\mathsf{H}}$ is satisfied for all $w \in \mathbb{R}^l$ and all $x \in \mathbb{R}^n$. Therefore, the storage function V is a ISS-Lyapunov function, and the existence of a smooth ISS-Lyapunov function is a consequence of proposition 5.10. □

5.12 Lemma *Given a system with strictly regular supply rate $s(w,z) = \alpha_w(|w|) - \alpha_z(|z|)$ and a performance function satisfying $\underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)|$, where α_w, α_z and $\underline{\alpha}_Z$ are functions of class $\mathcal{K}_\infty$, assume that there exists a positive $\mathcal{S}$-definite, proper and continuous solution to the HJI (13). Then there exists a smooth ISS-Lyapunov function satisfying the ISS-PDI*

$$\tfrac{d}{dt}V = \tfrac{\partial V}{\partial x}(x)X(x,w) \leq \alpha_w(|w|) - \alpha_x(|x|_{\mathcal{S}})$$

with $\alpha_x(|x|_{\mathcal{S}}) \leq (\alpha_z \circ \underline{\alpha}_Z)(|x|_{\mathcal{S}})$, where $\alpha_x \in \mathcal{K}_\infty$.

Proof: By the strict HJI (36) and the definition of the Hamiltonian (12) it follows that

$$\tfrac{d}{dt}V = \tfrac{\partial V}{\partial x}(x)X(x,w) \leq \alpha_w(|w|) - \alpha_z(|z|)$$

is satisfied for all bounded $w \in \mathbb{R}^l$ and all $x \in \mathbb{R}^n$. Now, $\underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)|$ implies $(\alpha_z \circ \underline{\alpha}_Z)(|x|_{\mathcal{S}}) \leq \alpha_z(|Z(x)|) = \alpha_z(|z|)$, hence $\frac{\partial V}{\partial x}X \leq \alpha_w - \alpha_x$. Therefore, the storage function V is a ISS-Lyapunov function, and the existence of a smooth ISS-Lyapunov function is a consequence of proposition 5.10. □

It is very desirable to have a similar result as proposition 5.10 for storage functions to prove that the existence of continuous storage functions implies the existence of smooth storage functions. Unfortunately, the lemmas 5.11 and 5.12 can not in general be inverted to give results on smoothness of storage functions given smoothness of ISS-Lyapunov functions. We have only the following partial result:

5.13 Corollary *(smooth storage functions) Given a system with strictly proper supply rate $s(w,z) = \alpha_w(|w|) - \alpha_z(|z|)$, assume that all requirements of theorem 4.5 and proposition 4.4 are satisfied using the strict HJI (36), and that there exists a compact set $\mathcal{S} \subset \mathcal{R}$, five functions of class $\mathcal{K}_\infty$, denoted $\underline{\alpha}_V$, $\overline{\alpha}_V$, α_{H}, $\underline{\alpha}_Z$, and $\overline{\alpha}_Z$, and a continuous and locally Lipschitz storage function V such that*

$$\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}}) \ , \ \mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \leq -\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \ , \text{ and}$$
$$\underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)| \leq \overline{\alpha}_Z(|x|_{\mathcal{S}}) \ .$$

Then there exists a smooth ISS-Lyapunov function $\widetilde{V}$ satisfying the ISS-PDI

$$\tfrac{d}{dt}\widetilde{V} = \tfrac{\partial \widetilde{V}}{\partial x}(x)X(x,w) \leq \alpha_w(|w|) - \widetilde{\alpha_x}(|x|_{\mathcal{S}})$$

where the inequalities

$$\widetilde{\alpha_x}(|x|_{\mathcal{S}}) \leq (\alpha_z \circ \underline{\alpha}_Z)(|x|_{\mathcal{S}}) + \alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \quad ,$$
$$\widetilde{\underline{\alpha}_V}(|x|_{\mathcal{S}}) \leq \underline{\alpha}_V(|x|_{\mathcal{S}}) \quad \text{and} \quad \overline{\alpha}_V(|x|_{\mathcal{S}}) \leq \widetilde{\overline{\alpha}_V}(|x|_{\mathcal{S}})$$

hold. Assume furthermore that there exists a $\mathcal{K}_\infty$ *function* $\widetilde{\alpha_{\mathsf{H}}}$ *satisfying*

$$\widetilde{\alpha_{\mathsf{H}}}(|x|_{\mathcal{S}}) \leq (\alpha_z \circ \underline{\alpha}_Z)(|x|_{\mathcal{S}}) - (\alpha_z \circ \overline{\alpha}_Z)(|x|_{\mathcal{S}}) + \alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \quad ,$$

then the smooth ISS-Lyapunov function $\widetilde{V}$ *is a smooth storage function satisfying the HJI*

$$\mathsf{H}^*(x, \tfrac{\partial \widetilde{V}}{\partial x}) \leq -\widetilde{\alpha_{\mathsf{H}}}(|x|_{\mathcal{S}}) \quad , \quad \text{and} \quad \widetilde{\underline{\alpha}_V}(|x|_{\mathcal{S}}) \leq \widetilde{V(x)} \leq \widetilde{\overline{\alpha}_V}(|x|_{\mathcal{S}}) \quad .$$

Proof: The existence of a smooth ISS-Lyapunov function as stated follows from lemmas 5.11 and 5.12. Then, $\widetilde{\alpha_{\mathsf{H}}} \in \mathcal{K}_\infty$ and $\widetilde{\alpha_{\mathsf{H}}} \leq \alpha_z \circ \underline{\alpha}_Z - \alpha_z \circ \overline{\alpha}_Z + \alpha_{\mathsf{H}}$ directly implies that $\alpha_w - \widetilde{\alpha_x} \leq \alpha_w - \alpha_z - \alpha_{\mathsf{H}}$, hence the HJI $\mathsf{H}^*(x, \frac{\partial \widetilde{V}}{\partial x}) \leq -\widetilde{\alpha_{\mathsf{H}}}$ is satisfied. □

We see that it may be difficult to prove the existence of a $\mathcal{K}_\infty$ function satisfying $\widetilde{\alpha_{\mathsf{H}}} \leq \alpha_z \circ \underline{\alpha}_Z - \alpha_z \circ \overline{\alpha}_Z + \alpha_{\mathsf{H}}$, the problems are hidden in the performance function satisfying only $\underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)| \leq \overline{\alpha}_Z(|x|_{\mathcal{S}})$. In case that $\underline{\alpha}_Z = \overline{\alpha}_Z$ we are able to formulate the desired smooth inverse storage function proposition.

5.14 Proposition *Given a system with strictly proper supply rate* $s(w,z) = \alpha_w(|w|) - \alpha_z(|z|)$*, a compact set* $\mathcal{S} \subset \mathcal{R}$*, and a performance function satisfying* $|Z(x)| = \alpha_Z(|x|_{\mathcal{S}})$*, where* α_Z *is any* C^∞ *function of class* $\mathcal{K}_\infty$*. Then any of the three following statements are equivalent:*

1. *The system (4) is dissipative and ISS with respect to* $\mathcal{S}$*.*
2. *The system (4) admits a continuous ISS-Lyapunov function which is also a positive* $\mathcal{S}$*-definite and proper viscosity solution to the strict HJI (36).*
3. *The system (4) admits a smooth ISS-Lyapunov function which is also a positive* $\mathcal{S}$*-definite and proper storage function satisfying the strict HJI (36) in the classic sense.*

Proof: Since $|Z(x)| = \alpha_Z(|x|_{\mathcal{S}})$ it follows directly that the HJI (36) and the ISS-PDI (39) are one and the same if we choose $\alpha_x = \alpha_z \circ \alpha_Z + \alpha_{\mathsf{H}}$. Then proposition 5.10 combined with corollary 5.13 gives the result as stated. □

Notice again that the set of comparison functions associated to the continuous storage function V is related to the set of comparison functions associated to the smooth $\widetilde{V}$. In fact, there holds $\widetilde{\underline{\alpha}_V} \leq \underline{\alpha}_V$, $\widetilde{\overline{\alpha}_V} \geq \overline{\alpha}_V$ and $\widetilde{\alpha_{\mathsf{H}}} \leq \alpha_{\mathsf{H}}$.

It is not at all clear at the moment if the condition $|Z(x)| = \alpha_Z(|x|_{\mathcal{S}})$ is a necessary condition for proposition 5.14 to hold. Also the question whether the continuity of storage functions can be relaxed to lower semicontinuity or not is part of the author's current research.

5.3 Performance envelopes and practical stability

As it has been pointed out in [SW94], any system which is ISS with respect to a compact set $\mathcal{S}$, that is, satisfies definition 5.8, satisfies also an inequality of the form

$$|x(t, x_0, w(\cdot))|_{\mathcal{S}} \leq \max\{\widetilde{\beta}(|x_0|_{\mathcal{S}}, t), \widetilde{\sigma}(\|w|_{[0,t]}(\cdot)\|_\infty)\} \tag{41}$$

for all $t \in \mathbb{R}^+$, where $\widetilde{\beta} = 2\beta \in \mathcal{KL}$ and $\widetilde{\sigma} = 2\sigma \in \mathcal{K}$. Indeed, since $\beta + \sigma \leq \max\{\widetilde{\beta}, \widetilde{\sigma}\}$, the result trivially holds. Hence, given a bounded disturbance $\|w(\cdot)\|_\infty \leq c \in \mathbb{R}^+$ we can define the set

$$\left\{ x \in \mathcal{R} \;\middle|\; |x|_{\mathcal{S}} \leq 2\sigma(c) \right\}$$

which is clearly positive invariant by equation (41), and therefore is a performance envelope.

On the other hand, given a positive $\mathcal{S}$-definite storage function V satisfying the ISS-PDI (39), it has been showed [Son95c, SW96] that

$$\limsup_{t\to\infty} |x(t, x_0, w(\cdot))|_{\mathcal{S}} \leq \sigma(\|w(\cdot)\|_\infty) \quad , \tag{42}$$

and even

$$\limsup_{t\to\infty} |x(t, x_0, w(\cdot))|_{\mathcal{S}} \leq \sigma(\limsup_{t\to\infty} |w(t)|) \tag{43}$$

holds for all $w(\cdot) \in \mathcal{L}_\infty$ with

$$\sigma \equiv \underline{\alpha}_V^{-1} \circ \overline{\alpha}_V \circ \alpha_x^{-1} \circ \alpha_w \quad . \tag{44}$$

Moreover, given a positive $\mathcal{S}$-definite storage function V satisfying a *slightly stronger* PDI than the ISS-PDI (39), also the $\mathcal{K}$ function σ of definition 5.8 can be chosen as defined in equation (44) [Son95c]. In light of equation (42) it seems reasonable to believe that also the smaller set

$$\left\{ x \in \mathcal{R} \;\middle|\; |x|_{\mathcal{S}} \leq \sigma(c) \right\}$$

is positive invariant for all $x(\cdot)$ generated by bounded disturbances $\|w(\cdot)\|_\infty \leq c \in \mathbb{R}^+$. Unfortunately, this is not true, as we will see during the course of proving the following theorem.

Nevertheless, it is possible to refine the properties hidden in equations (41), (42) and (43) to stronger results: given a continuous ISS-Lyapunov functions satisfying *exactly* the same ISS-PDI (39), the $\mathcal{K}$ function σ of definition 5.8 can be chosen as defined in equation (44). Also, the max-bound (41) can be achieved with $\widetilde{\sigma} = \sigma$. These refinements of the results mentioned in [Son95c, SW94, SW95b, SW95a, SW96] give immediately the performance envelope and practical stability results needed.

For convenience, we define a set of disturbances, denoted $\mathcal{W}_c$, by

$$\mathcal{W}_c \equiv \left\{ w(\cdot) \in \mathcal{L}_\infty \;\middle|\; \|w(\cdot)\|_\infty \leq c \right\} \quad , \quad c > 0 \quad . \tag{45}$$

5.15 Theorem (Performance envelope and practical stability) *Given a continuous (and positive $\mathcal{S}$-definite and proper) ISS-Lyapunov function V satisfying the ISS-PDI (39), the following properties hold with the $\mathcal{K}$ function σ defined by equation (44):*

1. *There is a $\mathcal{KL}$ function β satisfying the ISS-property (where $w(\cdot) \in \mathcal{L}_\infty^{loc}$)*
$$|x(t, x_0, w(\cdot))|_\mathcal{S} \leq \beta(|x_0|_\mathcal{S}, t) + \sigma(\|w|_{[0,t]}(\cdot)\|_\infty) \quad .$$
2. *There is a $\mathcal{KL}$ function $\tilde{\beta}$ such that for all $x(\cdot)$ generated by $w(\cdot) \in \mathcal{L}_\infty^{loc}$ there holds*
$$|x(t, x_0, w(\cdot))|_\mathcal{S} \leq \max\{\tilde{\beta}(|x_0|_\mathcal{S}, t), \sigma(\|w|_{[0,t]}(\cdot)\|_\infty)\} \quad .$$
3. *The level set $\Omega \subset \mathcal{R}$ defined by*
$$\Omega \equiv \{\, x \in \mathcal{R} \;\big|\; V(x) \leq (\overline{\alpha}_V \circ \alpha_x^{-1} \circ \alpha_w)(c) \,\}$$
is globally asymptotically $\mathcal{W}_c$-stable and positive $\mathcal{W}_c$-invariant, hence a performance envelope. That is, Ω is globally asymptotically stable and positive invariant for all $x(\cdot)$ generated by $w(\cdot) \in \mathcal{L}_\infty$ satisfying $\|w(\cdot)\|_\infty \leq c$.
4. *If $w(\cdot) \in \mathcal{L}_\infty$, then the set $\overline{\mathcal{B}}_\infty \subset \mathcal{R}$ defined by*
$$\overline{\mathcal{B}}_\infty \equiv \{\, x \in \mathcal{R} \;\big|\; |x|_\mathcal{S} \leq \sigma(c_\infty) \,\} \tag{46}$$
is a global attractor for all $x(\cdot)$ generated by $w(\cdot)$ satisfying $\limsup_{t\to\infty} |w(t)| \leq c_\infty$.

Before we prove theorem 5.15, we need an auxiliary result which is interesting on its own, since it gives the sharpest known bounds on positive $\mathcal{W}_c$-invariant sets and global attractors.

5.16 Lemma *Given a continuous (and positive $\mathcal{S}$-definite and proper) ISS-Lyapunov function V satisfying the ISS-PDI (39), the level set Ω defined in theorem 5.15 is positive $\mathcal{W}_c$-invariant, and Ω is globally asymptotically stable for all $x(\cdot)$ generated by $w(\cdot) \in \mathcal{W}_c$.*

Moreover, the level set
$$\Omega_\infty \equiv \{\, x \in \mathcal{R} \;\big|\; V(x) \leq (\overline{\alpha}_V \circ \alpha_x^{-1} \circ \alpha_w)(c_\infty) \,\}$$
is a global attractor for all $x(\cdot)$ generated by $w(\cdot) \in \mathcal{L}_\infty$ satisfying $\limsup_{t\to\infty} |w(t)| \leq c_\infty$.

Proof: Let $\chi(k) \equiv \alpha_x^{-1}(k\alpha_w(c))$, $k > 0$, then $|x|_\mathcal{S} \geq \chi(k)$ implies $\alpha_x(|x|_\mathcal{S}) \geq k\alpha_w(|w|)$ for all $|w| \leq c$. Hence, with any $k > 1$ we have
$$\alpha_w(|w|) - \alpha_x(|x|_\mathcal{S}) \leq (\tfrac{1}{k} - 1)\alpha_x(|x|_\mathcal{S}) \;<\; 0$$
for all x, w satisfying $|w| \leq c$ and $|x|_\mathcal{S} \geq \chi(k)$. By theorem 3.15 any lower semicontinuous function V satisfying (39) also satisfies (40). It follows that all trajectories generated by $w(\cdot) \in \mathcal{W}_c$ satisfy $V(x_T) < V(x_0)$ for all $T \geq 0$ if evolving in $\mathcal{R}\backslash\underline{\mathcal{B}}_k$, where
$$\underline{\mathcal{B}}_k \equiv \{\, x \in \mathcal{R} \;|\; |x|_\mathcal{S} \leq \chi(k) \,\} \quad .$$

Define now the family of level sets

$$\Omega_k \equiv \left\{ x \in \mathcal{R} \;\middle|\; V(x) \le (\overline{\alpha}_V \circ \chi)(k) \right\} \quad ,$$

then $\Omega \subset \Omega_k$ for all $k \ge 1$.

Clearly $\underline{\mathcal{B}}_k \subset \Omega_k$: by assumption $V(x) \le \overline{\alpha}_V(|x|_{\mathcal{S}})$, hence $(\overline{\alpha}_V^{-1} \circ V)(x) \le |x|_{\mathcal{S}}$. It follows trivially that any x satisfying $|x|_{\mathcal{S}} \le \chi(k)$ also satisfies $V(x) \le (\overline{\alpha}_V \circ \chi)(k)$.

Consequently, all trajectories generated by $w(\cdot) \in \mathcal{W}_c$ satisfy $V(x_T) < V(x_0)$ for all $T \ge 0$ such that $x|_{[0,T[}(\cdot)$ evolves in $\mathcal{R}\backslash\Omega_k$, hence Ω_k is a global attractor and a positive $\mathcal{W}_c$-invariant set for all $k > 1$. By continuity it follows that Ω is a global attractor and a positive $\mathcal{W}_c$-invariant set. We notice that the following properties hold:

$$\begin{gathered} \Omega_{k_1} \subset \Omega_{k_2} \;\text{ for all }\; 1 \le k_1 \le k_2 \quad , \\ \lim_{k \to 1} \Omega_k = \Omega \quad , \quad \text{and} \\ \lim_{k \to \infty} \Omega_k = \mathbb{R}^n \quad . \end{gathered}$$

By positive $\mathcal{W}_c$-invariance of each Ω_k, $k \ge 1$, it is easily deduced that each Ω_k, $k \ge 1$, is globally $\mathcal{W}_c$-stable. It follows that each Ω_k, $k \ge 1$, and in particular Ω, is globally asymptotically $\mathcal{W}_c$-stable.

Finally, we prove that Ω_∞ is a global attractor for all $x(\cdot)$ generated by $w(\cdot) \in \mathcal{L}_\infty$ satisfying $\limsup_{t\to\infty} |w(t)| \le c_\infty$: all such $x(\cdot)$ are bounded in $V^{-1}([0, (\overline{\alpha}_V \circ \alpha_x^{-1} \circ \alpha_w)(c)])$ for some $c > 0$, and are therefore defined in future. Take $c_\infty = \limsup_{t\to\infty} |w(t)|$ and any sequence of times $t_i \to \infty$, then the sequence defined by $c_i \equiv \sup_{t\in[t_i,t_{i+1}]} |w(t)|$ satisfies $c_\infty \le c_i \le c$ and $c_i \to c_\infty$ as $i \to \infty$. Since the flow of the system has the semigroup property (by causality), the behavior of $x(\cdot)$ with starting point x_0 for $t \to \infty$ is given by the behavior of $x(\cdot)$ with starting point $x_i \equiv x(t_i)$. It follows that the sets

$$\Omega_i \equiv \left\{ x \in \mathcal{R} \;\middle|\; V(x) \le (\overline{\alpha}_V \circ \alpha_x^{-1} \circ \alpha_w)(c_i) \right\}$$

are global attractors, and by continuity of the trajectory also $\Omega_\infty = \lim_{i\to\infty} \Omega_i$ is a global attractor. Notice however, that asymptotical stability can not be proven. □

The original proof printed in [SW94] used $\chi(k = \frac{1}{2}) = \alpha_x^{-1}(\frac{1}{2}\alpha_w(c))$ throughout the entire argumentation. Therefore, it was only possible to prove that $\Omega_{k=\frac{1}{2}}$ is a global attractor and a positive invariant set. The continuity argument showing that also $\Omega = \lim_{k\to 1} \Omega_k$ is a global attractor and a positive invariant set was not used there.

Proof of theorem 5.15: In the following we explicitly denote the dependence of the above defined sets on the constant $c > 0$, that is, we define the balls

$$\underline{\mathcal{B}}(c) \equiv \underline{\mathcal{B}}_{k=1} = \left\{ x \in \mathcal{R} \;\middle|\; |x|_{\mathcal{S}} \le \chi(k = 1, c) \right\} \quad , \quad \overline{\mathcal{B}}(c) \equiv \left\{ x \in \mathcal{R} \;\middle|\; |x|_{\mathcal{S}} \le \sigma(c) \right\}$$

and the level sets

$$\Omega(c) \equiv \left\{ x \in \mathcal{R} \;\middle|\; V(x) \le (\overline{\alpha}_V \circ \chi)(k = 1, c) \right\} \quad .$$

These sets satisfy $\underline{\mathcal{B}}(c) \subset \Omega(c) \subset \overline{\mathcal{B}}(c)$ for all $c \geq 0$. It follows from lemma 5.16 that $\Omega(\widetilde{c})$ is positive $\mathcal{W}_c$-invariant, and the sets $\Omega(\widetilde{c})$ and $\overline{\mathcal{B}}(\widetilde{c})$ are global attractors for all $x(\cdot)$ subject to $w(\cdot) \in \mathcal{W}_c$ in case that $\widetilde{c} \geq c$. By causality any point $x(t)$, $0 \leq t \leq T$ of a trajectory $x(\cdot)$ generated by some $w(\cdot) \in \mathcal{L}_\infty^{\text{loc}}$ satisfying $\|w|_{[0,T]}(\cdot)\|_\infty \leq c$ is bounded in $\Omega(\widetilde{c})$ if $0 \leq c \leq \widetilde{c}$. Hence $x_0 \in \underline{\mathcal{B}}(\widetilde{c})$ implies $x(t) \in \overline{\mathcal{B}}(\widetilde{c})$ for all trajectories subject to $\|w|_{[0,T]}(\cdot)\|_\infty \leq c \leq \widetilde{c}$ and all $0 \leq t \leq T$. It follows that $x(\cdot)$ subject to $\|w|_{[0,T]}(\cdot)\|_\infty \leq c$ can be bounded in two ways:

$$\begin{array}{lll} |x(t)|_{\mathcal{S}} \leq (\underline{\alpha}_V \circ \overline{\alpha}_V^{-1})(|x_0|_{\mathcal{S}}) & \text{for all } 0 \leq t \leq T & \text{if } x_0 \in \mathcal{R} \backslash \underline{\mathcal{B}}(c) \\ |x(t)|_{\mathcal{S}} \leq \sigma(c) & \text{for all } 0 \leq t \leq T & \text{if } x_0 \in \Omega(c) \supset \underline{\mathcal{B}}(c) \quad . \end{array} \tag{47}$$

Notice that $(\underline{\alpha}_V \circ \overline{\alpha}_V^{-1})(|x_0|_{\mathcal{S}}) \leq \sigma(c)$ for all $x_0 \in \underline{\mathcal{B}}(c)$, hence $x(\cdot)$ subject to $\|w|_{[0,T]}(\cdot)\|_\infty \leq c$ is bounded by

$$|x(t)|_{\mathcal{S}} \leq \max\{(\underline{\alpha}_V \circ \overline{\alpha}_V^{-1})(|x_0|_{\mathcal{S}}), \sigma(c)\} \quad \text{for all} \quad 0 \leq t \leq T \quad .$$

It follows that all $x(\cdot)$ subject to $w(\cdot) \in \mathcal{L}_\infty^{\text{loc}}$ are defined in future and satisfy $x(\cdot) \in \mathcal{L}_\infty^{\text{loc}}$, and moreover, that $x(\cdot)$ subject to $w(\cdot) \in \mathcal{L}_\infty$ are defined in future and satisfy $x(\cdot) \in \mathcal{L}_\infty$. Also, $w(t) \to 0$ implies $x(t) \to \mathcal{S}$ by lemma 5.16.

From now on, we follow the proof of the similar, but slightly weaker lemma 2.10 in [SW94] to conclude that a suitable $\mathcal{KL}$ function β exists: by the proof of lemma 5.16 it follows that V satisfies the PDI

$$\tfrac{d}{dt} V(x(t)) \leq -\frac{1}{2}(\alpha_x \circ \overline{\alpha}_V^{-1})(V(x(t)))$$

weakly on $\mathcal{R} \backslash \Omega(c)$, hence by a standard comparison principle [LSW96] there exists some $\mathcal{KL}$ function β such that

$$|x(t)|_{\mathcal{S}} \leq \beta(|x_0|_{\mathcal{S}}, t) \quad \text{for all} \quad 0 \leq t \leq T_* \quad , \tag{48}$$

where $T_* \leq \infty$ is the time where $x(\cdot)$ enters the set $\overline{\mathcal{B}}(c)$.

Hence, combining inequalities (47) and (48) shows that the max-bound

$$|x(t, x_0, w(\cdot))|_{\mathcal{S}} \leq \max\{\beta(|x_0|_{\mathcal{S}}, t), \sigma(\|w|_{[0,t]}(\cdot)\|_\infty)\}$$

is satisfied for all $x(\cdot)$ generated by $w(\cdot) \in \mathcal{L}_\infty^{\text{loc}}$ with σ as defined in equation (44). The ISS-property follows trivially.

The positive $\mathcal{W}_c$-invariantness and global asymptotically $\mathcal{W}_c$-stability of Ω has been proved in lemma 5.16.

Finally, $\overline{\mathcal{B}}_\infty \supset \Omega_\infty$ is a global attractor by lemma 5.16. □

Even in the case that smooth storage functions do not exists, we are able to exploit the performance envelopes and practical stability results of theorem 5.15 in the context of dissipative systems by using lemma 5.11 or lemma 5.12 partially to show that a continuous storage function V is a ISS-Lyapunov function. We want to stress the following combination:

5.17 Proposition (performance envelope and practical stability) *Given a system with strictly proper supply rate $s(w,z) = \alpha_w(|w|) - \alpha_z(|z|)$, assume that all requirements of theorem 4.5 are satisfied using the strict HJI (36), and that there exists a compact set $\mathcal{S} \subset \mathcal{R}$, five functions of class $\mathcal{K}_\infty$, denoted $\underline{\alpha}_V$, $\overline{\alpha}_V$, α_{H}, $\underline{\alpha}_Z$, and $\overline{\alpha}_Z$, and a continuous storage function (viscosity solution) V such that*

$$\underline{\alpha}_V(|x|_{\mathcal{S}}) \leq V(x) \leq \overline{\alpha}_V(|x|_{\mathcal{S}}) \quad , \quad \mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \leq -\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \quad , \text{ and}$$
$$\underline{\alpha}_Z(|x|_{\mathcal{S}}) \leq |Z(x)| \leq \overline{\alpha}_Z(|x|_{\mathcal{S}}) \quad .$$

Then V is a continuous ISS-Lyapunov function satisfying the ISS-PDI

$$\tfrac{\partial V}{\partial x}(x) X(x,w) \leq \alpha_w(|w|) - \alpha_x(|x|_{\mathcal{S}})$$

weakly, where the $\mathcal{K}_\infty$ function α_x is defined by $\alpha_x(|x|_{\mathcal{S}}) \equiv (\alpha_z \circ \underline{\alpha}_Z)(|x|_{\mathcal{S}}) + \alpha_{\mathsf{H}}(|x|_{\mathcal{S}})$.

Moreover, the level set $\Omega \subset \mathcal{R}$ defined by

$$\Omega \equiv \left\{ x \in \mathcal{R} \;\middle|\; V(x) \leq (\overline{\alpha}_V \circ \alpha_x^{-1} \circ \alpha_w)(c) \right\}$$

is positive $\mathcal{W}_c$-invariant, hence a performance envelope; and it is globally asymptotically $\mathcal{W}_c$-stable. The set $\overline{\mathcal{B}}_\infty \subset \mathcal{R}$ defined by

$$\overline{\mathcal{B}}_\infty \equiv \left\{ x \in \mathcal{R} \;\middle|\; |x|_{\mathcal{S}} \leq \sigma(c_\infty) \right\}$$

is a global attractor for all $x(\cdot)$ generated by $w(\cdot) \in \mathcal{L}_\infty$ satisfying $\limsup_{t\to\infty} |w(t)| \leq c_\infty$.

Proof: Combination of theorem 5.15 and corollary 5.13 □

5.18 Example: Nonlinear oscillator, continued
We saw in example 5.7 that the nonlinear oscillator described by the equations (25) satisfies the bounds $\alpha_Z(|x|_{\mathcal{S}}) = |Z(x)|$, where $\alpha_Z(|x|_{\mathcal{S}}) = |x|_{\mathcal{S}} \in \mathcal{K}$, and that there exists a solution to the strict HJI $\mathsf{H}^* \leq -\alpha_{\mathsf{H}}$ for all $\gamma > 1$, where $\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) = (1 - \frac{1}{\gamma^2})|x|_{\mathcal{S}}^4 \in \mathcal{K}$. Even if the requirements of corollary 5.13 or proposition 5.14 are not satisfied since the comparison functions are not of class $\mathcal{K}_\infty$, there exists a smooth, positive $\mathcal{S}$-definite and proper storage function (28) satisfying the strict HJI $\mathsf{H}^* \leq -\alpha_{\mathsf{H}}$ for all $\gamma > 1$.

Therefore it seems reasonable to conjecture that the existence of smooth storage functions may be implied by the existence of suitable $\mathcal{K}$ comparison functions α_{H} and $\underline{\alpha}_Z$ (which by lemma 4.15 imply the existence of suitable $\mathcal{K}$ comparison functions $\underline{\alpha}_V$ and $\overline{\alpha}_V$).

By properness of the storage function (28) the findings of proposition 5.17 are partially satisfied. For all disturbances with $\mathcal{L}_\infty$ bound c small enough there holds $\alpha_w(c) \leq \max_{0 \leq r \leq 2} \alpha_x(r)$, hence there exists a $\varepsilon > 0$ such that the set Ω is positive $\mathcal{W}_c$-invariant for all $0 \leq c \leq \varepsilon$. Then the set $\overline{\mathcal{B}}_\infty$ is still an attractor for all motions generated by $w(\cdot) \in \mathcal{W}_c$ satisfying $\limsup_{t\to\infty} |w(t)| \leq c_\infty$, where $0 \leq c_\infty \leq c \leq \varepsilon$. ⋆

6 Classes of dissipative problems

In order to get some structure on the vast complexity of interlinked theorems, lemmas and corollaries developed in the previous sections it is convenient to identify four different classes of dissipative problems. These classes are identified by the structure of the HJI to be solved and the data given; class 2 is a superset of class 1, class 3 a superset of class 2 and 1, and so forth. The first row of table 6 defines the key properties of a class, the second shows the main results stated in this paper, and the third row lists additional properties of the classes.

Class 1 contains dissipative problems which satisfy the HJI (13) in the weak sense. Moreover, assumption 3.11 is met. This is the class of dissipative problems extensively discussed in the literature [Wil72a, Jam93a], often together with the $\mathcal{S}$-detectability assumption [HM80b, Hil92], or the zero-detectability assumption for systems with a single equilibrium point [HM76, HM80a], or, finally, in the context of nonlinear $\mathcal{H}_\infty$ control of systems [Isi92, IA92b, IA92a, vdS92a, BHW93, Jam93b, BH96, IK95, CMPP97, CS97]. Mostly, assumption 3.11 is trivially met since systems which are global reachable from an equilibrium point or a single point of minimal storage are considered. The storage functions considered here are often assumed to be continuous or C^1, even if the property of locally bounded excitation often not will be satisfied in practical applications. In this class the existence of lower semicontinuous storage functions can be guaranteed together with some weak asymptotic properties of the states subject to zero disturbances. The rate of convergence towards an equilibrium state can not be estimated. Positive definiteness of storage functions can be obtained in case of $\mathcal{S}$-detectability.

Class 2 consists of systems which satisfy in addition the strict HJI (36), or which have positive $\mathcal{S}$-definite and proper performance measures. The work of [Mar56, Cro98a] can be applied to show convergence of motions subject to $w(\cdot) \in \mathcal{L}_2$, $w(t) \to 0$ as $t \to \infty$ towards some compact set $\mathcal{S}$. Hence, the stability of the system is robust with respect to non-zero, but decaying disturbances. Often the storage function can be proved to be positive $\mathcal{S}$-definite and proper, which implies that powerful performance envelopes can be explicitly stated. State trajectories can thus be bounded in compact subsets of the Euclidean space.

Class 3 encloses systems of classes 1 and 2 which have positive $\mathcal{S}$-definite and proper performance measures *and* where a solution to the strict HJI (36) can be found. Based on the work [Son89a, Son89b, Lin92, Son95c, SW95b, SW96] it is possible to show that practical stability of trajectories subject to $w(\cdot) \in \mathcal{L}_\infty$ is given if the supply rate is strictly regular. This is the widest concept of stability investigated in this paper, systems of class 3 are robustly stable with respect to non-zero, non-decaying, but bounded disturbances. The associated storage functions are always positive $\mathcal{S}$-definite and proper, hence performance envelopes can be found.

Class 4 contains systems where the performance measure equals a function of class $\mathcal{K}_\infty$. Besides of the nice properties of class 3 problems, the existence of smooth storage functions has been proved. It is the author's belief that the existence of smooth storage functions also can be proved for all class 3 problems, and class 4 is redundant. However, since this assertion has not been proved yet, it seems appropriate to make a distinction between class 3 and class 4 problems.

Class 1	**Class 2**	**Class 3**	**Class 4**
$\mathsf{H}^* \leq 0$ weakly and assumption 3.11	$\mathsf{H}^* \leq -\alpha_{\mathsf{H}}(\lvert x\rvert_{\mathcal{S}})$ or $\underline{\alpha}_Z(\lvert x\rvert_{\mathcal{S}}) \leq \lvert Z(x)\rvert$	$\mathsf{H}^* \leq -\alpha_{\mathsf{H}}(\lvert x\rvert_{\mathcal{S}})$ and $\underline{\alpha}_Z(\lvert x\rvert_{\mathcal{S}}) \leq \lvert Z(x)\rvert$	$\mathsf{H}^* \leq -\alpha_{\mathsf{H}}(\lvert x\rvert_{\mathcal{S}})$ and $\alpha_Z(\lvert x\rvert_{\mathcal{S}}) = \lvert Z(x)\rvert$ and $s(w,z)$ strictly regular
$\Rightarrow$ V satisfies dissipation inequ. (5) $x(t)_{w(\cdot)=0} \to \mathcal{A}$, $\mathcal{A} \subset \mathcal{V} \cap \Omega \cap \mathcal{R}$ the. 3.15 the. 4.5	$\Rightarrow$ $x(t)_{w(\cdot)=0} \to \mathcal{S} \subset \mathcal{A}$ cor. 4.10 cor. 4.11	$\Rightarrow$ V pos. $\mathcal{S}$-def. and proper $\mathcal{N} = \mathcal{V} = \ker V = \ker Z \supset \mathcal{A} \supset \mathcal{S}$ pro. 4.12 lem. 4.15	$\Rightarrow$ smooth V exists pro. 5.14
+ locally bounded excitation $\Rightarrow$ all $V \in C^0$ pro. 3.23 + $\mathcal{S}$-detectability $\Rightarrow$ $x(t)_{w(\cdot)=0} \to \mathcal{S} \subset \mathcal{A}$ cor. 4.9 + regular $s(w,z)$ $\Rightarrow$ $w_{\max}(x) = 0$ on $\Gamma^+_{w=0} \subset \mathcal{X}_*$ cor. 4.7 + regular $s(w,z)$ + $\mathcal{S}$-detectability $\Rightarrow$ trajectory of points of minimal storage lem. 4.15	+ $x(\cdot)$ bounded in $\Omega \subset \mathcal{A}^+_\infty$ for all $w(\cdot) \to 0$ $\Rightarrow$ $x(\cdot) \to \mathcal{S}$ for all $w(\cdot) \to 0$ pro. 5.6 +$\underline{\alpha}_Z(\lvert x\rvert_{\mathcal{S}}) \leq \lvert Z(x)\rvert$ + str. reg. $s(w,z)$ $\Rightarrow$ V positive $\mathcal{S}$-definite and proper, performance envelopes lem. 4.15 lem. 3.25 $\mathsf{H}^* \leq -\alpha_{\mathsf{H}}(\lvert x\rvert_{\mathcal{S}})$ $\Rightarrow$ V positive $\mathcal{S}$-definite and proper, performance envelopes lem. 4.15 lem. 3.25	+ $s(w,z)$ strictly regular $\Rightarrow$ performance envelopes and practical stability for all $w(\cdot) \in \mathcal{W}_c$ pro. 5.17	

Table 1: Classes of dissipative problems

Systems of the classes 2, 3 and 4 have not previously caught research interest in the community of nonlinear dissipative or $\mathcal{H}_\infty$ control, and nonlinear $\mathcal{H}_\infty$ control techniques based on the knowledge of class 1 problems seems not to have large support among technicians and engineers which face the problem of implementing these strategies in practical applications.

There are quite good reasons to avoid class 1 problems in applications: first of all, the regularity of storage functions is often restricted to lower semicontinuity since the property of locally bounded excitation is not naturally met (after all, real world plants are mostly designed to minimize the influence of disturbances, and therefore it is avoided to make them controllable from the terminals where disturbances act!).

Second, the HJI's to be solved may be very close to Hamilton-Jacobi *equalities*, which accounts for a great deal of trouble: solving PDI's or PDE's numerically implies that the problem at hand is discretized to a system of ODI's or ODE's, for example by the use of finite difference or finite volume schemes, finite element methods, Fourier- or spectral methods. Since usual control problems have a state space of higher dimension than two or three, the discretization must be rather coarse to allow computation in mortal time. The approximation error between numerical approximation and true storage function will therefore often lead to areas in state space where the HJI $\mathsf{H}^* \leq 0$ can not be guaranteed, even not with equality. It follows that the feedback implemented on basis of the approximated storage function may have very unpleasant destabilizing effects, and dissipation may be violated.

Third, besides lacking robustness to modelling errors, there is no guaranteed robustness margin for nonzero disturbances. Therefore, it is the author's opinion that dissipative problems of class 1 are only interesting from a system theoretical point of view.

On the other hand, systems of the classes 2, 3 and 4 have appealing robustness properties, which make successful implementations and applications to real world system possible. First of all, the existence of smooth storage functions can be shown in special cases of class 3, and is given for all class 4 problems. The author expects that similar smoothness results will be found in future for all class 3 problems, and other regularity results than continuity by locally bounded excitation may be available for class 2 problems in a couple of years. The seek of numerical approximations makes only sense in case that the existence of solutions with better regularity than lower semicontinuity can be proved.

Second, the above described destabilization effect of numerical approximations to the true storage function is less prone to take place when solutions to the strict HJI $\mathsf{H}^* \leq \alpha_{\mathsf{H}}$ are found. In case that the approximation error on the gradient $\frac{\partial V}{\partial x}$ is known, it is in principle a simple matter to trace the influence of the errors through the Hamiltonian to decide which $\mathcal{K}$ function α_{H} must be selected to avoid destabilization effects.

Third, dissipative control problems of the classes 2, 3 and 4 give often easy access to useful performance envelopes, and robustness results in relation to nonzero disturbances. The practically oriented engineer is always faced with finite strength of the individual components of a plant, and excessive stress has to be avoided by the choice of a suitable performance envelope not to be exceeded. Also, robustness and practical stability in the presence of nonzero disturbances will often be mandatory.

Shortly, to make nonlinear dissipative and $\mathcal{H}_\infty$ control techniques useful in practice, the community of control researchers has to provide better regularity estimates for class 2 and 3 problems, more resources must be used to find numerical methods which are capable to exploit the structure of nonlinear HJI (i.e first order problems of high state space dimension involving only the gradient $\frac{\partial V}{\partial x}$, in case of affine $\mathcal{H}_\infty$ control HJI's which are quadratic in $\frac{\partial V}{\partial x}$), and useful and tight approximation error bounds must be computed together with the numerical solutions to strict HJI's. It is the author's hope that this paper will trigger further research in the directions sketched above.

6.1 Summary

In this paper it is shown that state analysis of problems involving the stability of open loop invariant sets can successfully be recast as dissipative problems, that is, generalized formulations of nonlinear local state $\mathcal{H}_\infty$ analysis problems. The main tools are a combination of dissipative techniques with the La Salle's invariance principle, and a throughout analysis of the structure of the solution to a certain Hamilton-Jacobi inequality (HJI).

This HJI is not assumed to be solvable on the entire, unbounded state space $\mathbb{R}^n$; the solvability of the HJI is restricted to the reachable subspace $\mathcal{R}$ without loss of generality. A fundamental assumption of the structure of the reachable subspace is made, which is mostly satisfied in practice.

Given a solution V to the HJI, the generalized problem is solved regionally provided V is such that the some connected component of the preimage $V^{-1}([0, c))$ for some $c \in \mathbb{R}$ is bounded and includes the to-be-stabilized invariant set. That is, all state trajectories can be bounded inside a compact set $\Omega \subset \mathcal{R}$ called a performance envelope. Unsupportable strain and stress on real world plants can be avoided by this approach.

The plant is often assumed to have a certain detectability property (which is just the generalization of the standard zero-detectability assumption) to prove asymptotic stability of the undisturbed system with respect to the invariant set of concern.

In case that a slightly stronger property holds, namely that the performance measure is positive definite (and eventually proper), or/and the HJI of concern can be solved strictly negative outside the to be stabilized set, strong and new results are proved to hold: in some cases the existence of smooth storage functions is given, in others the system will be robustly stable with respect to nonzero, but decaying disturbances.

Moreover, combining the input-to-state-stability property with the regional dissipative analysis mentioned above, it is possible to archive asymptotic stability even under time-persistent disturbances. The methods described in this paper are showing robust stability of invariant sets not only with respect to modelling errors, but also with respect to $\mathcal{L}_\infty$ norm bounded disturbances. The performance envelope of nonlinear plants is estimated, and the dissipation of the system (or $\mathcal{L}_2$ gain from disturbance to to-be-controlled output) is guaranteed for all motions inside the performance envelope, and all disturbances bounded by a certain $\mathcal{L}_\infty$ norm. The dissipative analysis problem can be showed to be structurally stable in the following practical sense: a neighborhood of the to be stabilized invariant set is shown to be an attractor for all trajectories generated by disturbances of sufficient small $\mathcal{L}_\infty$ bound, and furthermore, the size of the above mentioned practical stability neighborhood is continuously dependent on the $\mathcal{L}_\infty$ bound of the disturbances.

The presented results constitute a natural extension of local $\mathcal{H}_\infty$ analysis which enables the control engineer to address general dissipative analysis problems including various beneficial robustness results and regularity properties of storage functions not investigated before.

A Comparison functions

A.1 Definition *A real valued function $\alpha(r)$ belongs to class $\mathcal{K}$ if it is defined, continuous, and strictly increasing on $0 \le r \le r_c$, or $0 \le r < \infty$, and satisfies $\alpha(0) = 0$.*

A real valued function $\alpha(r)$ belongs to class $\mathcal{K}_\infty$ if it is defined, continuous, and strictly increasing on $0 \le r < \infty$, and satisfies $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

A real valued function $\sigma(s)$ belongs to class $\mathcal{L}$ if it is defined, continuous, and strictly decreasing on $0 \le s_c \le s < \infty$, and satisfies $\sigma(s) \to 0$ as $s \to \infty$.

Finally, a real valued function $\beta(r, s)$ is of class $\mathcal{KL}$ if it is defined for all r satisfying $0 \le r \le r_c$, or $0 \le r < \infty$, all s with $0 \le s_c \le s < \infty$, is of class $\mathcal{K}$ for each fixed s, and for each fixed r is monotone (not necessarily strict monotone) decreasing to zero as $s \to \infty$.

Let us in the following denote the inverse of functions of class $\mathcal{K}$ by the exponent -1, then we have

$$\alpha^{-1}(\alpha(r)) = r \quad \text{for all} \quad 0 \le r \le r_c$$

and

$$\alpha(\alpha^{-1}(R)) = R \quad \text{for all} \quad 0 \le R \le \alpha(r_c) \quad .$$

The following properties can be proved:

1. $\alpha_1(\alpha_2(r)) \in \mathcal{K}$ if $\alpha_1(r),\ \alpha_2(r) \in \mathcal{K}$
2. $\alpha(\sigma(s)) \in \mathcal{L}$
3. $\alpha^{-1}(r) \in \mathcal{K}_\infty$ if $\alpha(r) \in \mathcal{K}_\infty$
4. If $\beta(r, s)$ is bounded with respect to r, then $\beta(r, s) \le \alpha(r)\sigma(s)$

Since this functions are exclusively used in inequalities, we can whenever convenient assume that they are smooth. They can always be replaced by a smooth function of the same type.

In case that the reachable space $\mathcal{R}$ is a proper subset of $\mathbb{R}^n$ we have to modify the notion of class $\mathcal{K}_\infty$ functions to make sense. Given a compact and connected set $\mathcal{S} \subset \mathcal{R}$, the inequalities

$$\underline{\alpha}_V(|x|_\mathcal{S}) \le V(x) \le \overline{\alpha}_V(|x|_\mathcal{S})$$

with $\underline{\alpha}_V, \overline{\alpha}_V \in \mathcal{K}_\infty$ is to be understood in the following way:

1. There exists a $\mathcal{K}$ function $\underline{\alpha}_V : [0, \max_{x\in\partial\mathcal{R}} |x|_{\mathcal{S}}] \to \mathrm{IR}^+$ such that $\underline{\alpha}_V(r) \to \infty$ as $r \to \max_{x\in\partial\mathcal{R}} |x|_{\mathcal{S}}$, $\underline{\alpha}_V \leq V$, and $V(x) \to \infty$ as $x \to \partial\mathcal{R}$.

2. There exists a $\mathcal{K}$ function $\overline{\alpha}_V : [0, \min_{x\in\partial\mathcal{R}} |x|_{\mathcal{S}}] \to \mathrm{IR}^+$ such that $\overline{\alpha}_V(r) \to \infty$ as $r \to \min_{x\in\partial\mathcal{R}} |x|_{\mathcal{S}}$, $V \leq \overline{\alpha}_V$, and $V(x) < \infty$ for all $x \in \mathcal{R}$.

It follows that V satisfies the following properties: $V(x) > 0$ on $\mathcal{R}\backslash\mathcal{S}$, $V(x) = 0$ on $\mathcal{S}$, $V(x) \to \infty$ as $x \to \partial\mathcal{R}$, and $V(x) < \infty$ on $\mathcal{R}$.

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Comments and References

The notion of dissipation has also been cast in the framework of indefinite quasimetric spaces by Eduardo Sontag [Son95a, Son95b]. This makes an abstract approach to dissipation, the starting observation is that the dissipation inequality is equivalent to the statement that $V(x_T) - V(x_0) \leq W(x_0, x_T)$, where $W(x_0, x_T) \equiv \inf \int_0^T s(w(t), z(t))\ dt$, and the infimum is over all $w(\cdot)$ that steer $x(\cdot)$ from x_0 to x_T in time T. It is easily shown that $W(x_0, x_T)$ defines a indefinite quasimetric, since $W(x, x) = 0$ and $W(x, z) \leq W(x, y) + W(y, z)$ for all $x, y, z \in \mathbb{R}^n$. In this framework the choice of control and trajectories is blurred, and the pure cost structure is abstracted. Several basic facts of dissipative systems are then consequences of the properties of indefinite quasimetric spaces.

Stability issues of nonlinear state feedback systems in relation to equilibrium points have been investigated before. Among others, D.J. Hill and P.J. Moylan are concerned with stability results for nonlinear feedback systems [HM77] and general instability results for interconnected systems [HM83].

Stability of equilibrium points in relation to dissipative systems has been described by P. J. Moylan and D.J. Hill, which focus on the use of dissipativety in the stability analysis of interconnected systems [MH78], see also the overview on dissipation, stability theory and some remaining problems, given by David J. Hill [Hil88].

In the following we describe the interaction between dissipation, stability issues with respect to general invariant sets, and state feedback algorithms based on differential games.

Storage functions which are continuous and locally Lipschitz

As we have seen in the preceding paper in section 3.2, all systems which have the property of locally bounded excitation, have certainly continuous storage functions in case that they are dissipative. Without the rather restrictive assumption of locally bounded excitation, the only regularity we can guarantee for storage functions is lower semicontinuity. On the other hand, the analysis in section 4 of the preceding paper requires continuous and locally Lipschitz storage functions.

At a first glance, there seems to be a wide gap between the guaranteed and the needed regularity properties of storage functions. The example of the nonlinear oscillator (which posses a smooth storage function even if the property of locally bounded excitation is not satisfied for $\gamma > 1$), as well as the following argumentation, shows that this gap in practice often can be closed.

We want to stress that this subsection does not contain rigerous mathematical proofs; it is included to give a preliminary feeling why the property of locally bounded excitation is not necessary for a system to posses continuous and locally Lipschitz storage functions.

First of all, let us investigate the continuity properties of the storage function along some trajectory $x(\cdot) = x(\cdot, 0, x_0, w(\cdot))$ driven by the disturbance $w(\cdot)$, that is, the continuity of the map $V(x(\cdot)) : \mathbb{R} \mapsto \mathbb{R}^+$.

We remember that $w(\cdot) \in \mathcal{L}_2^{\mathrm{loc}}$ is assumed, and for sufficiently small $T > 0$ it follows that $x(\cdot), z(\cdot) \in \mathcal{L}_2^{\mathrm{loc}}([0,T])$. In fact, this is a direct consequence of the regularity of the performance measure Z and the fact that if x_0 is bounded, then there exists a small time $T > 0$ such that $x(t)$ is bounded too for all $0 \le t \le T$.

Now, in case that the disturbances are $w(\cdot) : \mathbb{R} \mapsto \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^l$ is a compact set, the dissipation inequality

$$V(x_T) - V(x_0) \le \int_0^T s(w(t), z(t))\, dt$$

shows that, for sufficiently small $T > 0$, $V(x_T)$ is bounded from above by

$$V(x_T) \le V(x_0) + T \max_{0 \le t \le T} |s(w(t), z(t))| \quad . \tag{49}$$

It follows that, if the map $V(x(\cdot)) : \mathbb{R} \mapsto \mathbb{R}^+$ is not continuous at x_0, then

$$\lim_{t \to 0^+} V(x(t)) \le V(x_0) \quad ,$$

that is, the jump must be downwards along the trajectory $x(\cdot)$.

If we make the assumption that the available storage V_A has no jumps downwards along any trajectory $x(\cdot)$, and the function $V_A(x(\cdot))$ satisfies

$$V_A(x_T) - V_A(x_0) \ge -Tk \quad , \tag{50}$$

where $k > 0$ is the maximal amount of energy dissipated locally near x_0, then we make implicit the assumption that a finite amount of energy can not be dissipated instantly, that is, the physical system at hand has no such behavior like shock waves, instant phase shifts, or ideal energy consuming collisions. This assumption may be rather hard to prove in general, but seems not to be strange for many mechanical or electrical systems.

It follows from (49) and (50) that $V_A(x(\cdot)) : \mathbb{R} \mapsto \mathbb{R}^+$ is continuous and locally Lipschitz along any part of trajectories $x(\cdot)$ which satisfy $|\frac{dx}{dt}| \ge \epsilon > 0$. Or, to put it in another way, then the directional derivative of $V_A : \mathcal{R} \mapsto \mathbb{R}^+$ in the directions $\frac{dx}{dt}$ and $-\frac{dx}{dt}$ exists and is bounded for all $x \in \mathcal{R}$ which are not near stagnation points of the vector field $x(t) \mapsto X(x(t), w(t))$.

Assume now furthermore that for each $x_0 \in \mathcal{R}$ there exists n disturbances $w_i(\cdot)$ such that the n vectors given by $v_i \equiv x(T) - x(0)$ for all $0 < T \le \varepsilon$ sufficient small are a basis of $\mathbb{R}^n$ (that is, the v_i are linearly independent, but not necessarily orthonormal), then we can show that $V_A : \mathcal{R} \mapsto \mathbb{R}^+$ is continuous and locally Lipschitz near each $x_0 = x(0)$.

To do so, let $x_0 \in \mathcal{R}$ be any point, then we show that there is neighborhood $\mathcal{N}_0$ around x_0 such that, under the above stated assumptions, $V_A : \mathcal{N}_0 \mapsto \mathbb{R}^+$ is continuous and Lipschitz on $\mathcal{N}_0$. Define the maps $\tilde{v}_i : \tilde{x} \times]0, \varepsilon] \mapsto \mathbb{R}^n$ by $\tilde{v}_i(\tilde{x}, T) = x(T, 0, \tilde{x}, w_i(\cdot)) - \tilde{x}$, then clearly $\tilde{v}_i$ is a continuous map. Hence, for all $\tilde{x} \in \mathcal{N}_0$, a sufficiently small neighborhood around x_0, and all $0 < T \le \varepsilon$ sufficiently small, the set $\{\tilde{v}_i(\tilde{x}, T)\}$ is a basis of $\mathbb{R}^n$. Therefore, at each $\tilde{x} \in \mathcal{R}$ the trajectories $x(\cdot, 0, \tilde{x}, w_i(\cdot))$ are interlacing each other (like an n-dimensional fibre compound) at finite angles. Since $V_A(x(\cdot))$ is by (49) and (50) continuous and locally

Lipschitz along each of them, it follows that $V_A : \mathcal{N}_0 \mapsto \mathrm{IR}^+$ is continuous and locally Lipschitz on $\mathcal{N}_0$.

Let us briefly discuss the examples 3.21 and 5.7 of the nonlinear oscillator in the above sketched context. We see from the polar coordinate system

$$\begin{aligned} \dot{r} &= r(r^2-1)(r^2-4) + r(r^2-4)w \\ \dot{\theta} &= 1 \\ z &= r^2 - 1 \quad . \end{aligned} \tag{51}$$

that the property of locally bounded excitation is not satisfied for $\gamma > 1$, since there does not exist such a class $\mathcal{K}$ function according to definition 3.22 in the following case: set $x_0 = (r = 1.5; \theta = 0)$ an set $x_T = (r = 1.5; \theta = -\varepsilon)$ for any $\varepsilon > 0$, then clearly there exist plenty of trajectories $x_0 \mapsto x_T$, for example, we can counteract the drift towards $\mathcal{S}$ by applying a suitable constant $w(\cdot) = c$, and float $2\pi - \varepsilon$ radians to end at x_T. However, for $\varepsilon \to 0$ it follows that $|x_0 - x_T| \to 0$, but

$$\left| \int_{t_1}^{t_2} \gamma^2 |w(t)|^2 - |z(t)|^2 \, dt \right| \to 0$$

can not be obtained for any of these connecting trajectories if $\gamma > 1$ is chosen.

It follows from the structure of the $\mathcal{H}_\infty$ problem that any storage function with $\gamma = 1$ is also a storage function for all $\gamma > 1$, hence the continuous and smooth storage function (28) solves the HJI (27) for $\gamma > 1$, even if property of locally bounded excitation is not satisfied.

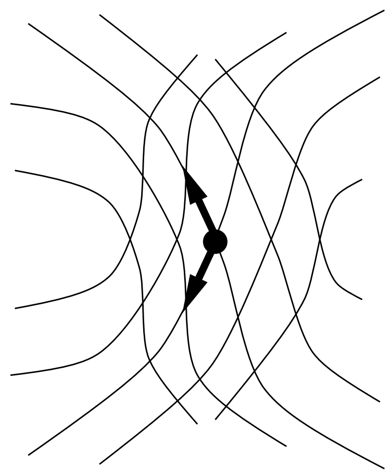

Figure 3: Interlacing trajectories

On the other hand, in any sufficient small neighborhood of $x_0 = (r = 1.5; \theta = 0)$ we can generate a basis by the disturbances $w_1(\cdot) = 0$ and $w_2(\cdot) = -\frac{5}{4}$, since $\dot{x_1} = (-\frac{75}{32}, 1)$ and $\dot{x_2} = (\frac{75}{32}, 1)$ span IR^2. Hence, in case that we can assure (50), continuity and the locally Lipschitz property follows near x_0 (see figure 3).

We see that the property of locally bounded excitation is a far stronger concept than the above proposed assumptions: locally bounded excitation implies that *any* point x_T in a neighborhood around each $x_0 \in \mathcal{R}$ can be reached by some trajectory which minimizes the exchange of energy with the surroundings by a function dependent on the distance between x_0 and x_T. On the other hand, the assumptions here imply only that n different, linear independent directions $x_T - x_0$ can be reached from x_0.

0.2 Example: Simple systems
It is not hard to prove that the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$
$$z = x_1$$

together with many supply rates has the property of locally bounded excitation: the key observation is that we can excite $\dot{x}$ to any desired direction. It follows that storage functions are continuous, if existing.

On the other hand, the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$
$$z = x_1$$

has not the property of locally bounded excitation, but has the property that the constant disturbances $w_1(\cdot) = 0$ and $w_2(\cdot) = 1$ span the directions $(-x_1^3, 0)$ and $(-x_1^3, 1)$, hence span $\mathbb{R}^2$ for all $x_1 \neq 0$. The discussion above can therefore be applied to conclude existence of continuous and locally Lipschitz storage functions on the left or right half plane, if we can show that the available storage does not jump downwards along trajectories, and the system is dissipative with respect to the chosen supply rate.

Finally, the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w$$
$$z = x_1$$

has neither of the two properties. However, in the case that

$$\ker V_A = \mathcal{X}_* = \left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 = 0 \right\} \quad ,$$

since then $\mathcal{R} = \mathbb{R}^2$ is an 2-dimensional set, the fundamental assumption on the reachable set may be satisfied. We can not by the above sketched approach determine whether a continuous storage function exists - if a storage function exists at all. ⋆

Differential games in dissipative state feedback control

Nine years ago Joseph A. Ball and J. W. Helton described some fundamental connections between time discrete $\mathcal{H}_\infty$ control and two-player, zero-sum differential games [BH89]. It turned out in the past that the interpretation of the best control strategy of $\mathcal{H}_\infty$ control problems as the minimizing player also is valid for time continuous nonlinear $\mathcal{H}_\infty$ state feedback control problems. See for example the paper "Structural properties of minimax policies for a class of differential games arising in nonlinear $\mathcal{H}_\infty$ control and filtering" by Garry Dinisky, Tamer Basar, and Pierre Bernhard [DBB93].

Some inspiration can also be found in the article [LS85] by P.-L. Lions and P.E. Souganidis, which describes the relations between differential games, optimal control, and directional derivates of viscosity solutions of Bellman's and Isaac's equation.

It seems therefore natural to make a similar generalization to dissipative systems which have an almost regular supply rate. Since this concept yet has not been throughout investigated, especially not in the context of general dissipativety where stabilization of invariant sets is considered, we present only some preliminary results along this line.

We assume that we are given some disturbed control system, together with a performance measure function $Z : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \mapsto \mathbb{R}^p$, and an almost regular supply rate $s : \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}$, both being continuous and locally Lipschitz. The dynamics of the differential game are in this context given by the two-player system

$$\begin{aligned} \dot{x} &= X(x, u, w) \\ z &= Z(x, u, w) \end{aligned} \tag{52}$$

with initial condition $x_0 \in \mathbb{R}^n$. The payoff is defined by

$$P(x_0, T, u(\cdot), w(\cdot)) = \int_0^T -s(w(t), z(t))\, dt \quad , \tag{53}$$

where T is the terminal time of the play, x_T the final point of the motion, and $u = u(\cdot)$ is the minimizing player strategy, whereas $w = w(\cdot)$ is the maximizing player strategy. The setup is a two-player, zero-sum, differential game, since we define the success for the minimizing player by $-P(x_0, u(\cdot), w(\cdot))$, and the success of the maximizing player by $P(x_0, u(\cdot), w(\cdot))$. Here, the integral (53) is taken along the trajectory $x(\cdot)$ of the dynamical system (52), with initial condition $x(0) = x_0$, and $P(x_0, u(\cdot), w(\cdot))$ represents the abstract "energy" which can be extracted from the system along the path $x(\cdot)$. Since

$$\sup_{w(\cdot)} \inf_{u(\cdot)} P(x_0, T, u(\cdot), w(\cdot)) \leq \inf_{u(\cdot)} \sup_{w(\cdot)} P(x_0, T, u(\cdot), w(\cdot)) \tag{54}$$

is always true for all possible initial points and play strategies, in the general case, the minimizing player $u(\cdot)$ has an advantage over the maximizing player $w(\cdot)$ if he is allowed to pick his strategy with knowledge of $w(\cdot)$. We will assume for the rest of this thesis that there exist a control strategy $u_{\min}(\cdot)$ such that the infimum is attained.

In the dissipative setup, both players act at the same time, but they have different play strategies: the minimizing player has to implement a state feedback law $u(\cdot) = u_{\min}(x)$, whereas the maximizing player uses an open loop strategy $w(\cdot) = w(t)$.

It is not reasonable to prescribe that the system (52) is such that the equality

$$\sup_{w(\cdot)} \inf_{u(\cdot)} P(x_0, T, u(\cdot), w(\cdot) = \inf_{u(\cdot)} \sup_{w(\cdot)} P(x_0, T, u(\cdot), w(\cdot) \tag{55}$$

holds - in fact, it suffices to consider the **upper value $P^* : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$** of the game. We define

$$P^*(x_0) \equiv \min_{u(\cdot)} \sup_{w(\cdot)} \lim_{T\to\infty} P(x_0, T, u(\cdot), w(\cdot)) \quad , \tag{56}$$

(in case that the integral is not convergent, we define $P^*(x_0) = +\infty$). We assume that the property of strong time consistency in the sense of T. Basar and P. Bernhard [BB95] is satisfied, that is, the minimizing control can be chosen as a state feedback law $u_{\min}(\cdot) = u_{\min}(x)$, independently of x_0. While this property is not satisfied in general for games, it is quite natural to assume in this context. We see by the following argumentation that the value of the game is closely connected to the available storage V_A with respect to the closed loop dynamics

$$\begin{aligned} \dot{x} &= X(x, u_{\min}(x), w) \\ z &= Z(x, u_{\min}(x), w) \quad . \end{aligned} \tag{57}$$

To show this, we remember that $V_A(x_0)$ is defined by the optimal path in the sense that maximal energy is extracted. Hence, given any path $x(\cdot)$ driven by some $w(\cdot)$ and by the minimizing control $u_{\min}(x)$, which leads from x_0 to some $x_T = x(T, 0, u_{\min}(\cdot), w(\cdot))$ with free terminal time T and terminal point x_T, the available storage is defined by

$$V_A(x_0) \equiv \sup_{\substack{T \in \mathbb{R}^+,\ w(\cdot) \\ u = u_{\min}(\cdot)}} \int_0^T -s(w(t), z(t))\, dt = \inf_{u(\cdot)} \sup_{w(\cdot)} \sup_{T \in \mathbb{R}^+} \int_0^T -s(w(t), z(t))\, dt \quad .$$

On the other hand, the value function is given by

$$P^*(x_0) \equiv \min_{u(\cdot)} \sup_{w(\cdot)} \lim_{T\to\infty} \int_0^T -s(w(t), z(t))\, dt \quad .$$

By almost regularity of the supply rate, $s(0, z(\cdot)) \geq 0$, it follows that $P^* \geq 0$, and moreover, that the map $T \mapsto \min_{u(\cdot)} \sup_{w(\cdot)} P(x_0, T, u(\cdot), w(\cdot))$ is monotone increasing in T. It follows that the value function can be written

$$P^*(x_0) \equiv \inf_{u(\cdot)} \sup_{w(\cdot)} \sup_{T \in \mathbb{R}^+} \int_0^T -s(w(t), z(t))\, dt \quad .$$

The equality $V_A = P^*$ follows in case that the system of concern is dissipative, that is, in case that $V_A < \infty$, or equivalently, in case that the indefinite integral converges.

There are obviously some open problems concerning the boundedness of state trajectories: the definition of the payoff function (53) implicitly assumes that the trajectories $x(\cdot, 0, u(\cdot), w(\cdot))$ do not have finite escape time, which is, after all, an assumption hard to prove. This could be fixed by showing that all such trajectories are bounded inside some compact set, but unfortunately, the boundedness or asymptotic stability issues of states have not gotten that much attention in the differential game circles.

The state feedback Hamiltonian-Jacobi inequality

One way to arrive at the state feedback Hamiltonian-Jacobi inequality is to assume that the payoff satisfies the equality (55), and that the resulting value function P^* is at least C^1. Then we can differentiate the dissipation inequality associated with the value $P^* = V_A$, which has to hold for all disturbances, to obtain the Hamilton-Jacobi inequality

$$\begin{aligned}\tfrac{d}{dt}P^* &- s(w_{\max}(x), Z(x, u_{\min}(x), w_{\max}(x))) \\ &= \tfrac{\partial P^*}{\partial x} X(x, u_{\min}(x), w_{\max}(x)) - s(w_{\max}(x), Z(x, u_{\min}(x), w_{\max}(x)) \le 0 \quad .\end{aligned}$$

This inequality is also often called Hamilton-Jacobi-Isaacs inequality to stress the relation to differential games. There are some problems hidden in this approach: first of all, it is not evident that the value function will be continuously differentiable, and secondly, it may not be defined since the existence of unbounded state trajectories is not a priori excluded, and the equation (55) is hard to prove in general.

Therefore, we would like to go the other way around: We define the state feedback **pre-Hamiltonian**

$$\mathsf{H}(x, p, u, w) \equiv pX(x, w, u) - s(w, Z(x, u, w)) \quad , \tag{58}$$

and notice that the map $(x, p, u, w) \mapsto \mathsf{H}(x, p, u, w)$ is continuous, locally Lipschitz, and affine in p, hence convex in p. We assume for simplicity that there exists a continuous minimizing control $u = u_{\min}(x, p)$ defined by

$$u_{\min}(x, p) \equiv \arg \min_{u \in \mathcal{U}} \left\{ \sup_{w \in \mathcal{W}} \mathsf{H}(x, p, u, w) \right\} \tag{59}$$

such that the map $(x, p, w) \mapsto \mathsf{H}(x, p, u_{\min}(x, p), w)$ is continuous, locally Lipschitz, and convex in p. Furthermore, we assume that the saddlepoint property

$$\mathsf{H}(x, p, u_{\min}(x, p), w) \le \sup_{w \in \mathcal{W}} \mathsf{H}(x, p, u_{\min}(x, p), w) \le \sup_{w \in \mathcal{W}} \mathsf{H}(x, p, u, w) \tag{60}$$

holds for all $u \in \mathcal{U}$ and all $w \in \mathcal{W}$. Then we define the **Hamiltonian**

$$\mathsf{H}^{**}(x, p) \equiv \sup_{w \in \mathcal{W}} \mathsf{H}(x, p, u_{\min}(x, p), w) \quad , \tag{61}$$

and it follows that the map $(x, p) \mapsto \mathsf{H}^{**}(x, p)$ is lower semicontinuous and locally bounded from below, but not necessarily convex in p.

At this point we are in the position to study the state feedback Hamilton-Jacobi inequality (also often called Hamilton-Jacobi Isaacs inequality)

$$\mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) \le 0 \quad , \tag{62}$$

or the strict state feedback Hamilton-Jacobi inequality

$$\mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) \le -\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \quad , \tag{63}$$

where the compact set $\mathcal{S} \subset \mathcal{R}$ denotes, as usual, any preferred mode of operation of the system of concern.

A weak formulation of the state-feedback HJI

When considering the solvability of a state feedback HJI, it is natural to take a closed look on the following subproblems: the solvability of non-feedback HJI in the weak sense, and the existence of weak solutions to the control Lyapunov inequality (21) of chapter 1.

We have seen in section 3.1 of the preceding paper that James proved the equivalence between the existence of a lower semicontinuous storage function and the existence of a lower semicontinuous viscosity solution to a HJI (in a weak sense), see theorem 3.15, or the original paper [Jam93a].

A similar property has been proved by Eduardo Sontag and Héctor J. Sussmann [SS95] in the context of non-smooth control Lyapunov functions. Consider the question: is the existence of a continuously differentiable control Lyapunov function *equivalent* to the possibility of steering every state asymptotically to zero (or, more general, to $\mathcal{S}$)? The answer is known to be negative. On the other hand, if the conditions on the control Lyapunov function is relaxed to continuity, and the generalized directional derivative is used to interpret the control Lyapunov inequality (21) of chapter 1, the answer is positive: A system of the form $\dot{x} = X(x, u)$ is asymptotically stabilizable if and only if it admits a weak control Lyapunov function.

While the power of the two above mentioned tools yet not has been fully combined, we have a rather simple equivalence between weak solutions to the state feedback HJI (62) and the dissipation properties of the controlled, closed loop system (57).

0.3 Corollary *If the closed loop system (57) is dissipative with (locally bounded) storage function $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$, then V satisfies the state feedback HJI (62) in the weak sense.*

Conversely, if a locally bounded function $V : \mathcal{R} \subset \mathbb{R}^n \mapsto \mathbb{R}$ is a viscosity solution to the state feedback HJI (62), then the controlled, closed loop system (57) is dissipative, and its l.s.c. envelope V_* is a l.s.c. storage function.

Proof: Since the state feedback Hamiltonian (61) is lower semicontinuous, locally bounded from below, and convex in p, and the closed loop system (57) is of the form of the perturbed system (4) in the preceding paper, the original proof in [Jam93a] applies without changes. □

See also Hitoshi Ishii's results on representations of solutions of Hamilton-Jacobi equations [Ish88].

Asymptotic behavior of controlled systems

We see immediately due to the above mentioned structural equivalence between the dissipation problem and the state feedback dissipation problem that **each and every lemma, corollary, proposition and theorem of the preceding paper which holds for the uncontrolled system (4) using the HJII (13), the weak HJI (16), or the strict HJI (36), also holds for the controlled, closed loop system (57) using the state**

feedback HJI (62), or the strict state feedback HJI (63). The proof of the equivalent properties is always a combination of the above mentioned game theoretic approach with the original proof of the properties concerning the uncontrolled case.

In principle, there is not much more to be said about the state feedback dissipative control problem. However, we would like to restate some of the most important results in the light of state feedback control. The following set definitions are appropriate in the new context:

0.4 Definition *Let $V : \mathcal{R} \subset \mathrm{I\!R}^n \mapsto \mathrm{I\!R}$ denote any continuous, locally bounded, locally Lipschitz, and non-negative function. We define the following subsets of $\mathcal{R} \subset \mathrm{I\!R}^n$:*
the ***control Hamiltonian null set***

$$\mathcal{N}^* \equiv \left\{ x \in \mathcal{R} \;\middle|\; \text{there exists } p(x) \in \partial V(x) \text{ with } \mathsf{H}^{**}(x, p(x)) = 0 \right\} \quad ,$$

the ***control storage null set***

$$\mathcal{V}^* \equiv \left\{ x \in \mathcal{R} \;\middle|\; \text{there exists } p(x) \in \partial V(x) \text{ with } p(x) X(x, u_{min}(x), 0) = 0 \right\} \quad ,$$

the ***storage kernel***

$$\ker V \equiv \left\{ x \in \mathcal{R} \;\middle|\; V(x) = 0 \right\} \quad ,$$

and the ***control performance kernel***

$$\ker^* Z \equiv \left\{ x \in \mathcal{R} \;\middle|\; Z(x, u_{min}(x), 0) = 0 \right\} \quad .$$

These definitions are essentially the same as in the analysis, they are always obtained by setting $u = u_{\min}$ and $w = 0$. We deduce directly from the invariance principle of theorem 4.5 the asymptotic properties of undisturbed, but controlled trajectories:

0.5 Corollary (Invariance principle) *Assume that the state feedback HJI (62) has a continuous and locally Lipschitz viscosity solution $V : \mathcal{R} \mapsto \mathrm{I\!R}$. Let $\Omega \subset \mathrm{I\!R}^n$ be any compact set. Assume that all $x(\cdot)$ with $x_0 \in \Omega$ generated by $u(\cdot) = u_{min}(x)$ and $w(\cdot) = 0$ are bounded in future inside Ω.*

Then all such $x(\cdot)$ approach the largest invariant set $\mathcal{A}$ contained in the intersection

$$\mathcal{A} \;\subset\; \mathcal{V}^* \cap \Omega \cap \mathcal{R} \quad .$$

Also in this case the control storage null set is related to the control Hamiltonian null set and the control performance kernel. There holds a corollary to theorem 4.6:

0.6 Corollary (Subsets) *Assume that the state feedback HJI (62) has a continuous and locally Lipschitz viscosity solution $V : \mathcal{R} \mapsto \mathrm{I\!R}$. Moreover, assume that all $x(\cdot)$ subject to $u(\cdot) = u_{min}(x)$ and $w(\cdot) = 0$ are bounded in future inside some compact $\Omega \subset \mathrm{I\!R}^n$. Then $\mathcal{V}^* \subset \mathcal{N}^*$ holds. If in addition the supply rate is regular, then $\mathcal{V}^* \subset \ker^* Z$ follows.*

There is, however, one important issue left to show: Given a compact invariant set $\mathcal{S} \subset \mathcal{R}$ of the uncontrolled and undisturbed system

$$\begin{aligned} \dot{x} &= X(x, 0, 0) \\ z &= Z(x, 0, 0) \quad , \end{aligned} \tag{64}$$

which represents the preferred modes of operation, we must assure that the controlled, but undisturbed system

$$\begin{aligned} \dot{x} &= X(x, u_{\min}(x), 0) \\ z &= Z(x, u_{\min}(x), 0) \end{aligned} \tag{65}$$

preserves the invariantness of the set $\mathcal{S}$, or even stronger, that the control is such that $u_{\min}(x) = 0$ on $\mathcal{S}$.

This resembles the situation in standard nonlinear $\mathcal{H}_\infty$ control theory where the stability of the equilibrium point zero is investigated: there it is known that the minimizing control $u_{\min}$ and the maximizing disturbance $w_{\max}$ vanish at the origin [IA92b, IA92a, vdS92a, IK95, BHW93].

A similar property holds for the generalized problem treated here. We denote the **the union of all positive limit sets** of bounded $x(\cdot, t_0, x_0, u(\cdot) = u_{\min}(x), w(\cdot) = 0)$ by the symbol $\cup\boldsymbol{\Gamma}^+_{\boldsymbol{u_{\min}, w=0}}$.

0.7 Corollary *Assume that the supply rate is regular, and the supremum in the definition of the state feedback Hamiltonian (61) is attained at each* $(x, p) \in \mathcal{R} \times \mathrm{IR}^n$*, and that the map* $u \mapsto Z(x_*, u, 0)$ *is one-to-one for all* $x_* \in \mathcal{X}_*$*, or equivalently, that* $\frac{\partial Z}{\partial u}(x_*, 0, 0)$ *has rank* m *for all* $x_* \in \mathcal{X}_*$*. Assume furthermore that the HJI (62) has a continuous and locally Lipschitz viscosity solution* $V : \mathcal{R} \mapsto \mathrm{IR}$*, and define the set-valued functions*

$$\begin{aligned} u_{min}(x) &\equiv u_{min}(x, \tfrac{\partial V}{\partial x}(x)) \quad , \quad \tfrac{\partial V}{\partial x} \in \partial V \quad , \\ w_{max}(x) &\equiv w_{max}(x, \tfrac{\partial V}{\partial x}(x)) \quad , \quad \tfrac{\partial V}{\partial x} \in \partial V \quad , \end{aligned} \tag{66}$$

where the minimizing control $u_{min}(x, p)$ *is defined by (59), and the maximizing disturbance* $w_{max}(x, p)$ *is given by (15). Moreover, assume the existence of some compact set* $\Omega \subset \mathrm{IR}^n$ *such that all* $x(\cdot)$ *with* $x_0 \in \Omega$*, and generated by* $u(\cdot) = u_{min}(x)$ *and* $w(\cdot) = 0$*, are bounded in future inside* Ω*, and assume that the intersection between* $\cup\Gamma^+_{u_{min}, w=0}$ *and* $\mathcal{X}_*$ *is non-empty. Then there holds*

$$u_{min}(x) = w_{max}(x) = 0 \quad on \quad \Gamma^+_{u_{min}, w=0} \subset \mathcal{X}_* \quad .$$

Proof: We follow exactly the same steps as in the proof of corollary 4.7, and arrive again at the equation

$$-s(w_{\max}, 0) = 0 \quad \text{on} \quad \Gamma^+_{u_{\min}, w=0} \subset \mathcal{X}_* \quad .$$

Therefore, $w_{\max} = 0$ on $\Gamma^+_{u_{\min}, w=0} \subset \mathcal{X}_*$ follows by regularity of the supply rate.

Now, by $w_{\max} = 0$ on $\Gamma^+_{u_{\min}, w=0}$ and by $\Gamma^+_{u_{\min}, w=0} \subset \ker^* Z$ it follows that $Z(x, u_{\min}(x), 0) = 0$ on $\Gamma^+_{u_{\min}, w=0}$. Finally, we remark that the map $u \mapsto Z(x_*, u, 0)$ is assumed to be one-to-one for all $x_* \in \mathcal{X}_*$, and therefore

$$u_{\min}(x) = 0 \quad \text{on} \quad \Gamma^+_{u_{\min}, w=0} \subset \mathcal{X}_*$$

follows immediately. □

It follows that all positive limit sets of the controlled, but undisturbed system (65) which are inside the set of minimal storage $\mathcal{X}_*$ also are compact invariant sets of the uncontrolled and undisturbed system (64).

We remark that the condition that $\frac{\partial Z}{\partial u}(x_*, 0, 0)$ has rank m for all $x_* \in \mathcal{X}_*$, is a natural generalization of the non-singular $\mathcal{H}_\infty$ condition for affine systems, namely that $D_u^T(x)D_u(x) > 0$ holds for all x (see comments and references in chapter 2). However, the later ensures also the saddlepoint property in the $\mathcal{H}_\infty$ case. In the general case treated here, the saddlepoint property must be imposed additionally to the rank condition of Z.

In addition, we want often to ensure that the set $\mathcal{S}$ is **asymptotically stabilizable** by the control $u(\cdot) = u_{\min}(x)$, that is, the compact set $\mathcal{S}$ is asymptotically stable under the dynamics (65). The state-feedback version of proposition 4.12 reads then:

0.8 Corollary (identity of sets) *Assume that the supply rate is regular and that a continuous and locally Lipschitz viscosity solution to the strict state feedback HJI (63) exists, such that*

$$\underline{\alpha}_V(|x|_{\mathcal{S}}) \le V(x) \le \overline{\alpha}_V(|x|_{\mathcal{S}}) \quad , \quad \mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) \le -\alpha_{\mathsf{H}}(|x|_{\mathcal{S}}) \quad ,$$
$$\text{and} \quad \underline{\alpha}_Z(|x|_{\mathcal{S}}) \le |Z(x, u_{min}(x), 0)| \quad .$$

Here $\mathcal{S} \subset \mathcal{R}$ is a compact set, and $\underline{\alpha}_V$, $\overline{\alpha}_V$, α_{H}, and $\underline{\alpha}_Z$ are four functions of class $\mathcal{K}$.

Then the identity

$$\mathcal{N}^* = \mathcal{V}^* = \ker V = \ker^* Z = \mathcal{S}$$

holds, and any trajectory $x(\cdot)$ subject to $u(\cdot) = u_{min}(x)$ and $w(\cdot) = 0$ which is bounded in future satisfies $x(\cdot) \to \mathcal{S}$. Moreover, $\mathcal{S}$ is locally asymptotically stable, and in case that V is proper, globally asymptotically stable.

Assume in addition that $\mathcal{S}$ consists only of one isolated trajectory of the undisturbed and uncontrolled system (64), then

$$\mathcal{N}^* = \mathcal{V}^* = \ker V = \ker^* Z = \mathcal{S} = \mathcal{A} = \Gamma^+ \quad .$$

We see that also lemma 4.15 has a straight forward translation to the state feedback case. It follows that the state feedback storage function V is positive $\mathcal{S}$-definite, and eventually proper, in case that the strict HJI (63) holds with $\mathcal{S} = \mathcal{X}_*$.

Smooth and robust state feedback control

The last results which we want to emphasize in the case of state feedback is the combination of the ISS-property with the above mentioned differential game approach to state feedback control. Analogous to proposition 5.14 the following holds:

0.9 Corollary *Given a system with strictly proper supply rate $s(w, z) = \alpha_w(|w|) - \alpha_z(|z|)$, a compact set $\mathcal{S} \subset \mathcal{R}$, and a performance function satisfying $|Z(x)| = \alpha_Z(|x|_{\mathcal{S}})$, where α_Z is any C^∞ function of class $\mathcal{K}_\infty$. Then any of the three following statements are equivalent:*

1. *The system (65) is dissipative and ISS with respect to $\mathcal{S}$.*
2. *The system (65) admits a continuous ISS-Lyapunov function which is also a positive $\mathcal{S}$-definite and proper viscosity solution to the strict state feedback HJI (63).*
3. *The system (65) admits a smooth ISS-Lyapunov function which is also a positive $\mathcal{S}$-definite and proper storage function satisfying the strict state feedback HJI (63) in the classic sense.*

Finally, the last result we want to stress in the context of state feedback control is the combination of proposition 5.17 with the theory of differential games. Here we have the possibility to estimate performance envelopes and sets of practical stability.

0.10 Corollary (performance envelope and practical stability) *Given a system with strictly proper supply rate $s(w,z) = \alpha_w(|w|) - \alpha_z(|z|)$, assume that all requirements of corollary 0.5 are satisfied using the strict state feedback HJI (63), and that there exists a compact set $\mathcal{S} \subset \mathcal{R}$, five functions of class $\mathcal{K}_\infty$, denoted $\underline{\alpha}_V$, $\overline{\alpha}_V$, α_{H}, $\underline{\alpha}_Z$, and $\overline{\alpha}_Z$, and a continuous storage function (viscosity solution) V such that*

$$\underline{\alpha}_V(|x|_\mathcal{S}) \leq V(x) \leq \overline{\alpha}_V(|x|_\mathcal{S}) \quad , \quad \mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) \leq -\alpha_{\mathsf{H}}(|x|_\mathcal{S}) \quad , \text{ and}$$
$$\underline{\alpha}_Z(|x|_\mathcal{S}) \leq |Z(x, u_{min}(x), 0)| \leq \overline{\alpha}_Z(|x|_\mathcal{S}) \quad .$$

Then V is a continuous ISS-Lyapunov function satisfying the ISS-PDI

$$\tfrac{\partial V}{\partial x}(x) X(x,w) \leq \alpha_w(|w|) - \alpha_x(|x|_\mathcal{S})$$

weakly, where the $\mathcal{K}_\infty$ function α_x is defined by $\alpha_x(|x|_\mathcal{S}) \equiv (\alpha_z \circ \underline{\alpha}_Z)(|x|_\mathcal{S}) + \alpha_{\mathsf{H}}(|x|_\mathcal{S})$.

Moreover, the level set $\Omega \subset \mathcal{R}$ defined by

$$\Omega \equiv \left\{ x \in \mathcal{R} \;\middle|\; V(x) \leq (\overline{\alpha}_V \circ \alpha_x^{-1} \circ \alpha_w)(c) \right\}$$

is globally asymptotically $\mathcal{W}_c$-stable and positive $\mathcal{W}_c$-invariant, hence a performance envelope. The set $\overline{\mathcal{B}}_\infty \subset \mathcal{R}$ defined by

$$\overline{\mathcal{B}}_\infty \equiv \left\{ x \in \mathcal{R} \;\middle|\; |x|_\mathcal{S} \leq \sigma(c_\infty) \right\}$$

is a global attractor for all $x(\cdot)$ generated by $w(\cdot) \in \mathcal{L}_\infty$ satisfying $\limsup_{t\to\infty} |w(t)| \leq c_\infty$.

Chapter 6

Solving Nonlinear Hamilton-Jacobi Inequalities

In case that only stability has our interest, we can use backstepping methods to find control Lyapunov functions [KKK95], but in general, it is not at all easy to find solutions to the partial differential inequalities (PDI's) arising in nonlinear control. Various efforts have been made to solve the Hamilton-Jacobi inequalities associated with nonlinear $\mathcal{H}_\infty$ control. We discuss here some of the main ideas previously applied to $\mathcal{H}_\infty$ control problems. We remember that the resulting HJI associated with the analysis, that is the bounded real lemma, is of the form

$$\begin{aligned} \mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) &= \tfrac{\partial V}{\partial x} X(x, w_{\max}(x, \tfrac{\partial V}{\partial x})) - \gamma^2 \left|w_{\max}(x, \tfrac{\partial V}{\partial x})\right|^2 + \left|Z(x)\right|^2 \\ &\leq 0 \quad \text{for all } x \in \Omega \quad , \end{aligned} \tag{1}$$

whereas HJI's belonging to state feedback problems are of the form

$$\begin{aligned} &\mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) \\ &\quad = \tfrac{\partial V}{\partial x} X(x, u_{\min}(x, \tfrac{\partial V}{\partial x}), w_{\max}(x, \tfrac{\partial V}{\partial x})) - \gamma^2 \left|w_{\max}(x, \tfrac{\partial V}{\partial x})\right|^2 + \left|Z(x, u_{\min}(x, \tfrac{\partial V}{\partial x})\right|^2 \\ &\quad \leq 0 \quad \text{for all } x \in \Omega \quad . \end{aligned} \tag{2}$$

In case that we are studying input affine systems, all HJI's are of the form

$$\begin{aligned} \mathsf{H}^{**}(\tfrac{\partial V}{\partial x}, x) &= \tfrac{\partial V}{\partial x} Q(x) \tfrac{\partial V}{\partial x}^T + \tfrac{\partial V}{\partial x} L(x) + K(x) \\ &\leq 0 \quad \text{for all } x \in \Omega \quad . \end{aligned} \tag{3}$$

Here the quadratic term $Q(x)$, the linear term $L(x)$, and the constant term $K(x)$ are defined by

$$\begin{aligned} Q(x) &\equiv \tfrac{1}{4\gamma^2} B B^T \\ L(x) &\equiv A \\ K(x) &\equiv C^T C \quad , \end{aligned} \tag{4}$$

in the case that we are investigating the bounded real lemma, or of the form

$$\begin{aligned} Q(x) &\equiv \tfrac{1}{4\gamma^2} B_w B_w^T - \tfrac{1}{4} B_u (D^T D)^{-1} B_u^T \\ L(x) &\equiv A - B_u (D^T D)^{-1} D^T C \\ K(x) &\equiv C^T (I - D(D^T D)^{-1} D^T)\, C \quad , \end{aligned} \tag{5}$$

for state feedback systems satisfying the DGKF simplifying assumptions, or more generally defined by (see chapter 2)

$$\begin{aligned} Q(x) &= -\frac{1}{4} \begin{bmatrix} B_u & B_w \end{bmatrix} H^{-1} \begin{bmatrix} B_u^T \\ \\ B_w^T \end{bmatrix} \\ L(x) &= A - \begin{bmatrix} B_u & B_w \end{bmatrix} H^{-1} \begin{bmatrix} D_u^T C \\ \\ D_w^T C \end{bmatrix} \\ K(x) &= C^T C - \begin{bmatrix} C^T D_u & C^T D_w \end{bmatrix} H^{-1} \begin{bmatrix} D_u^T C \\ \\ D_w^T C \end{bmatrix} \quad . \end{aligned} \tag{6}$$

Of course, the simplest approach is to try inserting a quadratic storage function in the HJI of concern. This approach equals solving the algebraic Riccati inequality

$$P^T \tfrac{\partial^2 \mathsf{H}}{\partial x \partial p}(0,0) + \tfrac{\partial^2 \mathsf{H}}{\partial x \partial p}(0,0) P + P^T \tfrac{\partial^2 \mathsf{H}}{\partial p^2}(0,0) P + \tfrac{\partial^2 \mathsf{H}}{\partial x^2}(0,0) \;\leq\; 0$$

associated to the linearized $\mathcal{H}_\infty$ control problem, and can therefore not regarded to be a non-linear approach.

Singular $\mathcal{H}_\infty$ control is not considered here, mainly because it is not mature in the nonlinear context (in contrast to linear singular $\mathcal{H}_\infty$ control theory, see for example Pascal Gahinet and Alan J. Laub [GL97] for numerical techniques).

1 Luke's approximation scheme

If we formally insert a polynomial expansion of higher degree than two at the place of the storage function in the HJI (2) or (3), and sort the polynomial coefficients according to their order, we use Lukes approximation scheme [Luk69], which was originally developed to HJI's belonging to optimal control problems with quadratic cost functions. Lukes approximation scheme has been proposed in [IK95] to solve the HJI's of general $\mathcal{H}_\infty$ control state feedback problems, and has been implemented in the symbolic language `MAPLE` in the masters thesis [MP95]. See also the application in [CMPP96, CMPP97] described in chapter 2. Lukes approximation scheme is easily implemented, either in a symbolic language, or numerically for larger problems. Since this approach consists of a polynomial expansion around the critical point of concern, it is a regional method restricted to a neighborhood, whose size can be estimated. However, it is not at all clear how the permuting terms $\Phi(x)$ should be chosen to maximize the valid region of the problem at hand. Another drawback of this method is that the linearized problem - that is the algebraic, permutated Riccati equation - must have a stabilizing solution to ensure a unique choice of the m-th order coefficients in equation (2.16). Despite these drawbacks, Lukes approximating scheme is an attractive and simple method suited for many problems where regionally control around some equilibrium point is needed.

2 The method of characteristics

There have also been multiple efforts to solve Hamilton-Jacobi equalities (HJE's). These are PDE's of first order, and the method of characteristics can therefore be applied. It follows that successive approximating solutions to the HJE's belonging to the $\mathcal{H}_\infty$ control problem can be found. I. Norman Katz and Jerry Markmann [KM96] proposed an iterative algorithm to solve HJE's arising in $\mathcal{H}_\infty$ control (see also [KS]) which takes the stabilizing solution of the linearized problem - that is the stabilizing matrix solving the algebraic Riccati equation - as starting point to iterate successive approximation solutions by the method of characteristics. As far as the author is informed, it seems to be difficult to prove stringent convergence bounds.

Another method for solving Hamilton-Jacobi equalities has been proposed and implemented by Kevin A. Wise and Jack L. Sedwick [WS94]. This approach is essentially a combination of Lukes approximation scheme with the method of characteristics. Only almost linear, input affine systems - the nonlinearities are only found in the driving term $A(x) = A_l x + \delta A(x)$ with $\delta A(x)$ being $O(x^2)$ - can be solved: the storage function is then of the form $V(x) = x^T X x + \delta V(x)$, where X is the solution to the linearized algebraic Riccati equation, and $\delta V(x)$ is of order $O(x^4)$. This ansatz results in a quadratic, first order PDE in δV which is then successively solved by the method of characteristics. This approach will probably soon be forgotten, since it combines elegantly the drawbacks of Lukes method with the drawbacks of the method of characteristics, and restricts the class of solvable problems efficiently to very special systems.

3 Nonlinear matrix inequalities

Linear $\mathcal{H}_\infty$ control problems have recently been transformed to Linear Matrix Inequalities (LMI's). These are equivalent to convex constrained optimizing problems which can be solved very efficiently to high accuracy, even if the state space has quite large dimensions. See the plenary lecture [Boy93] by S.P. Boyd, the overview article [BBFE93], and the Ph.D thesis by Ph.D Eric B. Beran [Ber97] and the references therein. A similar transformation to convex constrained optimizing problems via Nonlinear Matrix Inequalities (NLMI's) has been proposed by Wei-Min Lu and John C. Doyle in the technical reports [LD93a, LD93b], and in a short version in the conference paper [LD93c]. Also robustness analysis of uncertain nonlinear systems can be cast in the frame of NLMI's [LD94a, LD94b]. Let us sketch the approach briefly in the case of $\mathcal{L}_2$ gain analysis (the case of state feedback control is very similar, it consists of a combination of the here sketched tools with the differential game theoretic interpretation of the Hamiltonian; only the formulas are more complicated). The nonlinear system is assumed to have the form

$$\begin{aligned} \dot{x} &= A(x)x + B(x)w \\ z &= C(x)x + D(x)w \quad , \end{aligned} \tag{7}$$

where A, B, C and D are sufficient smooth matrix valued functions of suitable dimensions. Notice the slight abuse of notation in contrast to the standard form of the input affine system (2.8), there A and C are vector valued functions. Also, without loss of generality, the technical reports [LD93a, LD93b] consider only the $\mathcal{H}_\infty$ control problem with $\gamma = 1$. Let us assume that suitable reachability conditions are given, and assume that the available storage V_A, and any other storage function V are C^1 functions, and can be written

$$V_A(x) = x^T Q_A x + r_A(x) \quad , \quad V(x) = x^T Q x + r(x) \tag{8}$$

for some $Q_A, Q \geq 0$ and some C^1 functions $r_A, r : \mathbb{R}^n \mapsto \mathbb{R}$ satisfying

$$\lim_{x \to 0} \frac{r_A(x)}{|x|^2} = 0 \quad , \quad \lim_{x \to 0} \frac{r(x)}{|x|^2} = 0 \quad .$$

Since $V_A(0) = V(0) = 0$ and $\frac{\partial V_A}{\partial x}(0) = \frac{\partial V}{\partial x}(0) = 0$, there are continuous matrix valued functions P_A and P such that

$$\tfrac{\partial V_A}{\partial x}(x) = 2x^T P_A(x) \quad , \quad \tfrac{\partial V}{\partial x}(x) = 2x^T P(x) \quad .$$

Simple computations show now that the Hamiltonian can be rewritten in the form

$$\mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) = x^T \mathcal{H}^*(x, P) x$$

where the Hamiltonian matrix $\mathcal{H}^*$ is given by

$$\begin{aligned} &\mathcal{H}^*(x, P) \equiv \\ &\qquad P^T(A - BR^{-1}D^TC) + (A^T - C^TDR^{-T}B^T)P + P^TBR^{-1}B^TP + C^TR^{-1}C \quad . \end{aligned} \tag{9}$$

Here we suppress the dependency on x for notational ease, and we use the short hand $R(x) = I - D^T(x)D(x)$. It follows from the results published in [vdS92c, LD93a] that the existence of a C^1 function V satisfying $\frac{\partial V}{\partial x}(x) = x^T P(x)$, where P satisfies the nonlinear matrix inequality (NLMI) $\mathcal{H}^*(x, P) \leq 0$, implies that the HJI $\mathsf{H}^*(x, \frac{\partial V}{\partial x}) \leq 0$ holds, hence the system (7) has $\mathcal{L}_2$ gain less than or equal to one. Conversely, the $\mathcal{L}_2$ gain $\gamma \leq 1$ implies together with $\frac{\partial V}{\partial x}(x) = x^T P(x)$ or $\frac{\partial V_A}{\partial x}(x) = x^T P_A(x)$ that the NLMI $\mathcal{H}^*(x, P) \leq 0$, or even the nonlinear matrix equality (NLME) $\mathcal{H}^*(x, P_A) = 0$ holds.

In case of state feedback control, the system equations are given by

$$\begin{aligned} \dot{x} &= A(x)x + B_u(x)u + B_w(x)w \\ z &= C(x)x + D_u(x)u + D_w(x)w \quad , \end{aligned} \tag{10}$$

and a simple saddlepoint analysis shows that the Hamiltonian is given by

$$\mathsf{H}^{**}(x, \tfrac{\partial V}{\partial x}) = x^T \mathcal{H}^{**}(x, P) x$$

where the Hamiltonian matrix $\mathcal{H}^{**}$ is given by similar, but more complicated, formulae than the case of the bounded real lemma.

The NLMI's $\mathcal{H}^*(x,P) \leq 0$ and $\mathcal{H}^{**}(x,P) \leq 0$ are quadratic in P, hence not necessarily convex in P. Fortunately, the use of Schurs complement formula can be used to convexify the problem at hand. It is easy to show that the following are equivalent:

1. P satisfies
$$\mathcal{H}^*(x,P) \ \leq \ 0$$
2. P satisfies
$$\mathcal{M}^*(x,P) \equiv \begin{bmatrix} A^TP + P^TA + C^TC & P^TB + C^TD \\ B^TP + D^TC & D^TD - I \end{bmatrix} \ \leq \ 0$$
3. P satisfies
$$\tilde{\mathcal{M}}^*(x,P) \equiv \begin{bmatrix} A^TP + P^TA & P^TB & C^T \\ B^TP & -I & D^T \\ C & D & -I \end{bmatrix} \ \leq \ 0 \ ,$$

where the dependency on x has been suppressed toe ease the notation. The similar equivalent convex formulations in the case of state feedback are given by essentially the same matrix inequalities: the structure is not altered, only the complexity of the formulas is increased (see [LD93a, LD93b] for the details).

More general, the control problems at hand can be convexified to the following standard form: let $\mathcal{S}$ be the set of all symmetric n-dimensional matrices, and $\mathcal{P}$ the subset of positive semi-definite symmetric matrices. Then the solution set to the NLMI's considered so far can be regarded as the level set

$$\mathcal{L}(x) \equiv \left\{ P(x) \in \mathcal{P} \;\middle|\; \mathcal{M}(x,P(x)) \leq 0 \right\} \tag{11}$$

of a continuous matrix-valued map $\mathcal{M} : \mathcal{P} \times \mathbb{R}^n \mapsto \mathcal{S}$ which satisfies in addition

$$\mathcal{M}(x, \alpha P_1(x) + (1-\alpha)P_2(x)) = \alpha\mathcal{M}(x, P_1(x)) + (1-\alpha)\mathcal{M}(x, P_2(x)) \tag{12}$$

for all $P_1, P_2 : \mathbb{R}^n \mapsto \mathcal{P}$, all $x \in \mathbb{R}^n$ and all $0 \leq \alpha \leq 1$ (the later is a consequence of linearity in P), hence, $\mathcal{L}(x)$ is convex in $P(x)$.

It has been showed in [LD93a, LD93b] using arguments of lower semicontinuous set-valued functions and from Michael's selection theorem that the existence of a not necessarily continuous matrix valued map $P : \mathbb{R}^n \mapsto \mathcal{P}$ satisfying the strict NLMI

$$\mathcal{M}(x, P(x)) < 0$$

implies the existence of a continuous matrix valued map $\tilde{P} : \mathbb{R}^n \mapsto \mathcal{P}$ satisfying the NLMI

$$\mathcal{M}(x, \tilde{P}(x)) \leq 0 \ \ .$$

It seems therefore not to be very restrictive to assume that the members of the level set $\mathcal{L}(x)$ are continuous in x.

Unfortunately, we have to make the more restrictive assumption that the strict NLMI

$$\mathcal{M}(x, P(x)) < 0 \quad \text{for all} \quad x \in \Omega \tag{13}$$

can be solved by a continuous $P : \mathbb{R}^n \mapsto \mathcal{P}$, where $\Omega \subset \mathbb{R}^n$ is a suitable compact set. Since, by equation (12) the matrix-valued function $\mathcal{M}$ satisfies

$$\mathcal{M}(\sum_{i=1}^{N} \alpha_i P_i, x) = \sum_{i=1}^{N} \alpha_i \mathcal{M}(P_i, x) \quad \text{for all} \quad P_i \in \mathcal{P} \quad \text{and all} \quad \alpha_i \geq 0 \quad \text{with} \quad \sum_{i=1}^{N} \alpha_i = 1 \quad , \tag{14}$$

we are able to implement the following approximation scheme for one of the continuous solutions $P : \mathbb{R}^n \mapsto \mathcal{P}$: First, the compact subset Ω is discretized by choosing a suitable fine grid $\{x_i\}_{i=1}^N$, where $x_i \in \Omega$ for all $1 \leq i \leq N$. Then, a covering $\{\mathcal{N}_i\}_{i=1}^N$ satisfying $\Omega \subset \cup_{i=1}^N \mathcal{N}_i$, where each $\mathcal{N}_i$ is a neighborhood of the associated x_i, is constructed. This allows us to implement a suitable continuous (or even smooth) partition of the unity, that is a set of continuous (or smooth) functions $\{\alpha_i(x)\}_{i=1}^N$, $\alpha_i : \cup_{i=1}^N \mathcal{N}_i \mapsto \mathbb{R}$ which in addition satisfies

$$\alpha_i(x) = 0 \quad \text{for all} \quad x \notin \mathcal{N}_i \quad , \quad \text{and} \quad \sum_{i=1}^{N} \alpha_i(x) = 1 \quad \text{for all} \quad x \in \Omega \quad . \tag{15}$$

In case that the grid is fine enough, it follows from the strict NLMI (13) and from the continuity of the map $\mathcal{M} : \mathcal{P} \times \mathbb{R}^n \mapsto \mathcal{S}$ that there exists for each x_i a constant $P_i \in \mathcal{P}$ satisfying

$$\mathcal{M}(x_i, P_i) < 0 \quad \text{for all} \quad i = 1, 2, \cdots, N \quad , \tag{16}$$

and in addition, P_i can be chosen to be positive definite. Furthermore, in case that the grid $\{x_i\}_{i=1}^N$ is fine enough, the solutions P_i to the N linear matrix inequalities (LMI's) (16) are solutions to the N local NLMI's

$$\mathcal{M}(x, P_i) < 0 \quad \text{for all} \quad x \in \mathcal{N}_i \quad , \quad i = 1, 2, \cdots, N \quad , \tag{17}$$

because the map $\mathcal{M} : \mathcal{P} \times \mathbb{R}^n \mapsto \mathcal{S}$ is continuous. Finally, it follows from (14), (15) and (17) that the continuous (or smooth) approximation

$$\tilde{P}(x) \equiv \sum_{i=1}^{N} \alpha_i(x) P_i \tag{18}$$

solves the NLMI $\mathcal{M}(x, \tilde{P}(x)) \leq 0$, or equivalently, is a member of the level set (11).

Given a continuous $P(x) \in \mathcal{L}(x)$, we have to make sure that a C^1 storage function V satisfying $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ exists, since only then the HJI $\mathsf{H}^*(x, \frac{\partial V}{\partial x}) \leq 0$ holds. Therefore, let us consider the one-form (we use the convention that one-forms are denoted by row-vectors, whereas vector fields are denoted by column-vectors)

$$\omega \equiv 2x^T P(x) = 2x^T \sum_{i=1}^{N} \alpha_i(x) P_i \quad . \tag{19}$$

Then, there is a real function V satisfying $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ if and only if ω is exact, that is ω equals the differential dV, or equally, if ω is closed. Hence, as it has been pointed out in [LD93b], a sufficient condition is that ω is C^1, and satisfies

$$\sum_{i=1}^{N} \left(\frac{\partial \alpha_i}{\partial x_j}(x) \, \frac{\partial V_i}{\partial x_l}(x) - \frac{\partial \alpha_i}{\partial x_l}(x) \, \frac{\partial V_i}{\partial x_j}(x) \right) \; = \; 0 \;\; \text{for all} \;\; j, l = 1, 2, \cdots, N \;\; , \tag{20}$$

where $V_i(x) \equiv x^T P_i x$. It follows that the $\mathcal{H}_\infty$ control problem is solvable in case that the convex solution set of the N linear matrix equalities

$$\sum_{i=1}^{N} \left(\frac{\partial \alpha_i}{\partial x_j}(x) \, x_l - \frac{\partial \alpha_i}{\partial x_l}(x) \, x_j \right) P_i \; = \; 0 \;\; \text{for all} \;\; j, l = 1, 2, \cdots, N \tag{21}$$

intersects with the convex level set $\mathcal{L}(x)$ defined in (11). It is not at all clear under which circumstances this intersection will be non-empty.

To summarize, the algorithm used to solve $\mathcal{H}_\infty$ control problems of the form (9) or (10) via NLMI's is the following:

1. Construct a grid $\{x_i\}_{i=1}^N$ on Ω, and a covering satisfying $\Omega \subset \cup_{i=1}^N \mathcal{N}_i$
2. Construct a partition of the unity (15) over the grid $\{x_i\}_{i=1}^N$
3. Solve the N linear matrix inequalities (16) with positive definite P_i by convex optimization techniques - if possible
4. Verify that the positive definite P_i satisfy (21)
5. Construct an approximation $\tilde{P}(x)$ of a solution to the NLMI $\mathcal{M}(x, \tilde{P}(x)) \leq 0$ by (18)

Despite the fact that the above sketched approximation scheme does not answer clearly the question of the existence of a storage function satisfying $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$, it is very promising, and deserves further research. There are also strong connections to convex optimization techniques, which may be exploited successfully (see for example Stephen Boyd and Lieven Vandenberghe's course reader [BV97]).

4 A finite difference method for $\mathcal{H}_\infty$ problems

Recently M.R. James implemented a finite difference scheme to solve the HJI related to the general formulation of the bounded real lemma (See [Jam93b] and chapter 5 subsection 3.1.2). Since it is not implicitly assumed that a maximizing disturbance exists, the HJI (1) takes the following form: The plant (5.20) has $\mathcal{L}_2$ gain less than or equal to γ if and only if there exists a locally bounded nonnegative l.s.c. function $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ such that $V(0) = 0$ and such that the partial differential inequality

$$\mathsf{H}^*(x, \tfrac{\partial V}{\partial x}) = \sup_{w \in \mathcal{W}} \{ \tfrac{\partial V}{\partial x}^T X(x, w) - \gamma^2 |w|^2 + |Z(x)|^2 \} \leq 0 \tag{22}$$

is satisfied weakly on $\mathbb{R}^n$.

To motivate the approach used by M.R. James, let us consider the available storage

$$V_A(x) \equiv \sup_{T \geq 0 \ , \ w(\cdot) \in \mathcal{L}_2^{\mathrm{loc}}(\mathcal{W})} \int_0^T -\gamma^2 |w(t)|^2 + |Z(x(t))|^2 \, dt \quad ,$$

where the initial point is $x(0) = x$. Note that V_A is a solution to (22). To approximate the available storage, consider the finite time horizon problem

$$V_H(x, \tau) \equiv \sup_{w(\cdot) \in \mathcal{L}_2^{\mathrm{loc}}(\mathcal{W})} \int_0^\tau -\gamma^2 |w(t)|^2 + |Z(x(t))|^2 \, dt$$

with initial data $x(0) = x$. This function is a solution to the partial differential equality (PDE)

$$\frac{\partial}{\partial t} V_H = \sup_{w \in \mathcal{W}} \{ \frac{\partial V}{\partial x}^T X(x, w) - \gamma^2 |w|^2 + |Z(x)|^2 \} \tag{23}$$

with initial data $V(x, 0) = 0$. Since clearly by definition

$$\lim_{\tau \to \infty} V_H(x, \tau) = V_A(x) \quad ,$$

it follows that $V_H(x, \tau)$ for sufficient large τ will give an arbitrarily exact approximation of V_A.

Now, a finite difference scheme has been used in [Jam93b] to solve the time dependent PDE (23) on a compact subset $\Omega \subset \mathbb{R}^n$. More precisely, the finite difference scheme proposed by Kushner and Dupuis [KD92] to approximate solutions to dynamic programming equations arising in stochastic optimal control is described in the following:

Let $(\mathbb{R}^n)^\delta$ denote a coordinate grid of mesh size $\delta > 0$, centered at the origin. Define a system of discrete neighborhoods $\mathcal{N}_\delta(x)$ for all $x \in (\mathbb{R}^n)^\delta$ by

$$\mathcal{N}_\delta(x) \equiv \left\{ z \in (\mathbb{R}^n)^\delta \;\middle|\; z = x \ \text{ or } \ z = x \pm \delta e_i \right\} \quad ,$$

where e_i, for some $i = 1, 2, \cdots, n$ is the i-th unit vector in $\mathbb{R}^n$. We define the discretized disturbance space $\mathcal{W}^\delta \equiv \mathcal{W} \cap (\mathbb{R}^l)^\delta$, and the real positive number (norming factor)

$$\lambda_\delta \equiv \sup_{x \in \Omega \ , \ w \in \mathcal{W}} |X(x, w)|_1$$

where the one-norm is defined by $|X|_1 = \sum_{i=1}^n |X_i|$ for all $X \in \mathbb{R}^n$. Define furthermore the normed finite difference approximation

$$X_\delta(x, z, w) \ \equiv \ \begin{cases} 1 - \frac{1}{\lambda_\delta} |X(x, w)|_1 & \text{if } \ z = x \quad , \\[2ex] \pm \frac{1}{\lambda_\delta} X_i(x, w) & \text{if } \ z = x \pm \delta e_i \quad , \\[2ex] 0 & \text{if } \ z \neq \mathcal{N}_\delta(x) \quad , \end{cases}$$

then the finite difference analogy of the HJI (22) is given by the discrete inequality

$$V_\delta(x) \geq \sup_{w \in \mathcal{W}_\delta} \left\{ \sum_{z \in (\mathbb{R}^n)^\delta} X_\delta(x,z,w) V_\delta(z) - \frac{\delta}{\lambda_\delta}(\gamma^2|w|^2 - |Z(x)|^2) \right\} \quad \text{for all} \quad x \in (\mathbb{R}^n)^\delta \quad . \tag{24}$$

As it has been pointed out in [Jam93b], this discretization can easily be interpreted in terms of a controlled Markov chain with transition probability $X_\delta(x,z,w)$ and state space $(\mathbb{R}^n)^\delta$. Unfortunately, the discrete HJI (24) has in general more than one unique solution, if γ is chosen large enough. Therefore M.R. James considers the analogous finite difference approximation to the time variant PDE (23), where the finite time horizon available storage $V_H(x,\tau)$ is discretized by a time partition $t_k = k\frac{\delta}{\lambda_\delta}$, $k = 1, 2, \cdots$, the numerical scheme is given by

$$V_\delta^k(x) = \sup_{w \in \mathcal{W}_\delta} \left\{ \sum_{z \in (\mathbb{R}^n)^\delta} X_\delta(x,z,w) V_\delta^{k-1}(z) - \frac{\delta}{\lambda_\delta}(\gamma^2|w|^2 - |Z(x)|^2) \right\} \tag{25}$$

for all $x \in \Omega_\delta \equiv \Omega \cap (\mathbb{R}^n)^\delta$ and all $k = 1, 2, \cdots$. It follows that $V_\delta^k(x)$ approximates $V_A(x, t_k)$ for $\delta > 0$ sufficient small, and an approximated solution to the discretized HJI (24), and hence to the continuous time HJI (22), on a compact $\Omega \subset \mathbb{R}^n$ can be found in finite time by iterating (25) forward in time until a stationary solution is obtained. Appropriate boundary conditions, such as von Neuman type boundary conditions, must be imposed to ensure convergence of the iterations. To summarize shortly, the finite difference approximation scheme works as follows:

1. Select the discretization size $\delta > 0$
2. Choose a $\gamma_1 > 0$
3. Iterate (25)forward in time
4. If a stationary solution is obtained, choose a $\gamma_2 < \gamma_1$, otherwise choose a $\gamma_2 > \gamma_1$
5. Repeat step 2-4 until desired accuracy is obtained
6. If necessary, adjust the discretization size δ and repeat step 1-5

There are some implicit assumed properties hidden in this approximation scheme which have not been justified in [Jam93b]. First of all, any use of a difference scheme needs the existence of continuous solutions, otherwise there is no reason to try a pointwise approximation. While a certain acceptable degree of smoothness is probably given for any solution to a strict HJI, the available storage V_A is only known to be a viscosity solution to the associated HJE, and moreover, the existence of examples where the available storage is not continuous has been described in the literature (See for example [BH96]). Hence, this approximation scheme should only be invoked when the existence of continuous solutions to the HJE is known.

Next, in the paper [Jam93b] there is no proof, nor any indication that the proposed finite difference approximation (25) to the PDE (23) has a fixpoint $V_\delta = \lim_{k\to\infty} V_\delta^k$ for all γ solving the original $\mathcal{H}_\infty$ problem. It has however been showed in [Jam93b] that the discretized HJI (24) has a solution for each $\gamma > 0$ which admits a solution to the continuous time HJI (22), and the lacking proof of the existence of a fixpoint seems only a technical matter.

Third, while the interior point condition $V(0) = 0$ is mandatory, the determination of appropriate boundary conditions on $\partial\mathcal{R}$ may be very difficult, if not impossible. This very important step in the algorithm is not explained. Especially, the use of von Neuman boundary conditions is not justified. It is easily seen that the use of a homogeneous von Neuman boundary condition is not what we want, since then the directional derivative of the storage function in the inwards direction is forced to be zero. It follows that stability properties of disturbed trajectories are worse near $\partial\mathcal{R}$ than in the center of $\mathcal{R}$, thereby the system will be vulnerable for crash due to small disturbances in case that trajectories near $\partial\mathcal{R}$ are considered.

Finally, one should be very cautious when using any form of approximation scheme to solve a HJE: while the exact solution to a HJE also solves the HJI and therefore the feedback law computed from such an exact solution stabilizes the origin as intended, the implemented feedback law computed from an approximating solution to the HJE may de-stabilize the origin, no matter how fine an approximating grid size is chosen. On the other hand, there is a finite small grid size such that the feedback law implemented from an approximating solution to a strict HJI with certainty stabilizes the origin.

Despite these words of caution, the finite difference scheme proposed by M.R. James in the research paper [Jam93b] is worth to consider, since it is easy to implement and finite difference schemes are known to perform well under various circumstances. Another advantage of this method is that it can be easily generalized to problems involving general supply rates, and to application where the stabilization of other invariant sets than the origin is wanted.

5 A Galerkin spectral method for optimal control problems

Very recently R.W. Beard, G.N. Saridis and J.T. Wen presented, and proved the convergence, of an algorithm which computes approximated solutions to the time-invariant Hamilton-Jacobi-Bellmann equation arising in nonlinear optimal control [BSW98]. The algorithm contains two main steps: First, successive approximations are used to reduce the nonlinear Hamilton-Jacobi-Bellmann equation (HJB) into a sequence of *linear* partial differential equations, there named generalized Hamilton-Jacobi-Bellmann equation (GHJB). Second, these linear GHJB are then approximated via the Galerkin spectral method.

The presented algorithm has many advantages: the resulting control is in feedback form, and it's associated region of attraction is well defined and estimated. In addition, all computations can be made off-line, and only a coefficient set belonging to the control has to be stored to implement the algorithm in real time. Moreover, the successive approximation part of the algorithm ensures that the solution found to the HJB not only satisfies the optimal performance criterion, but on the same time is a stabilizing solution to the HJB. Finally, the use of a Galerkin spectral method gives very fast convergence in case that the optimal cost function is known to be smooth.

The theory of optimal control, and the methods of dissipative control are closely related to each other, which makes a successful application of a modified Galerkin spectral algorithm in general dissipative control probable. Indeed, R.W. Beard, G.N. Saridis and J.T. Wen announce in [BSW98] the application of a modified algorithm to the HJE arising in nonlinear $\mathcal{H}_\infty$ control.

This promised modified algorithm seems to have many advantages over other numerical schemes discussed here: first of all, the successive approximation results in a series of linear first order PDE which are less demanding to solve than the nonlinear counterparts of the competing numerical schemes. Secondly, appropriate stability properties of the closed loop system are ensured together with satisfaction of the performance criteria. Third, spectral methods are known to converge fast - if a smooth solution is known to exist.

The comparative overview of rates of convergence for approximation schemes in optimal control by Paul Dupuis and Matthew R. James [DJ98] is interesting reading in this context.

Without having studied the algorithm yet, some words of caution are probably worth to mention: again, it seems desirable to solve Hamilton-Jacobi *inequalities* instead of equalities, since these give better stability results of the closed loop dynamics. Moreover, HJI's are more likely to have smooth solutions (if solutions to HJI's exist!), and therefore the faster-than-polynomial convergence of spectral methods, which is a considerable advantage over other numerical schemes, can then be fully exploited.

Having said this on a rather weak knowledge of the modified algorithm, the author looks forward to see this promising spectral Galerkin method implemented.

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