Robust and Optimal Control: Robust Sampled-Data $\mathcal{H}_2$ and Fault Detection and Isolation

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Summary

This thesis truly exploits and builds on the advancement in digital technology in its treatment of two key issues in digital control. The robust control framework is applied and extended to tackle these issues.

The first issue concerns LQG controllers with robustness to norm bounded model uncertainty in the sampled-data setting so that the intersample behaviour is taken into account.

The second issue is fault detection and isolation/estimation which is a cornerstone in most (computerised) reliable and supervisory control systems.

Both issues are essentially digital implementation aspects for which optimisation based design and analysis methods, generally computational demanding, are developed.

The robust $\mathcal{H}_2$ (LQG) problem (worst-case over a set of systems) is approached in two ways by applying the lifting technique which converts the hybrid system with both a continuous-time plant and discrete-time controller into a system in one time set.

Firstly by elaboration on the well-known loop transfer recovery procedure to handle sample-data systems. The work includes design methods based on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimisation. Secondly by extending recent results on robust $\mathcal{H}_2$ performance to the sample-data setting. Resulting in convex conditions consisting of a linear matrix inequality and an averaging integral.

The first approach is a synthesis method, however, the recovery may not be exact, whereas the second approach is basically an analysis method. Consequently it makes sense to employ these in turn.

For completeness related multiobjective sampled-data designs are sketched and discussed in some detail.

The FDI (Fault Detection and Isolation) problem is formulated for both nominal and uncertain systems. An analysis of threshold selection leads to the case without false alarms. This mounts into a study of the smallest fault guaranteed to be detected where the main result is a (possible conservative) norm based index for those. However, it opens for optimisation based design methods and is given in various forms.

Further, the integrated FDI and controller design problem is investigated to exploit their interaction as well as sharing an observer like part. Thus a full FDI design is facilitated by optimisation methods.

Though the two issues are treated quite separately in this work it will take only a small extra effort to do a combined design since the same framework handles both.

Sammenfatning (Danish)

Denne afhandling bygger på, og udnytter i særdeleshed, udviklingen indenfor digital teknologi i studiet af to centrale emner i digital kontrol (regulering og styring).

Første emne behandler LQG regulatorer med robusthed mod normbegrænset model-usikkerhed i sampled-data opsætningen, således, at opførselen mellem samplingstiderne medregnes.

Andet emne er fejldetektering og isolering/estimering. Dette udgør en hjørnesten indenfor pålidelige og "supervisory" digitale kontrol systemer.

Begge emner er grundlæggende digitale implementeringsspørgsmål, for hvilke optimeringsbaseret dimensionerings- og analysemetoder udvikles. Disse er som oftest beregningstunge.

Det robuste $\mathcal{H}_2$ (LQG) problem (det værst tænkelige tilfælde i en mængde af systemer) behandles på to måder, hvor den såkaldte "lifting" teknik omdanner et system med både et kontinuert reguleringsobjekt og en diskret regulator til et ekvivalent system.

Der startes med videreudvikling af det velkendte "loop transfer recovery" princip til at omhandle samplede systemer. Denne inkluderer dimensioneringsalgoritmer baseret på $\mathcal{H}_2$ og $\mathcal{H}_\infty$ optimering.

Derefter behandles den fornylig fremkomne metode til robust $\mathcal{H}_2$ ydeevne i sampled-data opsætningen med de nødvendige udvidelser. Dette resulterer i konvekse betingelser bestående af lineære matrix-uligheder og et integral til gennemsnitsudligning.

Den første angrebsvinkel er basalt set en syntesemetode, hvor gendannelsen ikke nødvendigvis er eksakt. Den anden angrebsvinkel er en analysemetode, derfor er det fornuftigt, at anvende disse i nævnte rækkefølge. For at afrunde emnet er fler-kriterie sampled-data metoder skitseret og belyst.

Fejl Detekterings- og Isoleringsproblemet (FDI) formuleres i både det nominelle og ubestemte tilfælde. En analyse af valget af grænseværdi, fører til tilfældet uden falske alarmer. Dette udmunder i et studie af den mindste fejl, som det er muligt at detektere. Her er hovedresultatet et indeks, muligvis konservativt, der kvantificerer disse. Imidlertid åbner dette for anvendelsen af forskellige optimeringsbaserede dimensioneringsmetoder.

Desuden er det integrerede FDI og kontrol-dimensioneringsproblem undersøgt, med henblik på at udnytte vekselvirkningen. En fuldstændig FDI-dimensionering er således understøttet og simplificeret af optimeringsmetoder.

Selvom de to emner er behandlet næsten særskilt kræver det kun en begrænset merindsats, at udføre en kombineret dimensionering, da den samme metoderamme er anvendt til begge emner.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering at Technical University of Denmark.

The research presented is carried out at the Department of Automation at Technical University of Denmark. The Ph. D. study has been supervised by:

- Associate Professor Henrik Niemann, Main supervisor, Department of Automation, Technical University of Denmark.

- Professor Jakob Stoustrup, Department of Control Engineering, Institute of Electronic Systems, Aalborg University. Until January 1. 1997 at the Department of Mathematics, Technical University of Denmark.

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Finally, the weather has been perfect this summer for writing a thesis.

August 1998

Mike Lind Rank

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Notation

Acronyms

BIBO (Bounded Input Bounded Output) . . . 11
FD (Fault Detection) . . . 67
FDE (Fault Detection and Estimation) . . . 68
FDI (Fault Detection and Isolation) . . . 67
FDLTI (Finite Dimensional Linear Time Invariant) . . . 16
FIR (Finite Impulse Response) . . . 17
FWL (Finite Word Length) . . . 99
LFT (Linear Fractional Transformation) . . . 24
LQE (Linear Quadratic Estimator) . . . 33
LQG (Linear Quadratic Gaussian) . . . 1
LQR (Linear Quadratic Regulator) . . . 5
LTI (Linear Time-Invariant) . . . 11
LTR (Loop Transfer Recovery) . . . 31
LTV (Linear Time-Varying) . . . 22
MIMO (Multiple Input and Multible Output) . . . 1
NL (Non-Linear) . . . 11
NP (Nominal Performance) . . . 25
NS (Nominal Stability) . . . 24
PTV (Periodic Time-Varying) . . . 11
RHP (Right-Half Plane) . . . 15
RMS (Root Mean Square) . . . 33
ROC (Robust and Optimal Control) . . . 2
RP (Robust Performance) . . . 25
RS (Robust Stability) . . . 25
SD (Sampled-Data) . . . 35
SDS (Sampled-Data Systems) . . . 35
SISO (Single Input and Single Output) . . . 1
SLTV (Slowly Linear Time-Varying) . . . 22
TFM (Transfer Function Matrix) . . . 17

Notation

$\mathcal{B}$ Banach space 10
$\mathcal{L}_p$-stable 11
$\mathcal{E}$ Euclidian space 39
$\mathcal{L}_2^e$ extended bounded space 13
$\mathfrak{L}_{\mathcal{A}}(\mathcal{L}_2)$ 16
$\mathcal{H}_\infty$ 15
$\mathcal{R}\mathcal{H}_\infty$ real rational subspace of $\mathcal{H}_\infty$ 15
$\mathcal{A}_\mathbb{R}$ 15
$\mathcal{A}(\mathbb{D},\mathfrak{L}(\mathcal{K}))$ 15
$\mathcal{H}_\infty(\mathbb{D},\mathfrak{L}(\mathcal{K}))$ 15
$\mathcal{H}$ Hilbert space 11
$\|\cdot\|_{\mathbb{HS}}$ Hilbert-Schmidt norm 12
$\langle\cdot,\cdot\rangle$ inner product 10
$\mathcal{L}_p^m$ 13
ℓ_p^m 13
$\ell_{\mathcal{K}}$ 13
Λ Λ-transform 14
L Laplace transform 14
L lifting map 37
$\|\cdot\|_\infty$ 15
$\|\cdot\|_1$ 45
$\|M\|_-$ smallest gain - not a norm 74
$\|M\|_\Omega$ smallest gain - not a norm 18
$\|\cdot\|$ norm 10
$\|\cdot\|_{\mathcal{H}_2}$ or 2 as subscript 41
Δ_a 22
$\overline{\sigma}(\cdot)$, $\underline{\sigma}(\cdot)$ maximal and minimal singular value 18
$\star$ is for the Redheffer Star-product 24
$\mathrm{tr}(\cdot)$ trace class norm 12
$\|\cdot\|_{W_{\eta,B}}$ 43

Chapter 1

Introduction

Automatic control is roughly speaking a 20th century interdisciplinary of engineering and mathematics initiated by the study of feedback amplifiers for telecommunications, however, one can easily say that it goes all the way back to the Greeks building water clocks in around 300 B.C. or the 18th century for the Watt steam engine speed governor. Using Wiener's neologism Cybernetics (or control and communication in the animal and the machine) and thinking of control as forcing a system to have certain properties, it has certainly been around for quite a while, just think of the functions of your body. An easily accessed reference for a richer view of the control history is [Ben96].

We make a clean cut here and divide control theory into the following periods and give a few references and keywords; many of these techniques are still central in the current research.

1935-1960 The Classical period. Linear Time-Invariant (LTI) SISO (Single Input and Single Output) system design based on the frequency response techniques using Nyquist [Jen84], Bode [Bod45] plots etc. or in the time domain using the Laplace transform technique to obtain the wanted performance quantities based on rules of thumb.

1955-1980 Modern Control [KS72, ÅW84]. State-space LTI MIMO (Multiple Input and Multible Output) methods, LQG (Linear Quadratic Gaussian), digital computers, dynamical programming, adaptive control.

1975- Post Modern e.g. [ZDG95] Robust control. $\mathcal{H}_\infty$-control, Linear Matrix Inequalities (LMIs), the μ method (complex) structured sigular value.

In short one may summarise the realm of obstacles for (classical) control techniques to be that real systems are uncertain, time-varying, nonlinear and infinite-dimensional, moreover, measurements contain noise and errors. Besides, the application of digital hardware for implementation of controllers adds more obstacles. Systems with the latter two phenomenas being predominant are by themselves topics in control theory. Uncertain and time-varying systems are aimed at by both adaptive and robust control (described below).

The idea in adaptive control is to gather data on-line about the uncertain (or unknown) process or the feedback loop and based upon these to change/tune/adapt the controller. This is basically an algorithmic approach. It is difficult to show stability of the resulting closed-loop beforehand and convergence of the algorithm, therefore, the method is somewhat adhoc.

From an engineers point of view adaptive control can handle systems with slow and possible large uncertainty. Robust control can handle systems with small and possible fast uncertainty. Therefore it is no surprise that combining the underlaying ideas of these two theories/methods is an attractive research area, albeit a difficult one. Lately there has been progress in the area of gain scheduling.

Virtually, all tools from mathematics and computer science are thrown at control problems. For these reasons control theory has many branches which are in indeed interconnected, overlapping and often mixed.

Here we work in the broad framework of robust and optimal control as it has a good mathematical foundation and may be seen as a mixture of the best of the classical and modern periods.

1.1 Framework

ROC (Robust and Optimal Control) started around the mid-seventies and inspired by the small gain theorem [Zam66, Zam81], a little ahead of time, the problems were formulated using maps on function spaces. Most of the mathematical theory to solve the problems were not around, however, it have since been quite well established and now provides a solid foundation. The paradigm of thinking in maps and computing in state-space has been advantageous in the development.

ROC works on a class of plants, given by a nominal plant and an uncertainty set, the robustness phrase simply means that a property holds for a given class. This classifies it as a branch of model based control like most control theory. The two generic properties one is looking for are (internal) stability and performance which in short may be taken respectively as given an unforced system the states decay to zero by time and achieving certain specifications i.e. tracking or attenuation of noise. Feedback is the mean to obtain the properties; by analysis we check that they are met and by synthesis we design a controller with these. Essentially, feedback is a tradeoff since it has the ability to handle signal/model uncertainty and change dynamics at the price of higher noise sensitivity.

The celebrated result of parameterisations of all stabilising controllers [YJB76], makes it possible to start a brute force search based on trial and error. The main problem for this naive approach is the stopping criteria, which naturally will be time or rough bounds on the limitations of performance. Based on signal types methods for finding the optimal robust controller systematicly (by some algorithm) constitutes the cornerstones in ROC i.e. the so-called $\mathcal{H}_\infty$ and ℓ_1 theory.

1.2 Organisation and Motivation

In many ways the invention of the the 20th century is the transistor. The subjects in this thesis are closely related to the cheap and very powerful digital equipment available now in the turn of the century; most notably in form of the (personal) computer. The computer facilitates the main tasks of control engineering:

- *Design*. Such as solving algorithms and optimisation problems even for large complex systems. These calculations would not be attainably by pen and paper.

- *Implementation*. For instance by computing the control-law or filter interfaced via A/D and D/A converters. This is generally cheaper, simpler and more flexible than an analog implementation.

This thesis deals with two key aspects of control engineering namely digital controllers and filtering and the scope is analysis and design.

The control-law may be seen as the lowest level in the control hierarchy where higher levels count topics as FDI (Fault Detection and Isolation) filters and supervisory control (not studied here). However, in our prespective control and FDI are on an equal footing, see fig. 1.1.

More precisely we study, refering to fig. 1.1, 1) robust $\mathcal{H}_2$ for SD (Sampled-Data) Systems and fault detection and isolation in the robust control framework; 2) design regarding threshold and 3) integrated FDI and controller design.

Though both the SD and the FDI implementation issues often are essential parts of an actual design, they clearly have a right on their own and can be treated quite seperately which will be the case both in the next brief introductions and in the latter parts of the thesis.

The thesis is organised as follows with a general part on signals and systems modelling, the robust control framework followed by the SD systems part, the FDI part and conclusion with further directions.

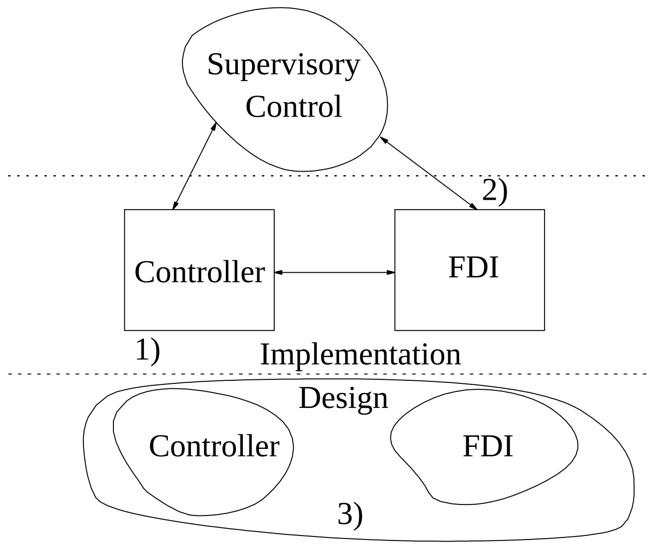


Figure 1.1: Organisation

1.2.1 Sampled-Data Systems

Due to the widespread use of digital hardware in control systems the overall system has commonly a continuous process and a discrete controller based on time-samples and this hybrid is called a sampled-data system. Besides, measurements in chemical and biological processes often only results in data at certain time instants. A sampled-data system we think of generically as an analog/continuous plant and a digital controller connected via A/D converter with analog prefilter and D/A converter. By standard idealisations we view the loop as an analog plant (appropriate prefilter absorbed therein), a sampler, a digital controller and a hold.

We want to shape the behaviour of the plant, hence to have continuous time specifications. Therefore studying the "at sample" behaviour is not enough; we need the intersample behaviour as well. Recent years research has established a theory for taking the intersample behaviour into account, a main tool is the lifting/raising framework which transforms a linear periodic problem into a LTI problem. Generally the intersample behaviour is of interest when the sampling rate is limited which to some extent is always the case. Even if the rate is free and therefore can be chosen corresponding to a frequency much higher than the bandwidth of the closed-loop, say a hundred times, there may be complications.

The common denominator in the SD part is to tackle the problem of getting a robust $\mathcal{H}_2$ (read LQG) controller in the SD sense.

The story starts with the quest for methods to perform MIMO designs. The state-feedback $\mathcal{H}_2$ controller, or LQR (Linear Quadratic Regulator), has good closed-loop robust stability properties and is found efficiently by a Riccati equation, however, it is often not possible in a practical system to measure the full state. An observer is hence used to estimate the state; the Kalman filter which is found by a Ricatti equation again. In fact the time-reversed equation of the LQR problem - the two problems are indeed "dual", see [KS72].

Looked upon in turn these have each an infinite gain (and a 1/2 reduction) margin and no less than 60° phase margin. Their combination, the LQG problem, results in a two ways striking abstract to the paper "Guaranteed Margins for LQG Regulators" quote "There are none" [Doy78].

Besides adding momentum to the aforementioned $\mathcal{H}_\infty$ and ℓ_1 theory; the paper initiated a still active line of work in robust $\mathcal{H}_2$: first by LTR (Loop Transfer Recovery) and later $\mathcal{H}_2/\mathcal{H}_\infty$ and robust $\mathcal{H}_2$ performance.

Here we pursue in the SD setting the LTR idea (chapter 5) since it is still commonly applied by practitioners and the robust $\mathcal{H}_2$ performance problem (chapter 6) as it solves the problem more satisfactorily.

1.2.2 Fault Detection and Isolation

FDI by observers also known as analytical redundancy contrary physical redundancy has been an active research area for two decades whereas the area of fault detection originated in close connection with adaptive control a decade earlier. The observer approach is easy to implement in software and clearly less expensive than physical redundancy, furthermore, it is a cornerstone in most reliable and supervisory control.

An observer based fault detector consists of a residual generator (the observer) and a residual evaluator (the threshold and a comparator e.g. with some time-window averaging). For accommodation of the fault it is necessary to isolate the sensor or actuator which failed which is often possible by the observer. Moreover, the faults degree of severeness can be estimated; known as FDE (Fault Detection and Estimation). This is truly what we aim at, but we still use FDI (or fault detection) as the topic term.

In this work we exploit the robust control framework to obtain a systematic way to select threshold via a performance index; a norm based design (chapter 8) and enhance design methods for simultaneous design of controller and fault detector (chapter 9). Thus the full FDI design is facilitated by widely available and efficient optimisation methods.

1.3 Contributions

Contributions in this work include related published material by the author and coauthors in:

- Loop Transfer Recovery for Sampled-Data Systems [NSRS96]. A journal version is under preparation.
- Simultaneous Design of Controller and Fault Detector [KRNS96].
- Robust $\mathcal{H}_2$ Performance for Sampled-Data Systems [RP97]. The technical report contains more details.
- Norm Based Threshold Selection for Fault Detectors [RN98]. A journal version is submitted.
- Multiobjective Sampled-Data Design. A conference version is under preparation.

Highlights are summarised below.

SDS:

- A setup for the SD LTR problem and design methods for $\mathcal{H}_2$/LTR and $\mathcal{H}_\infty$/LTR on page 52.
- Application of fast discretisation for SD LTR on page 53.
- In order to apply the lifting technique to the recovery error operator a composite description is needed in the $\mathcal{H}_\infty$ case see page 49.
- Obtain a controller of the same order as the plant using the recovery matrix on page 50.
- No duality between the input loop breaking point and the output loop breaking point for SD $\mathcal{H}_\infty$/LTR design on page 55.
- Sufficient conditions for robust $\mathcal{H}_2$ performance for sampled-data systems have been derived under three different cases of uncertainty:
  - periodic time-varying on page 59.
  - linear time-varying on page 61 with necessity when using set-based white noise modelling.
  - linear time-invariant on page 64.
- Multiobjective sampled-data designs are sketched and discussed on page 103

FDI:

- Given an induced norm the accompanying smallest gain is related to a matching problem on page 19.
- The formulation of the fault detection and isolation problem in terms of induced norms of transfer functions on page 75.
- A new performance index for the optimisation of FDI filters for both nominal systems on page 77 as well as for uncertain systems on page 81.
- The case with a fixed number of faults may occur simultaneously on page 79; particularly the case with one fault at a time.
- An iterative threshold design method on page 81.
- A setup and design for simultaneous design of controller and fault detector on page 90 with a more general filter structure than observer based ones.
- Low order filters/controllers on page 92.

Though, the SDS nature and FDI are treated quite separately in this work it will take only a small extra effort to do a combined design, basicly, since the framework of robust and optimal control is exploited and extended to handle both.

Chapter 2

Signals and Systems Modelling

This chapter and the following serve to introduce notation, main concepts and to give references to the comprehensive literature used.

Control theory deals with analysis and synthesis of dynamical systems. One way to view a system is as a device which takes some input signals and gives some output signals [DV75]. Another view is the behavioural approach [Wil91]. The signals may be digital/discrete or analogue/continuous.

As with all mathematical modelling of the real world one pays a price for getting the tools and abstraction of mathematics at hand by the simplifications or limitations of the chosen model/space. Some choices are explicit others implicit. In this thesis we study linear (finite dimensional) systems mainly because of the mathematical convenience.

The mathematical abstraction we use is as follows: Systems are maps or operators and signals are sequences or functions (of time). The time-set discrete (continuous) is given by the support assumed non-negative of the sequences (functions). Which is basic functional analysis [Kre89, Con90] and topics in complex functions theory [Con78, Con95] are also needed. We will use the terminology of the mathematical description and physical description interchangeably.

A key mathematical tool is optimisation [Lue68] and to perform this it is important to have measures for signals and systems.

2.1 Complete Normed Linear Spaces

The common underlaying structure of both signals and systems is the notion of a vector/linear space with the usual addition and scalar (field, $\mathbb{F}$) multiplication operations. We start with the requirements for a norm on a vector/linear space.

Definition 2.1 *A norm, $\|\cdot\|$, on a vector/linear space X maps $X \to \mathbb{R}$ and satisfies for all $x, y \in X$ and scalars $\alpha \in \mathbb{F}$*

N1 $\|x\| \geq 0$ *for all* $x \in X$, $\|x\| = 0 \Leftrightarrow x = 0$

N2 $\|\alpha x\| = |\alpha| \|x\|$

N3 $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality)*

Though, the zero vector/matrix/function/sequence/functional/operator differs depending on space we simply write 0 and the definition stays the same. A normed vector/linear space X is simply a vector space with a norm defined on it. A norm on X induces a metric by $d(x, y) = \|x - y\|$.

Definition 2.2 *A normed vector/linear space where the induced metric space is complete is called a Banach space, $\mathcal{B}$. By rewriting definitions:*

- *A sequence $(x_n) \in X$ is convergent with limit $x_0 \in X$ if*

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow \|x_n - x_0\| < \varepsilon$$

- *A sequence $(x_n) \in X$ is a Cauchy sequency if*

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} : n, m \geq n_0 \Rightarrow \|x_n - x_m\| < \varepsilon$$

- *A Banach is a normed vector space X where every Cauchy sequency is convergent*

Note that every finite dimensional normed vector space is a Banach space.

Besides the (distance) measure often an analogue of the dot product is desirable. Such a space is called an inner product space

Definition 2.3 *An inner product, $\langle \cdot, \cdot \rangle$, on a vector space X maps $X \times X \to \mathbb{F}$ and for all $x, y, z \in X$ and scalars $\alpha \in \mathbb{F}$ satisfies*

I1 $\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \Leftrightarrow x = 0$

I2 $\langle \alpha x, x \rangle = \alpha \langle x, y \rangle$

I3 $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

I4 $\langle x, y \rangle = \overline{\langle y, x \rangle}$

An inner product on X defines a norm on X by $\|x\| = \sqrt{\langle x, x\rangle}$ and hence a metric. A complete inner product space is called a Hilbert space, $\mathcal{H}$, and thus also a Banach space.

Finally, it is worth to mention that the main reason for working with complete spaces is that the optimisation problem is often solved by construction of a sequence of improving solutions and the optimal one is then the limit which is assured to be in the space.

2.2 Systems as Linear Operators

We will study systems with the properties of linearity, causality, time-invariance and BIBO (Bounded Input Bounded Output) stability also called external stability.

Let X (input) and Y (output) be normed spaces and $T : X \to Y$ be an operator. Then T is bounded if $\exists c \in \mathbb{R}, (\forall x)\ \|Tx\| \leq c\|x\|$ and the induced norm is given by

$$\|T\| = \inf\{c \in \mathbb{R} : (\forall x)\ \|Tx\| \leq c\|x\|\} = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \tag{2.1}$$

Note that the same symbol denotes different norms, however, the context/factors indicates the proper (normed) space, the same goes for inner products. Furthermore, let T be linear i.e.

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2), \quad \forall x_1, x_2 \in X,\ \alpha, \beta \in \mathbb{R} \tag{2.2}$$

and one can show that T (linear) is continuous if and only if T is bounded. The space of all bounded linear operators from X into Y, $\mathfrak{L}(X,Y)$ (if $X = Y$ just $\mathfrak{L}(X)$), is a vector space itself with the induced norm (2.1) which due to the linearity also equals $\sup_{\|x\|=1} \|Tx\|$.

In the system context this is precisely BIBO stability and the induced operator norm is the worst-case gain i.e. maximum gain for all possible inputs. If $T \in \mathfrak{L}(X)$ we call it X-stable e.g. $\mathcal{L}_p$-stable[1]. Further the multiplicative inequality holds i.e. for concatenation of systems S and T holds $\|TS\| \leq \|T\|\|S\|$.

Let a signal (continuous) have support in $\mathbb{R}$ then the delay operator of $\tau > 0$ is defined by $D_\tau : f(t) \mapsto f(t-\tau)$ for $t \geq \tau$ and 0 for $0 \leq t \leq \tau$. Likewise let D be the unit shift operator (discrete time delay). A linear operator is LTI (Linear Time-Invariant) if $D_\tau T = T D_\tau \quad \forall \tau > 0$ for continuous systems and if $DT = TD$ for discrete systems. If T commutes with D_h we call it h-periodic or PTV (Periodic Time-Varying).

[1]If T is a NL (Non-Linear) operator we still call it stable if it is bounded with gain (2.1)

Further we define the truncation operator by $P_\tau : f(t) \mapsto f(t)$ for $0 \le t \le \tau$ and 0 for $t \ge \tau$. A linear system is causal if $P_\tau T P_\tau = P_\tau T \quad \forall \tau > 0$ and $\tau \in \mathbb{R}$ or $\tau \in \mathbb{Z}$ for respectively continuous and discrete systems.

2.3 Hilbert Space - some Concepts

The added structure in a Hilbert space yields a richer theory of which we will use orthonormal sequences, primarily Fourier sequences, and the Hilbert-adjoint operator.

Let X be a Hilbert space and M an orthonormal subset i.e. let $x,y \in M$ then $\langle x,y\rangle = 1$ if $x = y$ else zero. If M is a countable set we can rearrange the elements into an orthonormal sequence (x_n). We call (x_n) an orthonormal basis if $span(M) = X$ and it is complete if the only element orthogonal to all x_n is the zero element. Only Hilbert spaces, $\mathcal{H}$, with a complete orthonormal basis (separable) are used later.

The advantage is that given a complete orthonormal basis (from now on basis) (x_k) every element, $x \in \mathcal{H}$ can be represented as a linear combination $x = \sum_{k=0}^{\infty} \alpha_k x_k = \sum_{k=0}^{\infty} \langle x, x_k\rangle x_k$.

Next we define the Hilbert-adjoint operator denoted T^*. Let $T \in \mathfrak{L}(\mathcal{H}_1, \mathcal{H}_2)$ then there exists an unique operator,

$$T^* \in \mathfrak{L}(\mathcal{H}_2, \mathcal{H}_1) \text{ s.t. for all } x \in \mathcal{H}_1, y \in \mathcal{H}_2 \quad \langle Tx, y\rangle = \langle x, T^* y\rangle$$

One can show that $\|T\| = \|T^*\| = \sqrt{\|T^*T\|} = \sqrt{\|TT^*\|}$.

The Hilbert-Schmidt norm [HS78] for a Hilbert space operator $T \in \mathcal{H}$ is

$$\|T\|_{\mathbb{HS}}^2 \triangleq \sum_{i=0}^{\infty} \|Te_i\|_{\mathcal{H}}^2 = \sum_{i=0}^{\infty} \langle Te_i, Te_i\rangle_{\mathcal{H}} \tag{2.3}$$

and the trace class norm

$$\mathrm{tr}(T) \triangleq \sum_{i=0}^{\infty} \langle Te_i, e_i\rangle_{\mathcal{H}}, \quad \|T\|_{\mathbb{HS}}^2 = \mathrm{tr}(T^*T) \tag{2.4}$$

Where $(e_i)_{i=0}^{\infty}$ is any basis for $\mathcal{H}$.

2.4 Signal Spaces

In the following the actual spaces we will use are defined, the notation is along the lines of [Dul95, CF95].

2.4.1 Time-Domain Signal Spaces

Let $\mathcal{L}_p^m$ denote the set of one-sided functions mapping $[0;+\infty[\to \mathbb{R}^m$,

$$\mathcal{L}_p^m \triangleq \{f : \|f\|_{\mathcal{L}_p^m}^p \triangleq \int_0^\infty |f(t)|_p^p dt < \infty\} \tag{2.5}$$

$$\mathcal{L}_\infty^m \triangleq \{f : \|f\|_{\mathcal{L}_\infty^m} \triangleq \operatorname*{ess\,sup}_{t\in[0;+\infty[} |f(t)|_\infty < \infty\} \tag{2.6}$$

used for continuous signals and for discrete signals ℓ_p^m mapping $\mathbb{Z}_+ \to \mathbb{R}^m$,

$$\ell_p^m \triangleq \{f : \|f\|_{\ell_p^m}^p \triangleq \sum_{k=0}^\infty |f(k)|_p^p < \infty\} \tag{2.7}$$

$$\ell_\infty^m \triangleq \{f : \|f\|_{\ell_\infty^m} \triangleq \sup_{k\in\mathbb{Z}_+} |f(k)|_\infty < \infty\} \tag{2.8}$$

The remainder on spaces relates mostly to the sampled-data part, see section 4.3.

Let $\mathcal{K}^m$ be the compressed space of $\mathcal{L}_2^m$ the square integrable functions mapping $[0;h[\to \mathbb{R}^m$

$$\mathcal{K}^m \triangleq \{f : \|f\|_{\mathcal{K}^m}^2 \triangleq \int_0^h |f(t)|_2^2 dt < \infty\} \tag{2.9}$$

Let $\ell_\mathcal{K}$ denote the set of one-sided sequences mapping $\mathbb{Z}_+ \to \mathcal{K}^m$

$$\ell_\mathcal{K} \triangleq \{\tilde{f} : \|\tilde{f}\|_{\ell_\mathcal{K}}^2 \triangleq \sum_{k=0}^\infty \|\tilde{f}(k)\|_\mathcal{K}^2 dt < \infty\} \tag{2.10}$$

We will mostly omit the dimensions. Superscript e is used to denote extended bounded space, e.g. $\mathcal{L}_2^e$

$$\int_0^h |f(t)|_2^2 dt < \infty, \quad \forall h > 0 \tag{2.11}$$

2.4.2 Lifting

Given a signal $u \in \mathcal{L}_2^e$ define $u_k(t) \triangleq u(hk+t)$, $0 \le t \le h$, a sequence with each element defined on a sampling interval $\tilde{u} \triangleq [\dots u_{-2} u_{-1} u_0 u_1 \dots]^T$. Then let $L : \mathcal{L}_2^e[0,\infty) \to \ell_\mathcal{K}$ be defined by $\tilde{u} \triangleq Lu$.

2.4.3 Frequency-Domain Signal Spaces

On $\mathcal{L}_2$ and $\ell_{\mathcal{K}}$ we can define the Laplace and the Λ-transform.

$$\hat{f}(\omega) = (\mathcal{L}f)(\omega) \triangleq \int_0^\infty f(t)e^{-j\omega t}dt \tag{2.12}$$

$$\check{f}(\lambda) = (\Lambda\tilde{f})(\lambda) \triangleq \sum_{k=0}^{\infty} \tilde{f}[k]\lambda^k \tag{2.13}$$

the inverse transforms are defined on the respective images i.e. $\mathcal{L}\mathcal{L}_2$ and $\Lambda\ell_{\mathcal{K}}$ which can be identified with $\mathcal{H}_2$ ($\mathcal{H}$ for Hardy space see [Fra87]) and $\mathcal{H}_2(\mathbb{D},\mathcal{K})$, which is defined by

$$\begin{aligned} \mathcal{H}_2(\mathbb{D},\mathcal{K}) \triangleq \{&\check{f} \text{ is analytic in } \mathbb{D}: \\ &\|\check{f}\|^2_{\mathcal{H}_2(\mathbb{D},\mathcal{K})} \triangleq \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \|\check{f}(re^{j\theta})\|^2_{\mathcal{K}} d\theta < \infty\} \end{aligned} \tag{2.14}$$

The relations (isomorphisms) between the time-domain and frequency-domain and their lifted versions are shown is the commuting diagram 2.15.

$$\begin{array}{ccc} \check{f} \in \mathcal{H}_2(\mathbb{D},\mathcal{K}) & \xleftarrow{\Lambda} & \tilde{f} \in \ell_{\mathcal{K}} \\ {\scriptstyle \Lambda L \mathcal{L}^{-1}}\uparrow & & \uparrow{\scriptstyle L} \\ \hat{f} \in \mathcal{H}_2 & \xleftarrow[\mathcal{L}]{} & f \in \mathcal{L}_2 \end{array} \tag{2.15}$$

Throughout this thesis we reserve $\mathcal{H}_2$ for system norms. To point out domains for signal f and system T we use accents table 2.1 and the corresponding variables though often omitted. However, for usual discrete-time we use $\hat{f}(e^{j\theta})$ and $f(k)$ as in the continuous case.

Domain	Frequency	Time
Discrete	$\check{f}(\lambda)$ $\check{T}(e^{j\theta})$	$\tilde{f}(k)$ $\tilde{T}$
Continuous	$\hat{f}(\omega)$ $\hat{T}$	$f(t)$ T

Table 2.1: Sampled-data Notation

2.5 System Spaces

Let $\mathcal{H}_\infty$ be a subspace of $\mathcal{L}_\infty$, (essentially) bounded on the imaginary axis, with functions being analytic and bounded in the open RHP (Right-Half Plane), with norm

$$\|\hat{T}\|_\infty \triangleq \sup_{Re(s)>0} \bar{\sigma}[\hat{T}(s)] \tag{2.16}$$

where $\bar{\sigma}$ is the largest singular value. $\mathcal{RH}_\infty$ is the real rational subspace of $\mathcal{H}_\infty$ and is simply all proper and real rational stable transfer matrices, in that case we have by the maximum modulus theorem,

$$\|\hat{T}\|_\infty = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[\hat{T}(j\omega)] \tag{2.17}$$

this also holds for functions in $\mathcal{H}_\infty$ given an identification of a boundary function in $\mathcal{L}_\infty$ (see e.g. [Con95]).

Further introducing $\mathcal{A}_\mathbb{R}$ as a subspace of $\mathcal{H}_\infty$ which is continuous in the closure of the RHP and real in the sense of $T(s)^* = T(s^*)^T$ in the closed RHP (in the matrix case * denotes transpose and complex conjugate). Then $\mathcal{RH}_\infty$ is a subspace of $\mathcal{A}_\mathbb{R}$, which also is the closure of $\mathcal{RH}_\infty$.

Likewise may be done for (usual) discrete-time (we use the same notation) and for "lifted" discrete-time the normed algebra $\mathcal{H}_\infty(\mathbb{D},\mathfrak{L}(\mathcal{K}))$ with functions being analytic and bounded in the unit open disc, with norm

$$\|\check{T}\|_\infty \triangleq \sup_{|\lambda|<1} \|\check{T}(\lambda)\|_{\mathcal{K}\to\mathcal{K}} \tag{2.18}$$

The space of causal LTI operators on $\ell_\mathcal{K}$ is isomorphic to $\mathcal{H}_\infty(\mathbb{D},\mathfrak{L}(\mathcal{K}))$. Again a subspace which is continuous in the closure of the disc is introduced and is denoted $\mathcal{A}(\mathbb{D},\mathfrak{L}(\mathcal{K}))$ or just $\mathcal{A}$ with norm

$$\|\check{T}\|_\infty \triangleq \sup_{\theta\in[0;2\pi[} \|\check{T}(e^{j\theta})\|_{\mathcal{K}\to\mathcal{K}} \tag{2.19}$$

The system spaces are related as shown in the commuting diagram 2.20,

$$\begin{array}{ccc}
\mathcal{A} \subset \mathcal{H}_\infty(\mathbb{D},\mathfrak{L}(\mathcal{K})) & \xleftarrow{\tilde{T}\mapsto\check{T}} & \mathfrak{L}_\mathcal{A}(\ell_\mathcal{K}) \subset \mathfrak{L}(\ell_\mathcal{K}) \\
\Big\uparrow & & \Big\uparrow {\scriptstyle T\mapsto\tilde{T}} \\
\mathcal{A}_\mathbb{R} \subset \mathcal{H}_\infty & \xrightarrow[\hat{T}\mapsto T]{} & \mathfrak{L}_{\mathcal{A}_\mathbb{R}} \subset \mathfrak{L}(\mathcal{L}_2)
\end{array} \tag{2.20}$$

If the operator is known in one system space it can be mapped to another using the relations appearent from the commuting diagram 2.21, e.g. the lifted system $\tilde{T} = LTL^{-1}$,

$$\begin{array}{ccc}
\mathcal{H}_2(\mathbb{D},\mathcal{K}) & \xrightarrow{\check{T}} & \mathcal{H}_2(\mathbb{D},\mathcal{K}) \\
\Lambda^{-1}\downarrow & & \uparrow\Lambda \\
\ell_{\mathcal{K}} & \xrightarrow{\tilde{T}} & \ell_{\mathcal{K}} \\
L^{-1}\downarrow & & \uparrow L \\
\mathcal{L}_2 & \xrightarrow{T} & \mathcal{L}_2 \\
\mathcal{L}\downarrow & & \uparrow\mathcal{L}^{-1} \\
\mathcal{H}_2 & \xrightarrow{\hat{T}} & \mathcal{H}_2
\end{array} \tag{2.21}$$

and one can show,

$$\|T\|_{\mathcal{L}_2\to\mathcal{L}_2} = \|\tilde{T}\|_{\ell_{\mathcal{K}}\to\ell_{\mathcal{K}}} = \|\check{T}\|_{\mathcal{H}_2(\mathbb{D},\mathcal{K})\to\mathcal{H}_2(\mathbb{D},\mathcal{K})} = \|\check{T}\|_{\infty} \tag{2.22}$$

Let $\mathfrak{L}_{\mathcal{A}}(\ell_{\mathcal{K}})$ be the operators on $\ell_{\mathcal{K}}$ with transfer functions in $\mathcal{A}$ and isomorphic to $\mathfrak{L}_{\mathcal{A}}(\mathcal{L}_2)$ given by

$$\mathfrak{L}_{\mathcal{A}} \triangleq \{T \in \mathfrak{L}(\mathcal{L}_2) : \exists \check{T} \in \mathcal{A}, T = L^{-1}\Lambda^{-1}\theta_{\check{T}}\Lambda L\} \tag{2.23}$$

These elements are h-periodic i.e. $D_hT = TD_h, \forall T \in \mathfrak{L}_{\mathcal{A}}$. Note that not all causal h-periodic operators on $\mathcal{L}_2$ are in $\mathfrak{L}_{\mathcal{A}}(\mathcal{L}_2)$.

2.6 Lumped Linear Systems

Linear Time Invariant dynamical systems lumped into Finite Dimensions (FDLTI) can be modelled by linear constant coefficient differential/difference equations. For in depth studies of linear systems we refer to [Kai80, Vid85].

2.6.1 FDLTI Representations

For continuous time FDLTI (Finite Dimensional Linear Time Invariant) systems we use the following representations given below in summary, first state-space

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = 0 \tag{2.24}$$

$$y(t) = Cx(t) + Du(t) \tag{2.25}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^p$ is the system output. If $m = p = 1$ the system is SISO otherwise it is MIMO.

Let $\hat{u}(s), \hat{y}(s)$ be the Laplace transforms of $u(t), y(t)$ then the corresponding TFM (Transfer Function Matrix), $\hat{G}(s)$, is defined by $\hat{y}(s) = \hat{G}(s)\hat{u}(s)$ which can be computed easily as

$$\hat{G}(s) = C(sI - A)^{-1}B + D =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.26}$$

the latter notation makes TFM calculations handy and the state-space form is still explicit.

The impulse response is given by

$$G(t) = \mathcal{L}^{-1}(\hat{G}(s)) = Ce^{At}B1(t) + D\delta(t) \tag{2.27}$$

where $1(t)$ is the unit step (Heavisides function) and $\delta(t)$ is the unit impulse (Dirac's delta function). The convolution equation is given by

$$y(t) = (G * u)(t) \triangleq \int_{-\infty}^{t} G(t - \tau)u(\tau)d\tau \tag{2.28}$$

For discrete time FDLTI systems we use the following similar representations

$$x(k+1) = Ax(k) + Bu(k), \quad x(k_0) = 0 \tag{2.29}$$

$$y(k) = Cx(k) + Du(k) \tag{2.30}$$

Here $\hat{u}(\lambda), \hat{y}(\lambda)$ is the Λ-transforms of $u(k), y(k)$ just like the common z-transform with $\lambda = z^{-1}$, however, with the advantage that FIR (Finite Impulse Response) signals (polynomials) are in $\mathcal{RH}_\infty$.

$$\text{TFM:} \qquad \hat{G}(\lambda) = \frac{\hat{y}(\lambda)}{\hat{u}(\lambda)} = C\lambda(I - \lambda A)^{-1}B + D =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.31}$$

$$\delta \text{ response:} \qquad G(k) = CA^{k-1}B1(k) + D\delta(k) = (D, CB, CAB, CA^2B, \ldots) \tag{2.32}$$

$$\text{Convolution equation:} \qquad y(k) = (G * u)(k) \triangleq \sum_{l=0}^{k} G(k - l)u(l) \tag{2.33}$$

$$\text{System matrix:} \qquad \begin{bmatrix} y(0) \\ y(1) \\ \vdots \end{bmatrix} = \begin{bmatrix} g(0) & 0 & \ldots \\ g(1) & g(0) & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \end{bmatrix} \tag{2.34}$$

which is a Toeplitz matrix.

We will assume representations of G are minimal so that stability is depending only on the eigenvalues of A being in open left half plan or inside the unit disk. If so we write $G \in \mathcal{RH}_\infty$.

2.7 The Notion of Smallest Gain

In accordance with the induced norm, the (largest) gain, the smallest gain of a transfer matrix $\hat{M}$ is defined by $\inf_{\|x\|=1}\|Mx\|$, however, this is not a norm.

The $\mathcal{L}_2 \to \mathcal{L}_2$ induced norm i.e. $\mathcal{H}_\infty$ the largest matrix gain over frequency is accompanied by smallest gain over frequency i.e. $\inf_{\omega\in\mathbb{R}} \underline{\sigma}\big(\hat{M}(j\omega)\big)$. $\overline{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ denotes the maximal and minimal singular value. Note that the minimal singular value, $\underline{\sigma}(\cdot)$, of a square matrix is the distance to a singular one. See [GvL89] for more relations.

We will need the notion of smallest gain over a frequency range Ω given by

$$\Omega = ([\omega_l^1, \omega_u^1], \dots, [\omega_l^n, \omega_u^n])$$

which is defined analogous by

$$\|M\|_\Omega := \inf_{\omega\in\Omega} \underline{\sigma}\big(\hat{M}(j\omega)\big) \tag{2.35}$$

This motivates a signal norm which is the restriction of the $\mathcal{H}_2$ signal norm to Ω:

$$\|f\|_\Omega^2 := \frac{1}{2\pi}\int_{-\Omega,\Omega} \|\hat{f}(j\omega)\|_2^2 d\omega \tag{2.36}$$

we have the following relation:

$$\|Mf\|_\Omega^2 = \frac{1}{2\pi}\int_{-\Omega,\Omega} \|\hat{M}(j\omega)\hat{f}(j\omega)\|_2^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\Omega,\Omega} \|\hat{M}(j\omega)\frac{\hat{f}(j\omega)}{\|f\|_2}\|_2^2\|f\|_2^2 d\omega \geq \|M\|_\Omega^2\|f\|_\Omega^2$$

2.7.1 Preliminary Results

In this section we give some preliminary results used in chapter 8. We assume that the system norm is an induced norm and relate the accompanying smallest gain to a matching problem.

First we take a look at the $\mathcal{H}_\infty$ system norm and let $\|\cdot\|$ for a moment denote the Euclidean norm.

Lemma 2.1 *Given a transfer function $M \in \mathcal{RH}_\infty$ s.t.*

$$\|(I-\hat{M})\|_\infty \leq \alpha \leq 1$$

then

$$\inf_{\omega\in\mathbb{R}} \underline{\sigma}\big(\hat{M}(j\omega)\big) \geq 1-\alpha$$

Proof. First note that

$$\underline{\sigma}(Q) = \underline{\sigma}(I - (I - Q)) := \min_{\|x\|=1} \|Ix - (I - Q)x\|$$
$$\geq \min_{\|x\|=1} (\|Ix\| - \|(I - Q)x\|) \geq 1 - \max_{\|x\|=1} \|(I - Q)x\| = 1 - \overline{\sigma}((I - Q))$$

Hence $\underline{\sigma}\big(\hat{M}(j\omega)\big) \geq 1 - \overline{\sigma}\big((I - \hat{M})(j\omega)\big)$ for all ω then by the assumption $\underline{\sigma}\big(\hat{M}(j\omega)\big) \geq 1 - \alpha$ for all ω then taking infimum over $\omega \in \mathbb{R}$ on both sides gives the case. ■

This is the case for any induced norm of which above lemma is special case. As usual the overload of $\|\cdot\|$ will be clear from the context.

Lemma 2.2 *Given a transfer function M in an induced norm space s.t.*

$$\|(I - M)\| := \sup_{\|x\|=1} \|(I - M)x\| \leq \alpha \leq 1$$

then

$$\inf_{\|x\|=1} \|Mx\| \geq 1 - \alpha$$

Proof.

$$\inf_{\|x\|=1} \|Mx\| = \inf_{\|x\|=1} \|Ix - (I - M)x\| \geq \inf_{\|x\|=1} (\|Ix\| - \|(I - M)x\|) \geq$$
$$1 - \sup_{\|x\|=1} \|(I - M)x\| \geq 1 - \alpha$$

■

Lemma 2.3 *Given transfer functions W, M in an induced norm space s.t.*

$$\inf_{\|x\|=1} \|Wx\| = \delta, \ \|(W - M)\| \leq \alpha \leq \delta$$

then

$$\inf_{\|x\|=1} \|Mx\| \geq \delta - \alpha$$

Further, we need the following lemma, which is a modification of Lemma 2.3.

Lemma 2.4 *Define the frequency range* Ω *by*

$$\Omega = \left([\omega_l^1, \omega_u^1], \ldots, [\omega_l^n, \omega_u^n]\right)$$

Given transfer functions W *and* M *s.t.*

$$\inf_{\omega \in \Omega} \underline{\sigma}\big(\hat{W}(j\omega)\big) = \delta$$

$$\|(\hat{W} - \hat{M})\|_\infty = \alpha$$

then

$$\underline{\sigma}\big(\hat{M}(j\omega)\big) \geq \delta - \alpha \quad \forall \omega \in \Omega$$

Proof.

$$\begin{aligned}\underline{\sigma}\big(\hat{M}(j\omega)\big) &\geq \underline{\sigma}\big(\hat{W}(j\omega)\big) - \overline{\sigma}\big((\hat{W} - \hat{M})(j\omega)\big) \\ &\geq \underline{\sigma}\big(\hat{W}(j\omega)\big) - \alpha \quad \forall \omega \in \Omega\end{aligned}$$

■

2.8 Notes and References

For computation of norms in state-space we refer to [DDB95, ZDG95].

Chapter 3

The Robust Control Framework

This chapter like the previous serves to introduce notation, main concepts and to give references of which the main ones are [ZDG95, DDB95, SP96] and [Doy82, Saf82] for the origination of exploiting the structured uncertainty.

The idea of the framework is to cope with the "engineering" tradeoff between:

- a "simple" model which opens the door for powerful analysis/design methods and
- a "complex" model making the mismatch between the real system and the model small.

It is obtained by using a nominal plant and an uncertainty set to form a class of systems. The nominal system is a "simple" FDLTI model and the uncertainty set is a "complex" model based on perturbations (LTI, LTV or NL) and FDLTI weigths. In other words the intention is that the real system belongs to the class of systems formed and due to the partitioning of the model analysis/design is tractable.

The nominal system may be found via first principles together with suitable approximations or by "blackbox" system identification as well as combinations thereof.

The origin of uncertainty may be seen as either parametric like tolerances on measured, quoted standard and estimated parameters or unmodelled/neglected dynamics. The uncertainty may amongst others be modelled by the multiplicative model, the additive model, the coprime factor model or direct injection of disturbance signals in this framework. These models or their combinations are found essentially the same way as the nominal model though with some extra work/knowledge needed.

Though the parametric uncertainty calls for real-valued perturbations they are often modelled by complex ones for mathematical convenience.

Most considerations/results in linear robust and optimal control can be given in this framework (for reference these are often implicitly included in the term). This is also the case for the control and filtering problems dealt with in this thesis. Since the notation for both continuous and discrete-time is analogous we will quote for one time set and assume the other is similar.

3.1 Perturbations

The structured uncertainty models we consider are with respect to perturbations in $\mathfrak{L}(\mathcal{L}_2^n)$ assumed to have spatial structure [PD93] of block diagonal form

$$\Delta_s \triangleq \{\Delta = \operatorname{diag}(\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}, \Delta_{S+1}, \ldots, \Delta_{S+F}) : \delta_i \in \mathbb{C}, \Delta_k \in \mathfrak{L}(\mathcal{L}_2^{m_k})\} \tag{3.1}$$

which must be compatible hence $\sum_1^S r_i + \sum_1^F m_k = n$. If we have only a full block we call it unstructured.

This set we also denote Δ_{LTV} as it is the set of causal structured LTV (Linear Time-Varying) perturbations. Non-Linear (NL) perturbations often gives results which resembles the LTV case. The uncertainty may also be restricted to be LTI

$$\Delta_{LTI} \triangleq \{\Delta \in \Delta_{LTV} : D_T\Delta = \Delta D_T, \forall T \in \mathbb{R}_0\} \tag{3.2}$$

Further we introduce SLTV (Slowly Linear Time-Varying) perturbations with rate $\nu > 0$

$$\Delta_\nu \triangleq \{\Delta \in \Delta_{LTV} : \sup_{T>0} \frac{\|D_T\Delta - \Delta D_T\|}{T} \leq \nu\} \tag{3.3}$$

If we simply mean any class we use Δ_a. The unit balls of uncertainty w.r.t. a norm for each class are denoted $B\Delta_{LTV}$, $B\Delta_{LTI}$ etc.

3.2 The Standard Problem Setup

In fig. 3.1 we have the standard setup which can handle a variety of uncertainty models and their combinations for both analysis and synthesis problems in an unified manner. In the setup the uncertainty is arranged so that $\Delta \in B\Delta_a$, K is the (FDLTI) controller and G is the (FDLTI) generalised 3×3 plant i.e. the known remainder of e.g. process(es), sensors/actuators and weights. The generalised plant may be found by using linear N-port theory on a block diagram.

Further $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the measurement output, $w(t) \in \mathbb{R}^k$ is the external input and $z(t) \in \mathbb{R}^l$ is the controlled output. (The system state is as usual denoted $x(t) \in \mathbb{R}^{n_x}$.)

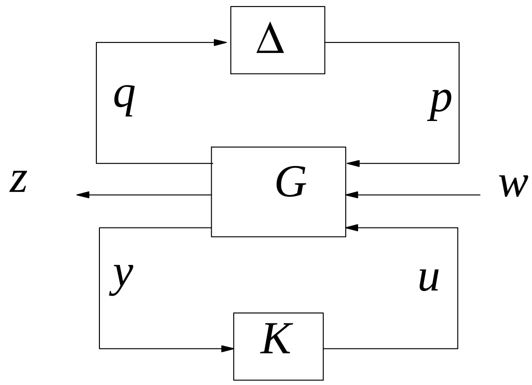


Figure 3.1: Standard Problem

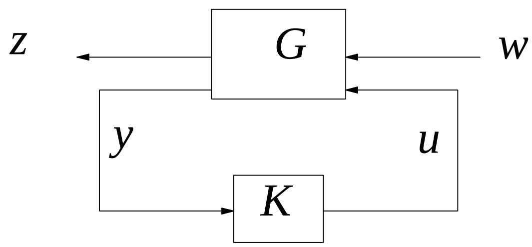


Figure 3.2: Controller Problem

Instead of the difficult synthesis problem the setup is often broken into the controller problem and the analysis problem. Partitioning G into a 2×2 system by absorbing q, p into w, z the system can be given in state-space by

$$\begin{array}{rcrcrcr} \dot{x} & = & Ax & + & B_1 w & + & B_2 u \\ z & = & C_1 x & + & D_{11} w & + & D_{12} u \\ y & = & C_2 x & + & D_{21} w & + & D_{22} u \end{array} \tag{3.4}$$

or in compact notation (as TFM)

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \tag{3.5}$$

By appropriate partitioning then the closed-loop controller problem in fig. 3.2 can be written as a lower LFT (Linear Fractional Transformation) denoted M

$$M = G \star K \triangleq G_{11} + G_{12} K (I - G_{22} K)^{-1} G_{21} \tag{3.6}$$

When the controller is given we have the analysis problem in fig. 3.3 where M defined as above the 2×2 system is assumed stable. The closed-loop denoted T_{zw} can be written as an upper LFT

$$T_{zw} = \Delta \star M \triangleq M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \tag{3.7}$$

Note that the interpretation of $\star$ depends on the position of the 2×2 block and $\star$ is given lower precedence than multiplication. The $\star$ is for the Redheffer Star-product and is more generally used for interconnection of LFTs which again gives another LFT.

Assuming proper partitioning of G (two cases) the synthesis map may be given as

$$T_{zw} = \Delta \star (G \star K) = (\Delta \star G) \star K \tag{3.8}$$

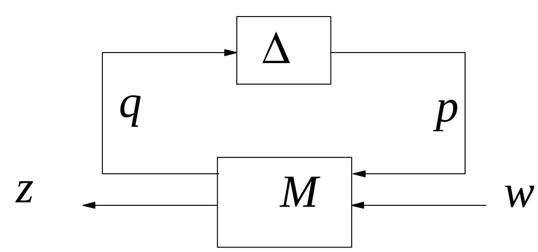


Figure 3.3: Analysis (RP) Problem

Performance specifications are given via choice of controlled output/external input and selection of norm in which the map from w to z is minimised in $\mathcal{H}_\infty$, $\mathcal{H}_2$ or ℓ_1(not continuous-time), without uncertainty these optimal control problems are fully solved [ZDG95, DDB95].

3.3 Robust Stability and Performance

Control problems, given a nominal model G in the uncertainty model set and some performance specifications resulting in a system with controller K, are in closed-loop classified as having:

- NS (Nominal Stability) if K stabilises the nominal model.
- NP (Nominal Performance) if the closed-loop satisfies the performance specifications and NS.
- RS (Robust Stability) if K stabilizes all plants in the uncertainty set.

- RP (Robust Performance) if RS and the performance specifications are satisfied for all plants in the uncertainty set.

The first two properties may be checked using “classical/modern control theory”, however, gain margin and phase margin may not be sufficient for robustness.

Hence in a sence the latter two properties are what robust (optimal) control is all about. Next we give more detailed definitions of those

Definition 3.1 *RS. Assume that* $M \in \mathfrak{L}(\mathcal{L}_p)$. *Then the uncertain system in fig. 3.3 has robust stability w.r.t. perturbations in* $B\Delta_a$ *if* $I - \Delta M_{11}$ *has a bounded inverse* $\forall \Delta \in B\Delta_a \in \mathfrak{L}(\mathcal{L}_p)$

Definition 3.2 *RP. The uncertain system in fig. 3.3 has robust* ($\mathcal{L}_p \to \mathcal{L}_p$) *performance if it has robust stability and*

$$\sup_{\Delta \in B\Delta_a} \|\Delta \star M\|_{\mathcal{L}_p \to \mathcal{L}_p} < 1 \tag{3.9}$$

When $p = 2$ we call it robust $\mathcal{H}_\infty$ performance (although this is only correct for the LTI case) and likewise when $p = \infty$ it is robust ℓ_1 performance.

3.4 Robustness Analysis

Robustness analysis is mainly based on and initiated by the small gain theorem [Zam66]. *Assume* M, Δ *stable in fig. 3.4 (possible NL) and the feedback interconnection is wellposed (i.e. for any injected signals* d_1, d_2 *there exist unique solutions* p, q*). Then the feedback interconnection is stable if* $\|M\|\|\Delta\| < 1$.

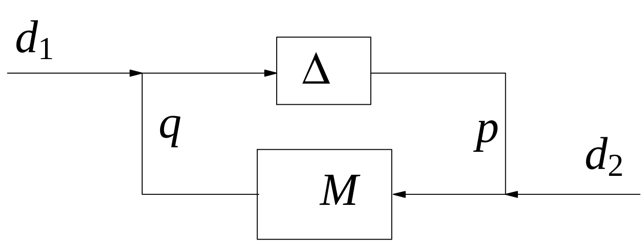


Figure 3.4: Feedback

For full block perturbations various conditions for RS may be found via the small gain theorem, however, these may be arbitrary conservative in the case of structured perturbations. Note that for RS it is only the M_{11} block of M (3.6) which matters so the index is omitted, however, it is clear from the interconnection which part is meant.

Next follows a recaption of some conditions for robust stability which aims at exploiting the structure to give non conservative results. First we define the structured singular value μ [PD93] at each frequency (a complex matrix) for a stable system M

$$\mu(\hat{M}(j\omega)) \triangleq \frac{1}{\inf\{\overline{\sigma}(\hat{\Delta}(j\omega)) : \Delta \in \Delta_{LTI|_{\mathcal{RH}_\infty}}, \det(I-\hat{M}(j\omega)\hat{\Delta}(j\omega)) = 0\}} \tag{3.10}$$

where the restriction of Δ_{LTI} is to the stable ones. A main result is that M is robustly stable for $\Delta \in B\Delta_{LTI|_{\mathcal{RH}_\infty}}$ if and only if $\sup_{\omega\in\mathbb{R}} \mu(\hat{M}(j\omega)) \leq 1$. This is really the core in the construction of μ.

Secondly this condition (without restriction) may be generalised as [DDB95, Dul95]

Definition 3.3 *Given stable system $M \in \mathfrak{L}(\mathcal{B})$.*

$$\mu_{\Delta_a}(M) \triangleq \frac{1}{\inf\{\|\Delta\|_i : \Delta \in \Delta_a, (I-M\Delta)^{-1} \textit{is not } \mathcal{B}\textit{-stable}\}} \tag{3.11}$$

Hence μ depends on the system M, the perturbation set Δ_a and the notion of stability. Again the RS theorem's condition is $\mu_{\Delta_a}(M) \leq 1$ against $\Delta \in B\Delta_a$. Thus it is also called the structured small gain theorem.

As one might imagine the μ condition is in general (non polynomial complexity) hard to compute and is instead approximated by upper and lower bounds. This is well known from the matrix case; the upper bound is only exact if $2S+F \leq 3$; although there may be a gap it turns out fairly moderate for most systems. Here we look only briefly at a upper bound which for some cases of perturbations is both sufficient and necessary.

Introducing commuting and invertible scales defined by

$$D_a \triangleq \{D \in \mathfrak{L}(\mathcal{B}) : D, D^{-1} \text{ stable and } D\Delta = \Delta D \quad \forall \Delta \in \Delta_a\} \tag{3.12}$$

gives an upper bound in form of $\inf_{D\in D_a} \|DMD^{-1}\|$. For the matrix case the scales are

$$D_{LTI} \triangleq \{D \in \mathbb{C} : D = \operatorname{diag}(d_1 I_{r_1}, \ldots, d_F I_{r_F}), 0 < d_i \in \mathbb{R}\} \tag{3.13}$$

and with the condition

$$1 \geq \inf_{\hat{D}\in\hat{D}_{LTI}} \|\hat{D}\hat{M}\hat{D}^{-1}\|_\infty = \sup_{\omega\in\mathbb{R}} \inf_{D\in D_{LTI}} \overline{\sigma}\big(D\hat{M}(j\omega)D^{-1}\big) \geq \sup_{\omega\in\mathbb{R}} \mu(\hat{M}(j\omega))$$

we have a bound with frequency dependent scales given as a $\mathcal{H}_\infty$-problem.

3.4.1 Robust Stability with SISO Blocks

Here we specialise to the cases of ℓ_2 and ℓ_∞ stability to give an overview [DK93, Sha94, PT95] and for simplicity we assume the perturbations to be F SISO blocks.

With M partitioned accordingly we may then define

$$\widehat{M} = \begin{bmatrix} \|M_{11}\|_1 & \dots & \|M_{1F}\|_1 \\ \vdots & & \vdots \\ \|M_{F1}\|_1 & \dots & \|M_{FF}\|_1 \end{bmatrix} \tag{3.14}$$

which is instrumental in the ℓ_∞ stability case. One may show the following inequalities for a stable M with $\widehat{M}$ positive

$$\begin{aligned} \mu(\hat{M}(e^{j\theta})) &\leq \inf_{\hat{D}\in\hat{D}_{LTI}} \|\hat{D}\hat{M}\hat{D}^{-1}\|_\infty \leq \inf_{D\in D_{LTI}} \|D\hat{M}D^{-1}\|_\infty \\ &\leq \inf_{\hat{D}\in D_{LTI}} \bar{\sigma}(D\widehat{M}D^{-1}) = \rho(\widehat{M}) \end{aligned} \tag{3.15}$$

The conditions for RS with SISO blocks are quoted in table 3.4.1. The extentions for MIMO blocks are also well known, see references.

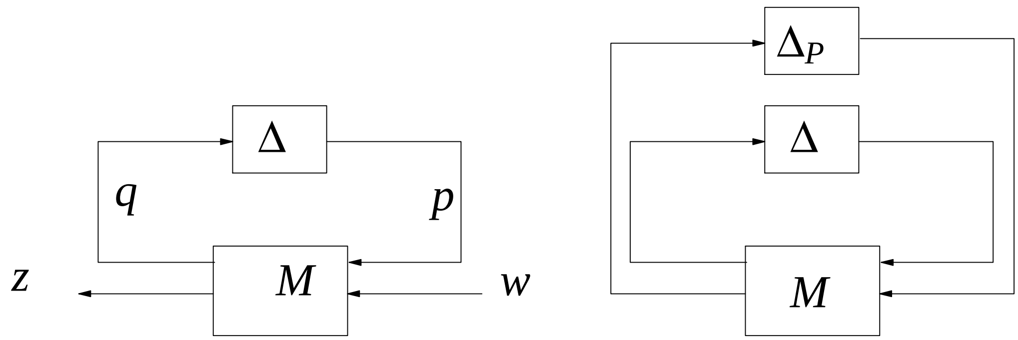
Condition (≤ 1) for RS with perturbations in								
	$\mu(\hat{M}(e^{j\theta})),\ \forall\theta$		$\inf_{\hat{D}\in\hat{D}_{LTI}} \|\hat{D}\hat{M}\hat{D}^{-1}\|_\infty$		$\inf_{D\in D_{LTI}} \|D\hat{M}D^{-1}\|_\infty$		$\rho(\widehat{M})$	
$B\Delta$	$\ell_\infty \to \ell_\infty$	$\ell_2 \to \ell_2$	$\ell_\infty \to \ell_\infty$	$\ell_2 \to \ell_2$	$\ell_\infty \to \ell_\infty$	$\ell_2 \to \ell_2$	$\ell_\infty \to \ell_\infty$	$\ell_2 \to \ell_2$
NL	**n**	**n**	**n**	**n**	**n**	**n&s**	**n&s**	**s**
LTV	**n**	**n**	**n**	**n**	**n**	**n&s**	**n&s**	**s**
SLTV	**n**	**n**	**n**	**n&s**	**s**	**s**	**s**	**s**
LTI	**n&s**	**n&s**	**s**	**s**	**s**	**s**	**s**	**s**

Table 3.1: RS SISO Blocks, **n** = necessary and **s** = sufficient

3.4.2 Robust Performance Stated as Robust Stability

RP problems can be stated as RS problems with an extra fictitious perturbation, see fig. 3.5.

The fictitious perturbation must be chosen as shown is table 3.2 to have equivalence, the sufficient part is by the the small gain theorem which we requote under



(a) RP

(b) RS with Δ_P

Figure 3.5: RP Stated as RS problem

the assumption that $M \in \mathcal{RH}_\infty$ then the closed-loop in fig. 3.4 is stable and well-posed in the cases of

ℓ_2) $\Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty < 1$ iff $\|M\|_\infty \leq 1$;

ℓ_∞) $\Delta : \ell_\infty \to \ell_\infty, \|\Delta\|_{\ell_\infty \to \ell_\infty} < 1$ iff $\|M\|_1 \leq 1$.

	Fictitious Perturbation Δ_P	
Plant Δ	$\ell_\infty \to \ell_\infty$-perf.	$\ell_2 \to \ell_2$-perf.
NL	NL	NL
LTV	LTV	LTV
SLTV	LTV	SLTV
LTI	LTV	LTI

Table 3.2: Fictitious Perturbation

The robust $\mathcal{H}_\infty$ performance is a μ problem (this explains the restriction to $\Delta_{LTI}|_{\mathcal{RH}_\infty}$). However, in cases with mixed perturbations the problems are difficult and still open.

3.5 Robust $\mathcal{H}_2$ Performance

The main contribution in [Pag96c] is the solution of the robust $\mathcal{H}_2$ performance problem, where the performance is specified in $\mathcal{H}_2$ which is often the more appropriate and the uncertainty in $\mathcal{H}_\infty$ (bounded but unknown).

In other words it solves: guaranteeing margins for LQG regulators or white noise rejection (a.k.a. $\mathcal{H}_2$) problem contrary nominal LQG [Doy78].

This paper initiated a still active line of work in robust $\mathcal{H}_2$: first by LTR, see section 3.7, and later $\mathcal{H}_2/\mathcal{H}_\infty$ [BH89, ZGBD94, DZGB94, KR91] and robust $\mathcal{H}_2$ performance [Pag96c, Pag96a, Fer97, Sto93] . This interest was also triggered by the result of finding the $\mathcal{H}_\infty$ solution [DGKF89] efficiently by Riccati equations in state-space.

Definition 3.4 *The uncertain system in fig. 3.3 has robust $\mathcal{H}_2$ performance if it has robust stability and*

$$\sup_{\Delta \in B\Delta} \|\Delta \star M\|_{\mathcal{H}_2} < 1. \tag{3.16}$$

Note that for the non-LTI cases a generalisation of $\mathcal{H}_2$ is needed see the formulation in [Pag96c], see 4.4.1. The solution is based on below condition 1 which is related to μ, (3.17) is the μ (scaled small gain) upper bound, the other part is a tradeoff/average over frequency i.e. $\mathcal{H}_2$-norm.

Denote by $\mathbb{X}$ the set of positive definite, continuous scaling functions $X(\omega) \in D_{LTI}$ with the above given structure (3.13).

Condition 1 *There exists $X(\omega) \in \mathbb{X}$, and a matrix function $Y(\omega) = Y^*(-\omega) \in \mathbb{C}^{m\times m}$, such that*

$$M(j\omega)^* \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} < 0 \tag{3.17}$$

holds for all $\omega \in \mathbb{R}$ and[1]

$$\int_{-\infty}^{\infty} \mathrm{trace}(Y(\omega)) \frac{d\omega}{2\pi} < 1. \tag{3.18}$$

Moreover the dependency on perturbation class is parallel to the scaled upper bound, see table 3.3 [2], which is also the case for the computational complexity [Pag96d].

Finally, note that robustness synthesis is feasible for the LTV and SLTV cases. But for the LTI case only the so-called D-K iteration is available which is ad hoc and has no guarantee to give a global optimal solution.

[1] if we ask for performance less than γ change 1 in (3.18) to γ^2.

[2] Remarks in section 3.6

	Robust performance	
	$\mathcal{H}_2$	$\mathcal{H}_\infty$
Δ	Cond. 1	$\inf_{\hat{D}\in\hat{D}_{LTI}} \|\hat{D}\hat{M}\hat{D}^{-1}\|_\infty$
LTV, constant scales	**n&s**	**n&s**
SLTV	**n&s**	**n&s**
LTI	**s**	**s**

Table 3.3: Robust performance, **n** = necessary and **s** = sufficient

3.6 Robust $\mathcal{H}_2$ Performance Revisited

We suggest defering this section until after reading section 4.4.1

It is claimed in [ST98][3] by a (counter)example that condition 1 is only sufficient, but for the SISO case. However, the example is for the LTI case where condition 1 is not necessary, see table 3.3 [Pag96a]. Though the μ upper bound is exact for $2S+F \leq 3$ the example demonstrates that this is not the case for condition 1[4].

A lesson from the example is that one can construct (LTI) perturbations so that $\|\Delta\star M\|_{W_{\eta,B}} \neq \|\Delta\star M\|_{\mathcal{H}_2}$.

In this work we only use the set-based approach for the LTV case in section 6.1.3. We proposed in [RP97] an averaging kind of generalisation (4.16) which is less conservative than the set-based one in (4.17); in other words the two norms are different and the example shows that this all the more also goes for the LTI case. To make the point fully clear (4.16) simplifies back correspondingly in the uncertain LTI case to the usual $\mathcal{H}_2$-norm, whereas (4.17) does not.

Nevertheless, the set-based approach is a sound way of modelling white noise[5], moreover, it provides necessary and sufficient conditions, table 3.3. This is at the price of being more conservative than some other $\mathcal{H}_2$ generalisations.

As argued in [ST98] this leaves the worst case robust $\mathcal{H}_2$ performance problem open in a sense. However, in [SAT][6] is given a nice proof for sufficiency and necessity for a slightly different condition for the MISO case. Besides, it is shown that the gap of condition 1 in the LTI case can be square root of k the number of external inputs.

[3]Obtained 1 of August, hence only this revisited section

[4]The authors of [ST98] say that condition 1 was claimed exact whenever the μ upper bound is so - (but this must be a misunderstanding; anyhow we leave the minor dispute here)

[5]That the causality of LTI perturbations may give a gap was pointet out in [Pag96c]

[6]This personal draft was kindly provided by M. Sznaier at the ACC'98

However, the idea in lemma 6.3 may be applied to show a proposition like 6.3 on page 62 based on [SAT]. Proposition 6.3 is still fully valid, albeit somewhat conservative.

Finally, we suggest that a further study includes a comparison between the various $\mathcal{H}_2$ generalisations and likewise robust $\mathcal{H}_2$ performance conditions, a first step may be found [PF97].

3.7 Loop Transfer Recovery

The procedure LTR (Loop Transfer Recovery) was originally introduced in [DS79, DS81]. Since then many papers with this topic have been published. The majority of these papers have been cited in the reference list of [SCS93].

The two stepped LTR procedure seeks to recover the closed-loop formed with a state-feedback controller by a dynamical (observer based) controller. This is in order to inherit the closed-loop robustness and performance properties from the state-feedback which may be designed by any method meeting the specifications; this is the first step. Likewise the second (LTR) step may be handled by any (optimisation) design method.

3.7.1 An Overview of LTR Design

Consider the system, in fig. 3.2, with state-space description

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{3.19}$$

It is assumed that (A, B_2) is stabilisable and that (C_2, A) is detectable. Suppose the LTR design methodology is applied at the input loop breaking point. We first design a target feedback loop with the static state feedback gain F, for the system described by:

$$\hat{G}_{SF}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right] \tag{3.20}$$

such that the design specifications are satisfied. It is assumed that the state feedback loop is asymptotically stable, i.e. all the eigenvalues of $A_F = A + B_2 F$ lie in the left half plane. The target loop transfer function is then given by:

$$\hat{T}_{zw,T}(s) = (C_1 + D_{12}F)(sI - A_F)^{-1}B_1 + D_{11} \tag{3.21}$$

which satisfies the closed-loop design specifications for the transfer function from w to z. Now, let the plant be controlled by a full-order observer based controller given by:

$$\hat{K}(s) = -F(sI - A - B_2F - LC_2)^{-1}L \tag{3.22}$$

where L is the observer gain. Then, the resulting closed-loop transfer function, in general, is not the same as the target loop transfer function $\hat{T}_{zw,T}(s)$. In the LTR step the observer based controller is designed so as to recover either exactly (perfectly) or asymptotically (approximately) the target loop transfer function.

For a more careful analysis, we define the closed-loop transfer recovery error as

$$\hat{E}_{cl}(s) = \hat{T}_{zw}(s) - \hat{T}_{zw,T}(s) \tag{3.23}$$

where $\hat{T}_{zw}$ is the closed-loop transfer function from w to z when a full order observer is applied. The closed-loop recovery error is related to the so-called recovery matrix $\hat{M}_I(s)$ given in [NSAS91] by the equation

$$\hat{E}_{cl}(s) = \hat{T}_{zu,T}(s)\hat{M}_I(s) \tag{3.24}$$

where $\hat{T}_{zu,T}(s)$ is the closed-loop transfer function from u to z under the target design given by:

$$\hat{T}_{zu,T}(s) = (C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_2 + D_{12} \tag{3.25}$$

and the recovery matrix $\hat{M}_I$ is given by

$$\hat{M}_I(s) = F(sI - A - LC_2)^{-1}(B_1 + LD_{21}) \tag{3.26}$$

We say that exact loop transfer recovery at the input point (ELTRI) is achieved if the closed-loop system comprised of $\hat{K}(s)$ and $\hat{G}(s)$ is asymptotically stable and $\hat{E}_{cl}(s) = 0$ or $\hat{M}_I(s) = 0$ when $\hat{T}_{zu,T}$ is left invertible, they are only equivalent in this case. For obtaining asymptotic LTR at the input point (ALTRI), [DS81], [SA87], we parameterise a family of controllers with a positive scalar q, and say that ALTRI is achieved if the closed-loop system is asymptotically stable and $\hat{E}_{cl}(s,q) \to 0$ pointwise in s as $q \to \infty$.

3.8 The Filtering Problem

The problem was first thoroughly treated under WWII by Weiner [Wei49], his approach on filtering noisy signals was based on minimising the RMS (Root Mean Square) value of the estimate error. The extension thereof is the Kalman filter or LQE (Linear Quadratic Estimator) [KS72]. Here we pose the filtering problem so that it fits in the framework of an optimal control problem in fig. 3.2.

3.8.1 Filtering Problem Stated as Controller Problem

Consider the FDLTI system G given by

$$\begin{array}{rcrcr} \dot{x} & = & Ax & + & B_1 w \\ z & = & C_1 x & + & D_{11} w \\ y & = & C_2 x & + & D_{21} w \end{array} \tag{3.27}$$

Let the filter be $F \in \mathcal{RH}_\infty$ as we want it to be stable and causal.

Problem 3.1 *The filtering problem, see fig. 3.6, is to estimate z using the filtered measurements $\hat{z}$ so that the error $\tilde{z} \triangleq z - \hat{z}$ is small in some sense.*

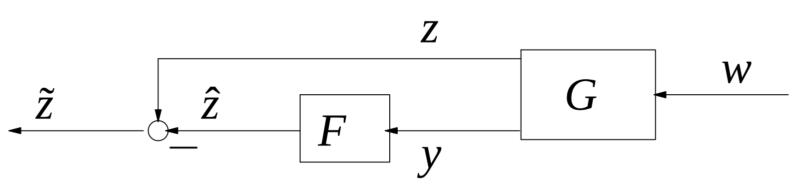


Figure 3.6: Filtering Problem

This can be redrawn as a controller problem, fig. 3.7 where P is seen to be

$$\hat{P}(s) = \left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & -I \\ C_2 & D_{21} & 0 \end{array}\right] \tag{3.28}$$

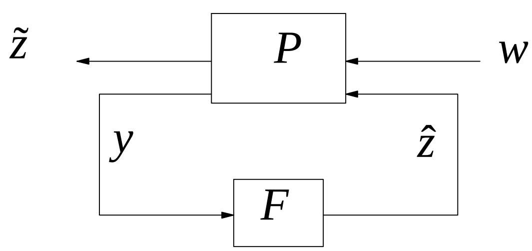


Figure 3.7: Filtering Problem stated as Controller Problem

If we want to minimise the induced $\mathcal{L}_2$-norm

$$\sup_{0 \neq w \in \mathcal{L}_2} \frac{\|\tilde{z}\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}} < \gamma \tag{3.29}$$

problem 3.1 is called $\mathcal{H}_\infty$-filtering and may be solved as a standard $\mathcal{H}_\infty$-problem, however, internal stability is not required [ZDG95]. Likewise formulations may be given for $\mathcal{H}_2$ and ℓ_1 filtering.

3.9 Notes and References

Some people talks about seeing the "big picture", this was a glipse of it.

One should pay attention to the worst-case strategy. This may seem somewhat conservative, however, a design based on stochastic distributions may again be too specialised and hard to handle hence not seizing good overall robustness.

At the same time we point out a few further interesting subjects and points of view by a few references and the ones therein. The following subjects are quite related: Limitations on performance (convex optimisation) [BB91]. Control based on covariance methods and Linear Matrix Inequalities (LMIs) [SIG97, BGFB94]. They also point out that reducing a problem to LMIs can be regarded a solution in sense of tractable computation (polynomial complexity) methods. Much in the spirit of these fixed-order control design is studied in [Ber97].

Mixed and/or multiobjective control such as $\mathcal{H}_2/\mathcal{H}_\infty$ [SGC97] problem refered above and $\ell_1/\mathcal{H}_\infty$ [SB98] and $\mathcal{H}_2/\ell_1$ with more in [ED97].

Fundamental limitations in control are treated in [SBG97]. Not all of these ideas have been used directly in this work, however, these have influenced the work. See also references in section 4.6.

Chapter 4

Sampled-Data Systems

Two comprehensive books (mostly complementing) with different viewpoints on SDS (Sampled-Data Systems), minding the intersample behaviour, are now available, suiteable for an indepth introduction to the matter. [FG96] has digital signal processing as its viewpoint and uses the Fourier transform to relate the different domains and the methods are primarily based on ("modern") discrete-time theory (as in [ÅW84]). Whereas [CF95] has (robust) optimal control as its viewpoint and aims at handling the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems with regard to the intersample behaviour. The latter viewpoint is somewhat closer to the approach taken in this treatise, which also treats uncertain systems as in the monograph [Dul95].

Hence the next sections only contain a brief tour through SD (Sampled-Data) Systems to establish notation and highlight main ideas.

4.1 Setup

We view a sampled-data system in fig. 4.1 as an analog plant (the prefilter is absorbed herein) and a digital controller interconnected by A/D and D/A converters. Thus a sampled-data system is a hybrid system with both a continuous-time and a discrete-time part. In our mathematical model we have

- G a LTI continuous-time generalised plant

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{4.1}$$

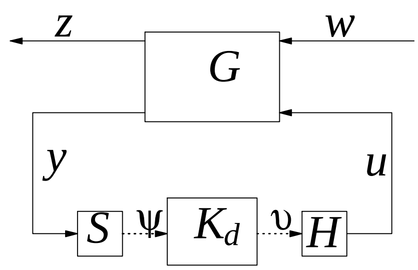


Figure 4.1: Standard digital control system

- K_d a LTI discrete-time controller with measurement input ψ and control output υ

$$\hat{K}_d(\lambda) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{4.2}$$

- S a sampler. A/D converter (prefilter absorbed into G so that $D_{21} = D_{22} = 0$).

- H a (zero order) hold device. D/A converter.

Assumming an ideal sample and hold with a syncronised (sampling) period of h, i.e. no delays, no quantization, modelled as

$$\psi(k) \triangleq y(kh) \tag{4.3}$$
$$u(t) = \upsilon(k) \quad \text{for} \quad kh \leq t < (k+1)h \tag{4.4}$$

Note that the interconnected system $G \star HK_dS$ is h-periodic since $K = HK_dS$ is so, hence the usual TFM approach cannot be applied as such.

We will assume non-pathological sampling (no pairs of eigenvalues with equal real part and imaginary parts differing only by an integer multiple of $\omega_s \triangleq \frac{2\pi}{h}$) in which case controllability and observability are preserved.

4.2 Design Approach

Say we want to solve the controller problem in fig. 4.1 i.e. find a $K = HK_dS$ which assures nominal stability and performance then we have two traditional ways which have been applied succesfully for many systems over the years. These are both indirect and summarised below:

- Find a continuous-time optimal controller, discretise K i.e. find an approximation of the form HK_dS. The simple choice is $K_d = SKH$, but often it is better to use bilinear transformation.
  - \+ Specifications in continuous-time are appropriate and are recovered as sampling becomes faster.
  - \- Sampling rate is (generally) limited.
- Discretise G, find discrete-time controller K_d.
  - \+ Simple, except reformulating specifications.
  - \- Ignores the intersample behaviour.

Both methods exclude a proper robust and optimal control design but with an extra analysis this can be adjusted/checked for, see section A.1. However, the analysis offers only little or no advise in case the specifications are not met.

Opposed to the standard ways a direct approach without approximations may be taken see section 4.3.

Sampling rate is limited due to:

- Only data at certain time instants. Often in chemical and biological processes.
- Slow hardware, e.g. microprocessor or network due to cost or technology.
- Quantization and truncation problems with fast sampling.
- Prespecified.

4.3 Direct Design via Lifting

Recent years research has established a theory for taking the intersample behaviour into account, a main tool is the lifting/raising framework [BPFT91, BJ92] which we briefly introduce next.

We recall the lifting map, see fig. 4.2. Given a signal $u \in \mathcal{L}_2^e$ define $u_k(t) \triangleq u(hk+t)$, $0 \leq t \leq h$ as a sequence with each element being a sampling interval denoted $\tilde{u} \triangleq [\ldots u_{-2}u_{-1}u_0u_1 \ldots]^T$. Then let $L : \mathcal{L}_2^e[0,\infty) \rightarrow \ell_{\mathcal{K}}$ be defined by $\tilde{u} \triangleq Lu$. L is a linear bijective norm preserving map. Algebraic operations are also preserved under lifting i.e. $(\widetilde{G_1+G_2}) = \tilde{G}_1 + \tilde{G}_2$ and $(\widetilde{G^{-1}}) = \tilde{G}^{-1}$ hence feedback stability is preserved.

The idea of lifting outlined below may be seen from the controller problem in fig. 4.3.

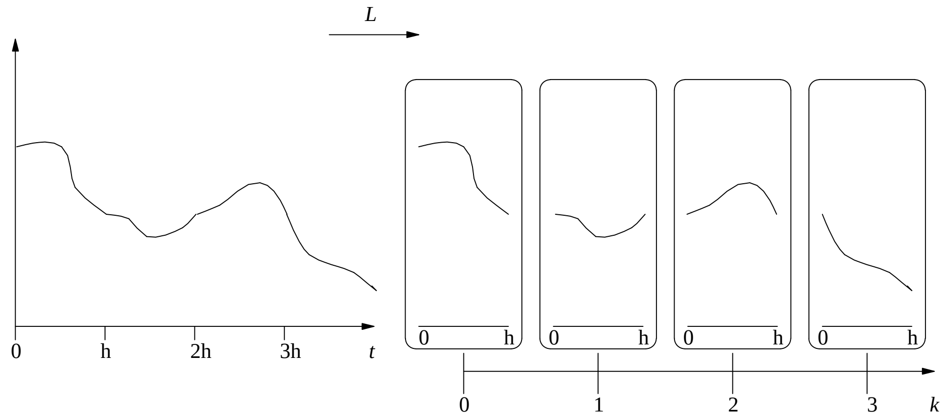


Figure 4.2: Lifting map

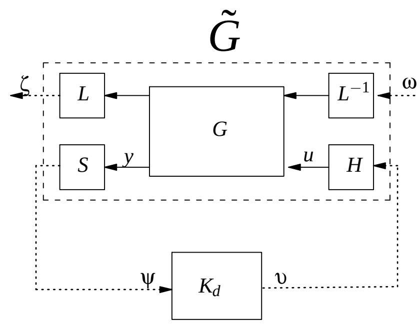


Figure 4.3: Lifted Controller Problem

The lifting map and its inverse are applied to z, w and the obtained map from ω to ζ is LTI,

$$\tilde{G} = \begin{bmatrix} L & 0 \\ 0 & S \end{bmatrix} G \begin{bmatrix} L^{-1} & 0 \\ 0 & H \end{bmatrix} = \begin{bmatrix} LG_{11}L^{-1} & LG_{12}H \\ SG_{21}L^{-1} & SG_{22}H \end{bmatrix} \tag{4.5}$$

and satisfies the following:

1. HK_dS stabilises G if K_d stabilises $\tilde{G}$
2. $\|G \star HK_dS\| = \|\tilde{G} \star K_d\|$

With (A_d, B_{2d}) being the usual discretisation (step invariant) i.e. the matrix representation of $SG_{22}H$ taking the other three blocks in turn we will below in section

4.3.1 show that

$$\tilde{G} = \left[\begin{array}{c|cc} A_d & \tilde{B}_1 & B_{2d} \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ C_2 & 0 & 0 \end{array}\right]$$

where

$$\tilde{B}_1 : \mathcal{K} \to \mathcal{E},\ \tilde{B}_1\omega = \int_0^h e^{(h-\tau)A} B_1 \omega(\tau) d\tau \tag{4.6}$$

$$\tilde{C}_1 : \mathcal{E} \to \mathcal{K},\ (\tilde{C}_1 x)(t) = C_1 e^{tA} x \tag{4.7}$$

$$\tilde{D}_{11} : \mathcal{K} \to \mathcal{K},\ D_{11}\omega(t) + C_1 \int_0^t e^{(t-\tau)A} B_1 \omega(\tau) d\tau \tag{4.8}$$

$$\tilde{D}_{12} : \mathcal{E} \to \mathcal{K},\ D_{12} v + C_1 \int_0^t e^{\tau A} d\tau B_2 v \tag{4.9}$$

with $\mathcal{E}$ for some Euclidian space with proper dimension.

Despite that these input/output spaces are infinite dimensional it is possible to obtain a finite dimensional discrete state space representation which is equivalent in norm, however, this depends on the choice of norm. This enables a proper SD design i.e. taking the intersample behaviour into account based on known and widely available algorithms for discrete-time.

The major observations in seeing why the representations becomes finite is that the the state-space remains the same and hence the B_1, C_1, D_{12} operators are of finite rank (dimension of image). Then decomposition of either the domain (or the co-domain) in kernel and its complement (or range and its complement) the infinite part can be left out without changing the closed-loop.

In a constructive way we see that by formal inspection of the Riccati equations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ and observing that operators enter like $\tilde{B}_1\tilde{B}_1^*$ mapping $\mathcal{E}$ to $\mathcal{E}$. This matrix may be found simply by matrix exponentials as

$$\tilde{B}_1\tilde{B}_1^* = \int_0^h e^{tA} B_1 B_1' e^{tA'} dt = P_{22}' P_{12}$$

with

$$\begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} = \exp\left\{ h \begin{bmatrix} -A & BB' \\ 0 & A' \end{bmatrix} \right\}$$

since for A_{11} and A_{22} both square and

$$\begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix} \triangleq \exp\left\{ t \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \right\}$$

then $F_{12}(t) = \int_0^t e^{(t-\tau)A_{11}} A_{12} e^{\tau A_{22}} d\tau$, see [GvL89].

One may use the Riccati equations directly or find an equivalent finite dimensional discrete-time system by above method combined with Cholesky factorisation, see [GvL89]. We will give some references to such representations in section 4.5.

4.3.1 Lifting Open-Loop Systems

Given we have $y = Gu$ with $\hat{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Let $\tilde{u} = Lu$ and $\tilde{y} = Ly$ then $\tilde{y} = \tilde{G}\tilde{u}$, where $\tilde{G} = LGL^{-1}$ is the lifted system. Because G is time-invariant then $\tilde{G}$ is too. $\tilde{G}$ may be modeled by a discrete system in a state-space form $\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$. To see this take as input $u(t) = u_0(t), \quad 0 \le t < h$ else 0 then the output is

$$y(t) = \begin{cases} 0, & t < 0 \\ Du_0(t) + \int_0^t Ce^{(t-\tau)A}Bu_0(\tau)d\tau, & 0 \le t < h \\ \int_0^h Ce^{(t-\tau)A}Bu_0(\tau)d\tau, & t \ge h \end{cases}$$

order as a sequence $\tilde{y} = [\ldots 00y_0y_1 \ldots]^T$ then

$$\begin{aligned} y_0(t) = y(t+0) &= Du_0(t) + \int_0^t Ce^{(t-\tau)A}Bu_0(\tau)d\tau =: \tilde{D}u_0 \\ y_1(t) = y(t+h) &= Ce^{tA}\int_0^h e^{(h-\tau)A}Bu_0(\tau)d\tau \qquad =: \tilde{C}\tilde{B}u_0 \\ y_2(t) = y(t+2h) &= Ce^{tA}e^{hA}\int_0^h e^{(h-\tau)A}Bu_0(\tau)d\tau \quad =: \tilde{C}\tilde{A}\tilde{B}u_0 \\ &\vdots \end{aligned}$$

where one identify

$$\begin{aligned} &\tilde{A} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x},\ \tilde{A}x = e^{hA}x \\ &\tilde{B} : \mathcal{L}_2[0,h] \to \mathbb{R}^{n_x},\ \tilde{B} = \int_0^h e^{(h-\tau)A}Bu(\tau)d\tau \\ &\tilde{C} : \mathbb{R}^{n_x} \to \mathcal{L}_2[0,h],\ (\tilde{C}x)(t) = Ce^{tA}x \\ &\tilde{D} : \mathcal{L}_2[0,h] \to \mathcal{L}_2[0,h],\ \tilde{D} = Du(t) + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau \end{aligned}$$

Lifting *SG*

G has no direct term. Only the input needs to be lifted. Lifted system $\tilde{SG} = SGL^{-1}$. Analogue we find a state model for $\widetilde{SG}$, $\left[\begin{array}{c|c} A_d & \tilde{B} \\ \hline C & 0 \end{array}\right]$.

Lifting *GH*

Only the output needs to be lifted hence we have lifted system $\tilde{GH} = LGH$. Again find state model for $\widetilde{GH}$, $\left[\begin{array}{c|c} A_d & B_d \\ \hline \tilde{C} & \tilde{D}_{res} \end{array}\right]$. Where $B_d = B_{2d}$ and the restriction of G is

$$\tilde{D}_{res} : \mathbb{R}^m \to \mathcal{L}_2[0,h],\ Dv + \int_0^t Ce^{\tau A} d\tau Bv$$

4.4 Performance Measures for SD Systems

The system maps w to z in continuous-time, see fig. 4.1. Hence, we stress that a proper performance measure must be continuous-time. Mainly generalisations of usual LTI norms are applied. Instead of working on matrix valued functions these norms are basicly redefined to take operator valued functions which simplifies back correspondingly in the LTI case.

When we write SD design we implicitly assume a continuous-time measure hence taking the intersample behaviour into account.

Given one may view HK_dS as a restriction of all continuous-time controllers, a clear rule is that restricted HK_dS controller cannot be better than the pure continuous-time controller (optimal w.r.t. a chosen continuous-time measure, by which the performance is measured of both controllers).

4.4.1 Generalised $\mathcal{H}_2$

Recall that in the LTI case the $\mathcal{H}_2$ norm is

$$\|\hat{T}\|^2_{\mathcal{H}_2} \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{trace}(\hat{T}(j\omega)^* \hat{T}(j\omega)) d\omega = \sum_{i=1}^{n} \|T\delta e_i\|^2_{\mathcal{L}_2} \tag{4.10}$$

$\delta(t)$ is the impulse function and (e_i) is the Euclidean basis in $\mathbb{R}^n$.

The exercise is now to find an expression in the lifted frequency space by walking trough the levels in commuting diagram 2.21.

We start with the PTV (Periodic Time-Varying) case and consider a continuous-time map $T \in \mathfrak{L}(\mathcal{L}_2^m, \mathcal{L}_2^n)$ described by the integral equation

$$(Tu)(t) = \int_0^\infty T(t,\tau)u(\tau)d\tau$$

where $T(t,s)$, a $n \times m$ matrix function (the impulse response), is the kernel of the time-varying operator T. Say it has support $[a;b]$ to $[c;d]$ then it is said to be a Hilbert-Schmidt operator if

$$\|T\|_{\mathbb{HS}}^2 = \int_a^b \int_c^d \operatorname{trace}\left(T'(t,\tau)T(t,\tau)\right) dt d\tau < \infty \tag{4.11}$$

is satisfied, which follows from the abstract definition (2.3). This class of operators forms a Hilbert space.

In the multi-variable case the generalised $\mathcal{H}_2$ [BP92] is given by

$$\|T\|_{\mathcal{H}_2}^2 = \frac{1}{h}\int_0^h \sum_{i=1}^n \|T\delta_\tau e_i\|_{\mathcal{L}_2}^2 d\tau \tag{4.12}$$

with $\delta_\tau = \delta(t-\tau)$ or expressed by the kernel

$$\|T\|_{\mathcal{H}_2}^2 \triangleq \frac{1}{h}\int_0^h \operatorname{trace}\left(\int_0^\infty T'(t,\tau)T(t,\tau)dt\right) d\tau \tag{4.13}$$

For periodic systems, when we look at the lifted system $\tilde{T} \in \mathfrak{L}(\ell_{\mathcal{K}})$ which is a discrete LTI system, this equals

$$\|T\|_{\mathcal{H}_2}^2 = \frac{1}{h}\sum_{i=0}^\infty \|\tilde{T}_i\|_{\mathbb{HS}}^2 \tag{4.14}$$

where $(\tilde{T}_i)$ a sequence of Hilbert-Schmidt operators is the impulse response of $\tilde{T}$. In the following we use $\mathbb{HS}$ to denote the class with support on $[0;h]$ to $[0;h]$.

By taking the Λ-transform of $(\tilde{T}_i)$ we have $\check{T}$ an element in the Hardy space $\mathcal{H}_2(\mathbb{D}, \mathfrak{L}(\mathcal{K}))$ of the Hilbert-Schmidt operator valued functions i.e. the transfer function $\check{T} = \check{T}(\lambda)$ maps into a Hilbert-Schmidt operator with the kernel denoted $\check{T}_\lambda(t,\tau)$, where the norm is defined by

$$\|\check{T}\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi}\int_0^{2\pi} \|\check{T}(e^{j\theta})\|_{\mathbb{HS}}^2 d\theta \tag{4.15}$$

Again we denote this the $\mathcal{H}_2$ norm since it can be shown to equal the other expressions. This class of operators again forms a Hilbert space. Note that we only need to evaluate $\check{T}$ on the unit circle. This is parallel to discrete-time with the trace norm replaced with the $\mathbb{HS}$ norm.

For LTV systems (4.12) may be generalised as follows given by the kernel representation.

$$\|T\|_{\mathcal{H}_2}^2 \triangleq \lim_{M\to\infty} \frac{1}{M} \int_0^M \operatorname{trace}\left(\int_0^\infty T'(t,\tau)T(t,\tau)dt \right) d\tau \tag{4.16}$$

Another generalisation is given in [Pag96b, Pag96a], see this for motivation. We will recall the SISO case continuous time and refer to the reference for the extention to the case of multivariable noise. The $W_{\eta,B}$ norm is defined by

$$\|T\|_{W_{\eta,B}} \triangleq \sup_{f\in W_{\eta,B}} \|Tf\|_{\mathcal{L}_2} \tag{4.17}$$

where the signals in $W_{\eta,B}$ have a cumulative spectrum approximating white noise up to bandwidth, B. The cumulative spectrum is

$$F_f(\beta) \triangleq \int_{-\beta}^{\beta} |\hat{f}(j\omega)|^2 \frac{d\omega}{2\pi} \tag{4.18}$$

which is a continuous and real function on $\mathbb{R}_+$ with a limit at infinity and should lie in the set of those constrained as

$$S_{\eta,B} \triangleq \left\{ g : \min\left(\frac{\beta}{\pi} - \eta, \frac{B}{\pi} - \eta \right) \leq g(\beta) \leq \frac{\beta}{\pi} + \eta \right\} \tag{4.19}$$

with accuracy, η, and bandwidth, B where the cumulative spectrum rolls of. So that white signals in this sense are

$$W_{\eta,B} \triangleq \left\{ f \in \mathcal{L}_2 : F_f(\cdot) \in S_{\eta,B} \right\} \tag{4.20}$$

It is shown in [Pag96a][1] that for any $T \in \mathcal{RH}_\infty$ then $\|T\|_{\mathcal{H}_2} \leq \|T\|_{W_{\eta,B}}$. Further, if $|\hat{T}(j\omega)|^2 \in \mathcal{F}$ then

$$\lim_{\eta\to 0, B\to\infty} \|T\|_{W_{\eta,B}} = \|T\|_{\mathcal{H}_2} \tag{4.21}$$

where

$$\begin{aligned} \mathcal{F} \triangleq \{ & Y \in BV(\mathbb{R}) : \exists G \in \mathcal{L}_1(\mathbb{R}_+), G \text{ monotone decreasing }, \\ & 0 \leq Y(\omega) = Y(-\omega) \leq G(|\omega|) \} \end{aligned} \tag{4.22}$$

and $BV(\mathbb{R})$ denotes bounded variation, see e.g. [Kre89].

[1] It is pointed out in [ST98] that this does not extend from the fixed to the uncertain case, see also sec. 3.6

4.4.2 Generalised $\mathcal{H}_\infty$

For the PTV case we refer to (2.19) and (2.22).

4.5 Controller Problem Representations

Here we give references to controller problem representations; in section 6.1.1 similar analysis problem representations are given.

For finite dimensional state space representations in the cases of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ we refer to [CF95] (denoted $G_{eq,d}$) (or [BP92, BJ92]) for computational procedure and derivation based on operator theory.

The equivalent/associated discrete system is here denoted $\bar{G}$ and it will have following form and depends on the (underlying) norm,

$$\bar{G}(\lambda) = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right] \tag{4.23}$$

For $\mathcal{H}_2$ optimisation it is required that $D_{11} = 0$ for the closed-loop to be strictly causal and the following relation may be derived,

$$\|\tilde{G} \star K_d\|^2_{\mathcal{H}_2} = \frac{1}{h}\left(\|\tilde{D}_{11}\|^2_{\mathbb{HS}} + \|\bar{G} \star K_d\|^2_{\mathcal{H}_2}\right) \tag{4.24}$$

which is an exact equivalence.

For $\mathcal{H}_\infty$ the relation between the lifted system $\tilde{G}$ and the equivalent finite dimensional discrete system $\bar{G}$, depends on γ, is given by the equivalence between,

1. $\|\tilde{G} \star K_d\|_\infty < \gamma$.
2. $\|\bar{G} \star K_d\|_\infty < \gamma$.

4.5.1 Fast Discretisation

The SD system in fig. 4.1 is changed by adding fast sample S_n and hold H_n, with period h/n, at the generalised output z and input w. A little reordering gives a two-rate discrete system, hence time-varying, this is handled by the (discrete) lifting technique. The fast discretisation idea is due to [KA92], formulas for discrete system G_n are nicely given in [CF95][2]. This is a way to improve the standard discretisation ($n = 1$) especially in $\mathcal{H}_\infty$ case as this procedure is much simpler than

[2]Based thereon a quote "readily" Matlab implementation is given in appendix A.2

the $\mathcal{H}_\infty$-discretisation. Note that the number of inputs and outputs for G_n grows linearly with n.

Fast discretisation gives an approximative solution the SD ℓ_1-problem [BDP93], actually it is argued therein that one probably cannot find an "exact" discretisation as the $\mathcal{H}_\infty$-discretisation.

The ℓ_1-norm is given by

$$\|G\|_1 = \max_{1\le i\le p} \sum_{j=1}^{m} \sum_{k=0}^{\infty} |g_{ij}(k)| \tag{4.25}$$

i.e. the elements in the system matrix (2.34).

Assume n is as least twice the number of states n_x. Then there exists K_0, K_1 so that

$$\|G_n \star K_d\|_1 \le \|\tilde{G} \star K_d\|_1 \le \frac{K_1}{n} + \left(1 + \frac{K_0}{n}\right) \|G_n \star K_d\|_1 \tag{4.26}$$

4.6 Notes and References

A brief discussion of the setup for which the main advantage is that the lifting technique leads to tractable problems, see also section A.1, where the solution is based on the known discrete-time methods. The disadvantages are numerous, however, these also go for discrete-time methods. Delays in S and H, non syncronous and real-time control [Nil98]. Finite wordlength considerations give the lesson of not sampling too fast [Bam96]. Finite wordlength (discrete-time) problems are covered indepth in [GL93].

The lifting framework is extended to handle generalised sampled-data hold ([Kab87]) in [MP95, MP97]. In some applications the aim is the "at sample" performance and the framework shows the tradeoff [MP98].

Multirate SD systems are studied in [CQ94] using lifting. Inherent design limitations are given in [FMB95, SBG97].

Other related approaches count solving the Ricatti eqautions directly [SNK93], by frequency response [YK93, AI93] and by game theory (multirate SD systems) [Lal95] much in the line of [BB95]. Earlier attacks used conic sectors [Zam66] as in [Nie88].

For more points of view on generalised $\mathcal{H}_2$ see [Sto93, ZGBD94] and [SAT, Wil89].

Chapter 5

LTR for Sampled-Data Systems

The majority of papers on LTR, see comprehensive list in [SCS93], deal either with continuous-time systems or with discrete-time systems. The LTR design of sampled-data systems has not been tackled in the literature except the paper [SFdS94]. The approach taken by [SFdS94] is based on the result by [SNK93], which is distinctly different from the lifting approach proposed in [BPFT91, BJ92]. In the former case, the controller turns out to be linear time-varying and it generates continuously varying input signals rather than a piecewise constant input signal.

The solution to the LTR problem for sampled-data systems given here is based on the lifting approach, see section 4.3. The disadvantage of this approach is that it is slightly more difficult to formulate than the purely continuous-time or discrete-time cases. However, as it will be shown, the LTR design of sampled-data systems can directly be tied to the conventional discrete-time LTR design methods.

When the LTR step is not exact or almost so the achieved robustness should be analysed for with the methods in chapter 6 in the $\mathcal{H}_2$ case. In other words one may view the LTR procedure as a synthesis method for robust $\mathcal{H}_2$ performance to be compared with the so-called D-K iteration.

5.1 LTR Design for Sampled-Data Systems

Using the formulations in section 4.5 and section 3.7, the LTR design problem for sampled-data systems can be solved by using the equivalent discrete-time system. In the following, we describe the SD LTR problem and suggest $\mathcal{H}_2$/LTR and $\mathcal{H}_\infty$/LTR design methods. Further, we discuss application of fast discretisation.

The discrete time controller with measurement input ψ and control output υ is described by

$$\hat{K}_d = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{5.1}$$

To apply the LTR design methodology on sampled-data systems, we consider the system

$$\hat{G}_{SD} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{5.2}$$

with state x, for state feedback design (3.20).

Let the target design be a state feedback controller given by

$$\upsilon = F\psi = FSx \tag{5.3}$$

with the resulting target closed-loop operator given by:

$$T_{zw,T} = G_{SF} \star HFS \tag{5.4}$$

It is assumed that the target closed-loop is stable and it satisfies the design specifications.

As in the continuous-time case, the target controller cannot be implemented, so we need to recover the target operator by using a dynamic controller $\hat{K}_d(\lambda)$. The closed-loop operator with the controller $\hat{K}_d(\lambda)$ is then given by

$$T_{zw} = G_{SD} \star HK_d S \tag{5.5}$$

Based on these two closed-loop operators, we can define the recovery error operator

$$E = T_{zw} - T_{zw,T} \tag{5.6}$$

From the recovery error operator in (5.6), we can now define the LTR design problem for sampled-data systems.

Problem 5.1 *Let the target loop operator, the full loop operator and the recovery operator be given by $T_{zw,T}$, T_{zw} and E respectively. The LTR design problem is then to design a dynamic controller $\hat{K}_d(\lambda)$ that stabilises the sampled-data system and makes a suitable norm of the recovery operator small in some sense.*

It is not possible by standard methods to minimise a suitable norm of the recovery operator as it is time-varying. Instead, by using lifting of the sampled-data system, the design problem can be transformed into an equivalent discrete-time design problem as described in the previous section. In order to apply the lifting technique to the recovery error operator given by (5.6), we need to construct a composite state space description before the system is lifted. If the system is lifted directly, we will not get the right equivalent discrete-time system to work with in the $\mathcal{H}_\infty$ case, due to the loop-shifting procedure. A state space description of the recovery error is given by

$$G_{E_{SD}} = \left[\begin{array}{c|cc} A_E & B_{E,1} & B_{E,2} \\ \hline C_{E,1} & 0 & D_{E,12} \\ C_{E,2} & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|ccc} A & 0 & B_1 & B_2 & 0 \\ 0 & A & B_1 & 0 & B_2 \\ \hline -C_1 & C_1 & 0 & -D_{12} & D_{12} \\ I & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \end{array}\right] \tag{5.7}$$

with the controller given by

$$\upsilon_E = \hat{K}_E(\lambda)\psi_E = diag(F, \hat{K}_d(\lambda)) \begin{bmatrix} Sx \\ \psi \end{bmatrix} \tag{5.8}$$

The recovery error described by (5.7) is now given in the standard description, which makes it possible to find an equivalent finite dimensional discrete time system by using the lifting technique. Performing this task, the following equivalent discrete time system for the recovery error is obtained:

$$\bar{G}_E = \left[\begin{array}{c|cc} \bar{A}_E & \bar{B}_{E,1} & \bar{B}_{E,2} \\ \hline \bar{C}_{E,1} & 0 & \bar{D}_{E,12} \\ \bar{C}_{E,2} & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|ccc} \bar{A} & 0 & \bar{B}_1 & \bar{B}_2 & 0 \\ 0 & \bar{A} & \bar{B}_1 & 0 & \bar{B}_2 \\ \hline -\bar{C}_1 & \bar{C}_1 & 0 & -\bar{D}_{12} & \bar{D}_{12} \\ I & 0 & 0 & 0 & 0 \\ 0 & \bar{C}_2 & 0 & 0 & 0 \end{array}\right] \tag{5.9}$$

It is important to note that the equivalent discrete-time state space description for the recovery error has exactly the same structure as the sampled-data description. This structure allows to rewrite the recovery error as a target loop transfer function multiplied by a recovery matrix as described in section 3.7. Using the controller $\hat{K}_E(\lambda)$ given by (5.8), we can express the recovery error

$$\bar{E}(\lambda) = \bar{G}_E(\lambda) \star \hat{K}_E(\lambda) \tag{5.10}$$

in the standard form as

$$\bar{E}(\lambda) = \bar{G}_R(\lambda) \star \hat{K}_d(\lambda) - \bar{G}_T(\lambda) \star F \tag{5.11}$$

where

$$\bar{G}_R(\lambda) = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right], \bar{G}_T(\lambda) = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ I & 0 & 0 \end{array}\right] \tag{5.12}$$

The lifting results, see section 4.5, guarantees that the norm (in consideration) of the recovery error is preserved. $\bar{G}_R(\lambda)$ is the data we continue with, however, it is only different from $\bar{G}(\lambda)$ in the $\mathcal{H}_\infty$ case, besides the $\bar{D}_{11}$ term.

Again consider a full-order prediction observer based controller given by:

$$\hat{K}_d(\lambda) = -F(\lambda I - \bar{A} - \bar{B}_2 F - L\bar{C}_2)^{-1} L \tag{5.13}$$

Based on the above description of the recovery error $\bar{E}$, and the result from section 3.7, we get

$$\bar{E}(\lambda) = \hat{T}_{zu,T}(\lambda)\bar{M}_I(\lambda) \tag{5.14}$$

where $\hat{T}_{zu,T}(\lambda)$ is the closed-loop transfer function from u to z under the target design given by:

$$\hat{T}_{zu,T}(\lambda) = (\bar{C}_1 + \bar{D}_{12}F)(\lambda I - \bar{A} - \bar{B}_2 F)^{-1}\bar{B}_2 + \bar{D}_{12} \tag{5.15}$$

and $\bar{M}_I$ is the recovery matrix given by

$$\bar{M}_I(\lambda) = F(\lambda I - \bar{A} - L\bar{C}_2)^{-1}\bar{B}_1 \tag{5.16}$$

To calculate the SD LTR observer gain, we consider the recovery matrix with the following state space realisation:

$$\bar{M}_I(\lambda) = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & I \\ \hline F & 0 & 0 \\ \bar{C}_2 & 0 & 0 \end{array}\right] \tag{5.17}$$

It is important to note that the design of the target gain F is free. It may be derived by e.g. a (SD) optimisation (design) method.

5.1.1 Recovery Conditions

Based on the recovery error (5.14), it is possible to give conditions for obtaining exact recovery. As in the continuous-time case, exact recovery is obtained if $\bar{E}(\lambda) = 0$. Thus, the following result can be deduced from [NSAS91, SCS93]:

Lemma 5.1 *Let $\hat{T}_{zw,T}(\lambda)$ be an admissible closed-loop target transfer function and let $\hat{T}_{zu,T}(\lambda)$ be left invertible. Exact LTR, i.e. $\bar{E}(\lambda) = 0$, can be obtained if and only if $\bar{M}_I(\lambda) = 0$.*

Proof. Lemma 5.1 follows immediately from (5.14). ■

Lemma 5.1 gives a necessary and sufficient condition for exact recovery. However, it is not in general possible to obtain exact recovery with a free target design when a full-order prediction observer is applied, see e.g. [ZF93, SCS93]. Moreover, it is not possible at all (independent of observer type) to obtain exact recovery with a free target design for discrete-time systems resulting from sampled-data systems. The reason is that non minimum phase zeros appear in the equivalent discrete-time system, see e.g. [FG96]. Since exact recovery is generally not achievable, we can look at asymptotic recovery as for continuous-time systems. Unfortunately, asymptotic recovery does not exist in discrete-time systems. It has been pointed out in [SCS93], that every asymptotically recoverable target loop can also be exactly recoverable. So, for discrete-time systems, there exists only two possibilities in connection with LTR design: Exact recovery is achievable or it is impossible, a finite recovery error will appear.

In the rest of this chapter we will concentrate on the $\mathcal{H}_2$/LTR and $\mathcal{H}_\infty$/LTR design methods. It is possible to minimise the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$ norm of the recovery error directly or indirectly by minimisation of the recovery matrix $\bar{M}_I(\lambda)$, [NSAS91, SN93]. This is equivalent to the standard LQG/LTR design as described in [NSAS91]. Here, we consider the minimisation of the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$ norm of the recovery matrix, which is based on the following norm inequality:

$$\|\bar{E}\| = \|\hat{T}_{zu,T}\bar{M}_I\| \leq \|\hat{T}_{zu,T}\|\|\bar{M}_I\| \tag{5.18}$$

As a direct consequence of the above norm inequality, the norm of the recovery matrix should satisfy:

$$\|\bar{M}_I\| \leq \|\bar{E}\|/\|\hat{T}_{zu,T}\| \tag{5.19}$$

when the norm of the recovery error is prespecified.

5.1.2 $\mathcal{H}_2$/LTR Design

Let the equivalent discrete-time state space description of the recovery matrix be given by (5.16). Then, the $\mathcal{H}_2$/LTR design problem is formulated as follows.

Problem 5.2 *Let the recovery matrix, $\bar{M}_I(\lambda)$, for the observer design be given by (5.16). Find an observer gain L such that $\underline{A} + L\bar{C}_2$ is stable and the $\mathcal{H}_2$ norm of $\bar{M}_I(\lambda)$ is minimised.*

The design of observer gain $\upsilon = L_2\psi$ can be obtained by using the discrete-time $\mathcal{H}_2$ design method of [TS93]:

Lemma 5.2 *Consider the system given by (5.17). It is assumed that* $(\bar{C}_2, \bar{A})$ *is detectable. Then there exist an observer* $\upsilon = L_2\psi$ *which stabilises the system (5.17) and minimises the* $\mathcal{H}_2$ *norm of the closed loop transfer function* $\bar{M}_I$ *if and only if there exists a symmetric matrix positive semidefinite matrix* Q_2 *such that*

$$Q_2 = \bar{A}Q_2\bar{A}^T + \bar{B}_1\bar{B}_1^T - \bar{A}Q_2\bar{C}_2^T(\bar{C}_2Q_2\bar{C}_2^T)^{-1}\bar{C}_2Q_2\bar{A}^T \tag{5.20}$$

Moreover, the observer gain L_2 *is given by:*

$$L_2 = -\bar{A}Q_2\bar{C}_2^T(\bar{C}_2Q_2\bar{C}_2^T)^{-1}. \tag{5.21}$$

It is important to note that, in general, the requirement of $D_{11} = 0$ for the original system is necessary due to the condition of strict causality. However, this is not a condition in connection with the $\mathcal{H}_2$/LTR design method, because the D_{11} term does not appear in the recovery error equation.

5.1.3 $\mathcal{H}_\infty$/LTR Design

Now, apply an $\mathcal{H}_\infty$ optimisation method instead. In this case, it is assumed that the equivalent discrete-time system (5.17) preserves the $\mathcal{H}_\infty$ norm. Then, we have the following $\mathcal{H}_\infty$/LTR design problem.

Problem 5.3 *Let* $\gamma > 0$ *be given. Design, if possible, an observer* $\upsilon = L_\infty\psi$ *which stabilises the system (5.17) and makes the* $\mathcal{H}_\infty$ *norm of the closed-loop transfer function* $\bar{M}_I$ *smaller than* γ*.*

This design can be performed by using the approach in [Sto92]:

Lemma 5.3 *Consider the system given by (5.17). Assume that* $(\bar{A}, \bar{B}_1, \bar{C}_2, 0)$ *is left invertible and has no invariant zeros on the unit circle. Then, there exists an observer* $\upsilon = L_\infty\psi$ *which stabilises the system (5.17) and makes the* $\mathcal{H}_\infty$ *norm of the closed-loop transfer function from w to z less than* γ*, if and only if there exists a symmetric matrix* $Q \geq 0$ *such that:*

$$\begin{aligned} R &= \gamma^2 I - FQF^T > 0 \\ Q &= \bar{A}Q\bar{A}^T + \bar{B}_1\bar{B}_1^T \\ &\quad - \begin{bmatrix} \bar{C}_2Q\bar{A}^T \\ FQ\bar{A}^T \end{bmatrix}^T G(Q)^\dagger \begin{bmatrix} \bar{C}_2Q\bar{A}^T \\ FQ\bar{A}^T \end{bmatrix} \end{aligned} \tag{5.22}$$

where

$$G(Q) = \begin{bmatrix} \bar{C}_2Q\bar{C}_2^T & \bar{C}_2QF^T \\ FQ\bar{C}_2^T & FQF^T - \gamma^2 I \end{bmatrix} \tag{5.23}$$

and the eigenvalues of $\bar{A}_{cl}$, where

$$\bar{A}_{cl} = \bar{A} - \begin{bmatrix} \bar{A}Q\bar{C}_2^T & \bar{A}QF^T \end{bmatrix} G^T(Q)^{-1} \begin{bmatrix} \bar{C}_2 \\ -F \end{bmatrix} \tag{5.24}$$

are inside the unit circle.

Moreover, an observer gain L_∞ is given by:

$$L_\infty = -(\bar{A}Q\bar{C}_2^T + \bar{A}QF^T R^{-1} FQ\bar{C}_2^T)H^{-1} \tag{5.25}$$

where $H = \bar{C}_2 Q\bar{C}_2^T + \bar{C}_2 QF^T R^{-1} FQ\bar{C}_2^T$.

5.1.4 Fast Discretisation LTR Design

From a practitioner's point of view it may seem like overdoing things to perform a SD $\mathcal{H}_\infty$-discretisation for doing the LTR step. Fast discretisation, see section 4.5.1, offers a simpler (approximative) way. In parallel to the SD $\mathcal{H}_2$/LTR case lifting preserves structure, moreover, $\bar{G}_R(\lambda)$ equals $\bar{G}(\lambda)$ but for the $\bar{D}_{11}$ matrix. Hence the data for (5.12) and the recovery matrix (5.17) can be found directly by fast discretisation of $\hat{G}_{SD}$ without construction of the composite system (5.7) etc.

Since the SD $\mathcal{H}_2$-discretisation is also simple to perform and exact it is better always apply that for SD $\mathcal{H}_2$/LTR.

5.1.5 ℓ_1/LTR Design

Ditto - for the setup, see above 5.1.4. However, finding an observer gain is in general not possible in ℓ_1, it will be dynamic. Likewise finding a state-feedback is neither in general possible. But for the trivial case with a minimum phase plant so that a pole-zero cancellation can be performed [DDB95].

We do again point out that F may steam from any method. In which case we may close the loop with a given F in (5.7) and then find K_d by ℓ_1 optimisation. Finding the controller for an order $2n$ system; typically results in an even higher order controller.

5.2 Example

An LTR design example for sampled-data systems is given in this section. The $\mathcal{H}_\infty$/LTR design method is applied for both a traditional discrete-time LTR design and for a sampled-data LTR design. The sampled-data system is given by:

$$\hat{G}_{SD} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|cc} -1000 & 0 & 39.478 & 0 & 0 & 1 \\ 0 & -.62832 & -39.478 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 39.478 & -100 & 1 & 1 \\ \hline 1000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The sample period is 0.1 sec. The target design is given by

$$F = \left[\begin{array}{cccc} 0.00 & -3.5495 & -32.2333 & 0 \end{array}\right]$$

The target loop $T_{zw,T}$ has a $\mathcal{H}_\infty$ norm of 3.61.

When we apply the $\mathcal{H}_\infty$/LTR design method on the discretised system of the continuous-time system, we get the following controller:

$$\left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & -.041112 & -.00041112 \\ .75596 & -3.5795 & -4.4684 & -.044684 \\ .09067 & .81293 & -1.0459 & -.010459 \\ -.0028312 & .01149 & -.39358 & -.0039362 \\ \hline 3.5495 & 32.233 & 0 & 0 \end{array}\right]$$

Using this controller to recover the target loop, the $\mathcal{H}_\infty$ norm of the sampled-data recovery error is 5.48 and the final closed loop transfer function from w to z has a sampled-data $\mathcal{H}_\infty$ norm of 5.49. If we instead apply the $\mathcal{H}_\infty$/LTR design method on the equivalent discrete-time system based on lifting, we get the following sampled-data LTR controller:

$$\left[\begin{array}{c|c} A_{SD} & B_{SD} \\ \hline C_{SD} & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & -.098694 & -.00098694 \\ .75596 & -3.5795 & 4.0535 & .040535 \\ .09067 & .81293 & -2.496 & -.02496 \\ -.0028312 & .01149 & -.99748 & -.0099753 \\ \hline 3.5495 & 32.233 & 0 & 0 \end{array}\right]$$

When we apply this sampled-data LTR controller, the $\mathcal{H}_\infty$ norm of the sampled-data recovery error is reduced to 2.18 and the sampled-data $\mathcal{H}_\infty$ norm of the closed loop transfer function from w to z is reduced to 4.51 as compared to the discrete-time design of LTR controller, see bode plots in fig. 5.1. In this example it is possible to reduce the $\mathcal{H}_\infty$ norm of the closed loop operator by 18% using the lifting technique.

Figure 5.1: SD LTR (upper plot) and Discrete LTR controller

5.3 Notes and References

There is not a straightforward duality between the input loop breaking point and the output loop breaking point for SD $\mathcal{H}_\infty$/LTR design. This is seen by writing down the state space description for the recovery error for the output loop breaking point alike (5.7). When lifting is performed the special structures in $\bar{B}_{E,2}$ and $\bar{D}_{E,12}$ vanish. Hence it is not possible to perform a standard recovery design for the output loop breaking point using a standard full order observer based controller as it is possible for the input loop breaking point. Minimising the recovery error for the output loop breaking point instead gives rise to controllers of order $2n$.

For SD $\mathcal{H}_2$/LTR design lifting does not obstruct the structure, hence it is possible to obtain recovery at the output loop breaking point by using a controller of order n.

Chapter 6

Robust $\mathcal{H}_2$ Performance for SD Systems

Based on results in [Pag96a] summarised section in 3.5 and the framework for uncertain sampled-data systems given in [Dul95] we derive conditions for robust $\mathcal{H}_2$ performance in the SD setting[1]. The motivation is to perform the analysis test in the more realistic SD setting. The uncertain SD feedback system, see fig. 6.1 (c.f.

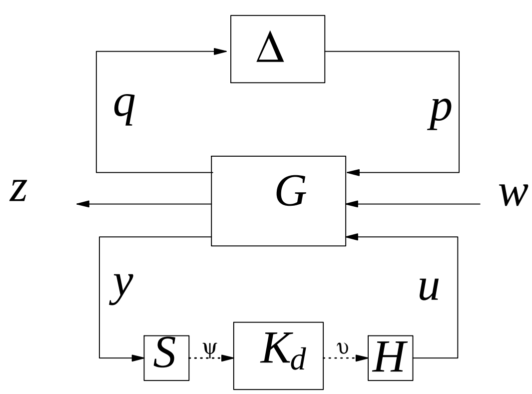


Figure 6.1: Uncertain Sampled-data System

4.1) is altogether h-periodic, which we refer to as Periodic Time-Varying (PTV) so that the resulting nominal system will be PTV, instead of LTI as usual in robust control.

In the treatment of uncertain systems it is common to study LTI perturbations

[1]This is joint work with F. Paganini. Initiated during my visit at LIDS, MIT.

as the system model is lumped into a LTI system, however, the larger class of LTV uncertainty as studied in e.g. [Sha94] and the case of slowly varying uncertainty in [PT95] are also of general interest. Here we focus firstly on PTV perturbations, into which the uncertainty may be lumped notably when system identification is performed on a SD system, and secondly on LTV and LTI perturbations. The purpose of the test is to check for robust stability (energy to energy) towards the uncertainty set and at the same time to have a certain level of $\mathcal{H}_2$ performance.

In the lifted frequency space we state the conditions for robust $\mathcal{H}_2$ performance for SD in the case of PTV and LTV perturbations. Further in the case of LTI perturbations we use a space, where the LTI structure of the perturbations is clear, the so-called frequency response for SD, to obtain the wanted conditions.

Before getting into the nitty-gritty details we like to point out that simple the fast discretisation, section 4.5.1, gives an approximative solution to the SD problems treated below when combined with the results in [Pag96c]. Here we use "exact" discretisation.

6.1 Robust $\mathcal{H}_2$ Performance SD for TV Uncertainty

In the following the properties of condition 1 on page 29 will be extended to the SD setting.

6.1.1 Analysis Problem Representations

Denote the nominal system $M \triangleq G \star HK_dS$, which is h-periodic since $K = HK_dS$ is so, however, the lifted operator $\tilde{M} \triangleq L(G \star HK_dS)L^{-1}$ is an LTI operator on $\ell_{\mathcal{K}}$. Therefore, it has a Λ-transform denoted $\check{M}$ see the diagrams 2.20 and 2.21 on page 16. Note that the state space representations of $\tilde{M}$ and $\check{M}$ are found in appendix A and section 6.2 [Dul95]. From which we quote the following two results for reference.

Lemma 6.1 *[Dul95] Let $\mathfrak{D}$ be a subspace of $\mathfrak{L}(\mathcal{L}_2)$. Then the system is robustly stabilized to $B\mathfrak{D}$, iff, for all $\tilde{\Delta} \in B\mathfrak{D}$, the $(I - \tilde{M}\tilde{\Delta})^{-1}$ exists in $\mathfrak{L}(\ell_{\mathcal{K}})$.*

Lemma 6.2 *[Dul95] Suppose $\Delta \in \mathfrak{L}_{\mathcal{A}}$. Then the map $(I - \tilde{M}\tilde{\Delta})^{-1}$ exists and is bounded, iff, for all $\lambda \in \mathbb{D}$, the operator $I - (\check{M}\check{\Delta})(\lambda)$ is nonsingular.*

6.1.2 PTV Perturbation Case

Consider a PTV uncertain system, see fig. 6.2, with $M \in \mathfrak{L}_{\mathcal{A}}(\mathcal{L}_2)$ and perturbations in Δ_{PTV} which may arise when system identification is performed on a SD system.

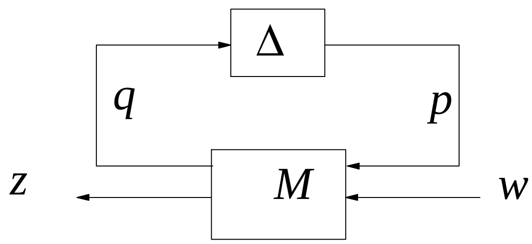


Figure 6.2: Uncertain System

The set of structured PTV operators is $\Delta_{PTV} \triangleq \mathfrak{L}_{\mathcal{A}} \cap \Delta_s$. These have the following structure in $\mathcal{A}$

$$\check{\Delta}(\lambda) \triangleq \begin{bmatrix} \check{\Delta}_1(\lambda) & & 0 \\ & \ddots & \\ 0 & & \check{\Delta}_F(\lambda) \end{bmatrix} \tag{6.1}$$

Let $\mathbb{X}$ commute with $\check{\Delta}_{PTV}(\lambda)$ given by

$$\mathbb{X} \triangleq \{X \in \mathfrak{L}(\mathcal{K}) : X = \mathrm{diag}[x_1 I_{m_1}, \ldots, x_F I_{m_F}], 0 < x_k \in \mathbb{R}\} \tag{6.2}$$

which is isomorphic to a set in Euclidean space.

Condition 2 *There exists $X(\theta) \in \mathbb{X}$, and an operator-valued function $Y(\theta) = Y(\theta)^* \in \mathfrak{L}(\mathcal{K}^m)$, such that*

$$\check{M}(e^{j\theta})^* \begin{bmatrix} X(\theta) & 0 \\ 0 & I \end{bmatrix} \check{M}(e^{j\theta}) - \begin{bmatrix} X(\theta) & 0 \\ 0 & Y(\theta) \end{bmatrix} < 0 \tag{6.3}$$

for all $\theta \in [0; 2\pi[$ *and*

$$\int_0^{2\pi} \mathrm{tr}\, Y(\theta) \frac{d\theta}{2\pi} = \int_0^{2\pi} \|Y^{\frac{1}{2}}(\theta)\|_{\mathbb{HS}} \frac{d\theta}{2\pi} = \int_0^{2\pi} \int_0^h \mathrm{trace}\, Y_\theta(t,t) dt \frac{d\theta}{2\pi} < 1. \tag{6.4}$$

Proposition 6.1 *If condition 2 holds for $X(\theta), Y(\theta) \in \mathfrak{L}(\mathcal{K})$ and $\Delta \in B\Delta_{PTV}$, then the uncertain system is robustly stable and*

$$\sup_{\Delta \in B\Delta_{PTV}} \|\Delta \star M\|_{\mathcal{H}_2} < 1 \tag{6.5}$$

i.e. the system has robust $\mathcal{H}_2$ performance; with the $\mathcal{H}_2$-norm given by (4.12).

Proof. The first block of (6.3) gives for all $\theta \in [0; 2\pi]$

$$\|X(\theta)^{\frac{1}{2}}\check{M}_{11}(e^{j\theta})X(\theta)^{-\frac{1}{2}}\|_{\mathcal{K}\to\mathcal{K}} < 1 \tag{6.6}$$

and analogue to [PT95] 2.3 we get

$$\|\check{X}(e^{j\theta})^{\frac{1}{2}}\check{M}_{11}(e^{j\theta})\check{X}(e^{j\theta})^{-\frac{1}{2}}\|_{\infty} < 1 \tag{6.7}$$

hence $I - \check{M}_{11}\check{\Delta}$ is nonsingular and from lemma 6.1 and 6.2 the system is robustly stable.

Let

$$\bar{M} = \begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \check{M} \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \tag{6.8}$$

then

$$\bar{M}^*\bar{M} - \begin{bmatrix} I & 0 \\ 0 & Y(\theta) \end{bmatrix} \leq 0, \forall\theta \in [0; 2\pi] \tag{6.9}$$

Since $\check{\Delta}(e^{j\theta})$ and $X^{\frac{1}{2}}(\theta)$ commute we can swap $\check{M}$ with $\bar{M}$ i.e. $\check{\Delta}\star\check{M} = \check{\Delta}\star\bar{M}$. From this system we now observe

$$\|z(e^{j\theta})\|^2_{\mathcal{K}} + \|q(e^{j\theta})\|^2_{\mathcal{K}} \leq \|p(e^{j\theta})\|^2_{\mathcal{K}} + \int_0^h w(e^{j\theta})^* Y(\theta) w(e^{j\theta}) dt \tag{6.10}$$

and since $\check{\Delta}$ is contractive

$$\|z(e^{j\theta})\|^2_{\mathcal{K}} \leq \int_0^h w(e^{j\theta})^* Y(\theta) w(e^{j\theta}) dt \tag{6.11}$$

Let $\check{T} = \check{\Delta}\star\check{M}$ and $z = \check{T}w$ we now have with θ suppressed

$$\|\check{T}w\|^2_{\mathcal{K}} \leq \langle Yw, w\rangle_{\mathcal{K}} \tag{6.12}$$

Since this holds for all w,

$$\sum_{i=0}^{\infty} \|\check{T}b_i\|^2_{\mathcal{K}} = \|\check{T}\|^2_{\mathbb{HS}} \leq \sum_{i=0}^{\infty} \langle Yb_i, b_i\rangle_{\mathcal{K}} = \int_0^h \mathrm{trace}\, Y(t,t) dt \tag{6.13}$$

Where $\{b_i\}_{i=0}^{\infty}$ is a basis on $\mathcal{K}$. Where we use the following (scalar case) relation

$$\begin{aligned} \sum_{i=0}^{\infty} \int_0^h Q(t,\tau)\phi_i(\tau) d\tau \phi_i^*(t) &= \int_0^h Q(t,\tau) \sum_{i=0}^{\infty} \phi_i(\tau)\phi_i^*(t) d\tau \\ &= \int_{0^-}^{h^-} Q(t,\tau)\delta(\tau - t) d\tau = Q(t,t) \end{aligned} \tag{6.14}$$

as $\sum_{k=-\infty}^{\infty} h^{-1} e^{jh^{-1}2\pi tk} = \sum_{k=-\infty}^{\infty} \delta(t-kh)$. Finally we integrate across frequency

$$\|\check{\Delta} \star \check{M}\|_{\mathcal{H}_2} \le \left(\int_0^{2\pi} \int_0^h \operatorname{trace} Y_\theta(t,t) dt \frac{d\theta}{2\pi} \right)^{\frac{1}{2}} < 1 \tag{6.15}$$

■

Remark 1 *This is a convex sufficient condition. The "frequency" and "time" dependency of* $\check{M}$ *enters via* $\operatorname{trace} Y_\theta(t,t) \in \mathbb{C}^{m \times m}$. *A finite dimensional approximation can be obtained by gridding. Clearly, this condition also holds for LTI perturbations, however, in section 6.2 the LTI behaviour is also exploited. The kernel of* $Y^{\frac{1}{2}}$ *is not easily obtained in general.*

6.1.3 LTV Perturbation Case

Consider an uncertain system with $M \in \mathfrak{L}_{\mathcal{A}}(\mathcal{L}_2)$.

Proposition 6.2 *If condition 2 holds for a constant matrix* $X \in \mathbb{X}$, *an operator* $Y(\theta) \in \mathfrak{L}(\mathcal{K})$ *and* $\Delta \in B\Delta_{LTV}$, *then the uncertain system is robustly stable and*

$$\sup_{\Delta \in B\Delta_{LTV}} \|\Delta \star M\|_{\mathcal{H}_2} < 1 \tag{6.16}$$

i.e. the system has robust $\mathcal{H}_2$ *performance; with the* $\mathcal{H}_2$*-norm given by (4.16).*

Remark 2 *Proof of sufficiency follows by using the small gain theorem and the previous proof. This is a convex condition.*

We will next show that the condition is also necessary and hence non-conservative when we use the set-based definition of $\mathcal{H}_2$ in (4.17). The idea is to reuse the proof for continuous time robust $\mathcal{H}_2$ [Pag96a]. A main step is to relate the LTI multipliers $\hat{Y}$ and $\check{Y}$ in the two domains.

In the LTI case the following formulations of $\mathcal{H}_2$ norm are equivalent

$$\begin{aligned}\|T\|^2_{\mathcal{H}_2} &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}(\hat{T}(j\omega)^*\hat{T}(j\omega))d\omega \\ &= \sum_{i=1}^{n}\|T\delta e_i\|^2_{\mathcal{L}_2} \\ &= \frac{1}{h}\int_0^h\sum_{i=1}^{n}\|T\delta_\tau e_i\|^2_{\mathcal{L}_2}d\tau \\ &= \frac{1}{h}\int_0^h\text{trace}\left(\int_0^\infty T'(t,\tau)T(t,\tau)dt\right)d\tau \\ &= \frac{1}{h}\sum_{i=0}^{\infty}\text{tr}(\tilde{T}_i^*\tilde{T}_i) \\ &= \frac{1}{h}\frac{1}{2\pi}\int_0^{2\pi}\text{tr}(\check{T}^*(e^{j\theta})\check{T}(e^{j\theta}))d\theta\end{aligned} \tag{6.17}$$

Using the map between $\mathcal{A}_{\mathbb{R}} \to \mathcal{A}$, see commuting diagram 2.21 on page 16, given by

$$\Pi : \hat{Y} \mapsto \Lambda L\mathcal{L}^{-1}\hat{Y}\mathcal{L}L^{-1}\Lambda^{-1} \tag{6.18}$$

we denote the image of Π on $\mathcal{F}$, (4.22), by $\mathcal{A}_{\mathcal{F}}$. As $\hat{Y}(\omega) = \hat{Y}^*(-\omega)$ we have likewise that $\check{Y}^*(e^{j\theta}) = \check{Y}(e^{j\theta})$. Moreover, from (6.17) we have,

Lemma 6.3 *Given a pair $\hat{Y},\check{Y}$ by Π (6.18) then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}(\hat{Y}(j\omega))d\omega = \frac{1}{2\pi}\int_0^{2\pi}\text{tr}(\check{Y}(e^{j\theta}))d\theta \tag{6.19}$$

Proposition 6.3 [2] *Assume $M_{12}, M_{22} \in \mathcal{RH}_2$. The condition 2 holds for a constant matrix $X \in \mathbb{X}$, an operator $Y(\theta) \in \mathfrak{L}(\mathcal{K})$ and $\Delta \in B\Delta_{LTV}$ and $B > 0, \eta > 0$, iff the uncertain system is robustly stable and*

$$\sup_{\Delta \in B\Delta_{LTV}} \|\Delta \star M\|_{W_{\eta,B}} < 1 \tag{6.20}$$

i.e. the system has robust $\mathcal{H}_2$ performance; with the $\mathcal{H}_2$-norm given by (4.17).

Proof. From lemma 6.3 and a little inspection it is clear that (6.3) and (6.4) in condition 2 are lifted frequency versions of (3.17) and (3.18) in condition 1. Now both sufficiency and necessity follows from the proof in [Pag96a]. For the necessity

[2] Remarks in section 3.6

part it is important to note that the S-produre step does not demand time-invariance for M and hence carriages through with it being PTV. ■

One may gain more insight to the problem if an appropriate characterization of $\mathcal{A}_{\mathcal{F}}$ is found in terms of $\check{Y}$.

6.2 Robust $\mathcal{H}_2$ Performance SD for LTI uncertainty

To exploit the LTI behaviour of the perturbation the domain is changed to the SD frequency domain. First by changing to $\mathcal{A}$ and then by a maximum modulus like result for the spectral radius function on $\mathcal{A}$ and making a (discrete) Fourier expansion, the representation is obtained (see [Dul95]). First some notation and results for robust performance for SD systems are given.

With a structure in accordance with Δ_s define

$$\mathcal{X} \triangleq \{\mathrm{diag}(\underline{\Delta}_1, ..., \underline{\Delta}_F) : \underline{\Delta}_k \in \mathbb{C}^{m_k \times m_k}\}$$

the set of LTI perturbations is now refined to be

$$\mathcal{A}_{\mathbb{R}}^{\mathcal{X}} \triangleq \{\hat{\Delta} \in \mathcal{A}_{\mathbb{R}} : \hat{\Delta}(s_0) \in \mathcal{X}, \forall s_0 \in \overline{\mathbb{C}}_+\} \tag{6.21}$$

6.2.1 Frequency Response Functions for SD

It is advantageous to change the domain again so that the structure of the LTI perturbations becomes visible. The representation of the elements in $\mathcal{A}$ on the unit disc is called the frequency response.

Given the complete orthonormal basis $\{\psi_k\}_{k=0}^{\infty}$ on $\mathcal{K}$, with

$$\psi_k(t) \triangleq h^{\frac{1}{2}} e^{jh^{-1}(2\pi\nu_k - \theta_0)t} \text{ for } t \in [0; h[\tag{6.22}$$

and $\theta_0 \in]-\pi; \pi]$ and ν the sequence $\{0, 1, -1, 2, -2, \dots\}$. Introduce $J_{\theta_0} : \mathcal{K} \mapsto \ell_2$ to be the coefficients of the expansion

$$\phi(t) = \sum_{k=0}^{\infty} a_k \psi_k(t) \overset{J_{\theta_0}}{\mapsto} (a_0, a_1, a_2, \dots) \tag{6.23}$$

and define on the unit disc $J(e^{j\theta}) \triangleq J_\theta$.

The frequency response functions are now defined as

$$M(e^{j\theta}) \triangleq J(e^{j\theta})\check{M}(e^{j\theta})J(e^{j\theta})^* \tag{6.24}$$

$$\Delta(e^{j\theta}) \triangleq J(e^{j\theta})\check{\Delta}(e^{j\theta})J(e^{j\theta})^* \tag{6.25}$$

where the latter is block diagonal at each frequency, it takes value in the set:

$$\nabla_{LTI} \triangleq \{\mathrm{diag}(\Delta_0, \Delta_1, \Delta_2, \ldots) : \Delta_k \in \mathcal{X}\}$$

See [Dul95] for state space expressions of M.

Given the frequency response $M(e^{j\theta})$ and $\Delta(e^{j\theta})$ mapping $\partial\mathbb{D}$ into $\mathfrak{L}(\ell_2)$. A main result in [Dul95] states:

Proposition 6.4 *[Dul95] The system M has robust stability to $\Delta \in B\Delta_{LTI}$ iff*

$$\sup_{\theta \in]-\pi;\pi]} \mu_{B\nabla_{LTI}}(M_{11}(e^{j\theta})) < 1$$

6.2.2 LTI Perturbation Case

Let

$$\mathbb{X}_{\mathcal{X}} \triangleq \{X = \mathrm{diag}[x_1 I_{m_1}, \ldots, x_F I_{m_F}], 0 < x_k \in \mathbb{R}\} \tag{6.26}$$

which commutes with $\mathcal{X}$ and the commutator of ∇_{LTI} be

$$\mathbb{X} \triangleq \{\mathrm{diag}(X_0, X_1, X_2, \ldots) : X_k \in \mathbb{X}\} \tag{6.27}$$

Condition 3 *There exists $X(\theta) \in \mathbb{X}$, and an operator-valued function $Y(\theta) = Y^*(\theta) \in \mathfrak{L}(\ell_2)$, such that*

$$M(e^{j\theta})^* \begin{bmatrix} X(\theta) & 0 \\ 0 & I \end{bmatrix} M(e^{j\theta}) - \begin{bmatrix} X(\theta) & 0 \\ 0 & Y(\theta) \end{bmatrix} < 0 \tag{6.28}$$

for all $\theta \in [0; 2\pi]$ and

$$\int_0^{2\pi} \|Y^{\frac{1}{2}}(\theta)\|^2_{\mathbb{HS}} \frac{d\theta}{2\pi} = \int_0^{2\pi} \mathrm{tr}\, Y_\theta \frac{d\theta}{2\pi} < 1. \tag{6.29}$$

Proposition 6.5 *If condition 3 holds for $X(\theta), Y(\theta)$ and $\Delta \in B\Delta_{LTI}$, then the system is robustly stable and*

$$\sup_{\Delta \in B\Delta_{LTI}} \|\Delta \star M\|_{\mathcal{H}_2} < 1 \tag{6.30}$$

Remark 3 *Recall that $\|T\|^2_{\mathbb{HS}} = \mathrm{tr}\, T^*T = \sum < Te_i, Te_i >_{\ell_2}$ then the proof follows the PTV case. This is a convex sufficient condition, however, infinite dimensional in X, Y and frequency.*

A finite dimensional approximation can be found by using the computational framework in [Dul95] analogue to the one deduced for proposition 6.4. The idea is based on extending the uncertainty set to be full block after some n and using loop-shifting in connection with the main-loop theorem. The main obstacle when using the same procedure on proposition 6.5 is that the original perturbation and the fictive performance perturbation are mixed up, however, this is overcomed by reordering and careful bookkeeping.

6.3 Notes and References

Future research may include finding computable state-space conditions as in [Pag96d] for the LTV case. It is expected that necessity results will follow if one replaces PTV uncertainty by a "quasi-PTV" notion (see [Dul95]) and adopts the notion of set-based $\mathcal{H}_2$ performance.

A synthesis method for robust $\mathcal{H}_2$ performance similar to the so-called D-K iteration can be applied, however, this approach will only converge to a local minimum. Hence good starting points are of interest; the $\mathcal{H}_2/\mathcal{H}_\infty$ methods may exploited to obtain such ones, this is studied in appendix B with an brief overview of other multiobjective methods in the SD setting.

Chapter 7

Fault Detection and Isolation

The problem of FDI (Fault Detection and Isolation) has been an active research area in the last two decades. Today there exists various design methods some are based on eigenstructure assignment methods as in [PFC89, PC91, JPC95]. Other methods are based on statistics as in [PFC89]. Also different optimisations techniques as $\mathcal{H}_\infty$ and $\mathcal{L}_1$ optimisation have been applied [MAVV95, EBK94, QG93, NS96, AK93]. These are by far the only fault detection methods; detailed surveys can be found in [Fra96, KT93, Pat94].

In general for FDI the hurdle is to distinguish failures from other disturbances in the presence of model uncertainties in the system. The idea in this work is to formulate the FDI problem in the framework of robust and optimal control; in order to take advantage of the optimisation methods in the aim at obtaining systematic design methods.

In the part on FDI we work with continuous time but it will take only minor changes to apply the results for discrete time or sampled-data systems. To simplify the notation we will omit accents to point out the domain since it will be clear from the context, however, the overload may be indicated by the argument.

7.1 Design Outlook

In the surwey paper [Wil76] covering the initial results in the topic; the three levels constituting the hierarchy of FDI (note the general abuse of FDI as the overall topic) are named

- FD (Fault Detection). A fault is seen in the residual.
- FDI (Fault Detection and Isolation). A fault is seen in the residual in an unique manner.

- FDE (Fault Detection and Estimation). A fault is seen in the residual in an unique manner and its degree of severeness can be estimated.

In the case of zero or almost zero threshold this gives rise to a serie of fundamental problems respectively denoted exact and almost exact; this requires that the filter can be designed so that no or almost no disturbance enters in the residual.

The former classes of problems are studied in [MVW89] where necessary and sufficient conditions are given. The punchline is that given a number of possible faults the range of the map from each fault wanted to be dectected and isolated to the residual must not overlap the unobservable subspaces of the likewise maps of the other faults; all taken in turn.

The latter are studied in [NSSS99] also with some extentions to the results in [MVW89]. We note that these geometric results resemble the ones of decoupling; details are given in the references, see also [Won85]. Furthermore, one can at most detect and isolate as many faults as the number of measurements.

However, the residuals are often contaminated by noise and disturbances which cannot be removed by a filter, therefore, a nonzero threshold is needed. The subtlety of finding a threshold is studied in chapter 8. The approach is based on formulating the FDE problem as a filtering problem, see section 3.8, to be solved by optimisation methods. The degree of solvability is merely reflected by the sizes of performance indices. This also holds for the study of simultaneous design of controller and fault detector in chapter 9.

We have the standing assumption that the faults allowing for a zero or almost zero threshold are handled as such. Thus these are excluded from the original problem before the methods given in this work are applied. So that only a minimal number of thresholds has to be found. Therefore the full solution is the combination. We do note that the methods also work without the exclusion.

The FDE problem is studied since it generates the more infomation and shows the full potential of this observer alike optimisation approach. The method given here can also be applied to the simpler FD and FDI problems by introducing appropriate logic functions of the estimates.

7.2 The Nominal FDI Setup

The FDI setup will be given in the following. Consider the following system G given by:

$$\begin{array}{rclcl} \dot{x} & = & Ax & + & B_d d & + & B_f f \\ y & = & C_y x & + & D_{yd} d & + & D_{yf} f \end{array} \tag{7.1}$$

or as transfer functions:

$$\begin{aligned} y &= (C_y(sI-A)^{-1}B_w + D_{yd})d &+& (C_y(sI-A)^{-1}B_f + D_{yf})f \\ &= G_{yd}(s)d &+& G_{yf}(s)f \\ &= \begin{bmatrix} G_{yf} & G_{yd} \end{bmatrix} \begin{bmatrix} f \\ d \end{bmatrix} \end{aligned}$$

Where d is a disturbance signal vector and f is a fault signal vector. It is without loss of generality to consider a system without a control input in connection with FDI design. We will assume ”compatible” dimensions of vectors and matrices.

It will be assumed that the fault signal f and the disturbance signal d are scaled such that the norm of these two vectors are 1, i.e.

$$\|f\| \leq 1, \;\; \|d\| \leq 1$$

and the scaling functions, S_f, S_d, are included in the two transfer functions G_{yf} and G_{yd}. This scaling is only done for simplification of the following analysis and will not affect the quality of fault detection.

The general system formulation given in (7.1) will be used throughout this and the following chapter. By the selection of B_f and D_{yf}, actuator, sensor and internal fault can be handled in this setup. E.g. actuator and sensor faults can be described in the general form in (7.1) by using

$$\begin{aligned} f &= \begin{bmatrix} f_a \\ f_s \end{bmatrix} \\ B_f &= \begin{bmatrix} B_u & 0 \end{bmatrix} \\ D_{yf} &= \begin{bmatrix} 0 & I \end{bmatrix} \end{aligned}$$

where f_a is the actuator fault signal, f_s is the sensor fault signal and B_u is the injection matrix for the feedback control signal, see [NS97].

A filter is now applied to give a residual vector r from the measurement signal vector y. Let the filter be given by $F(s)$, then the residual vector is given by $r = F(s)y$ or

$$r = F(s)\,(G_{yd}(s)d + G_{yf}(s)f) \tag{7.2}$$

A block diagram of (7.2) is shown in fig. 7.2.

If an optimisation design method is to be applied for the design of $F(s)$, it will in general be more convenient to use the residual error given by:

$$e_{nom} = W(s)f - F(s)\,(G_{yd}(s)d + G_{yf}(s)f) \tag{7.3}$$

where $W(s)$ is the desired transfer function from f to r. Scaling S_f must also be included in W. A block diagram of (7.3) is shown in fig. 7.2. Note that the usual fault estimation problem is a special case of the above with $W(s) = IS_f$.

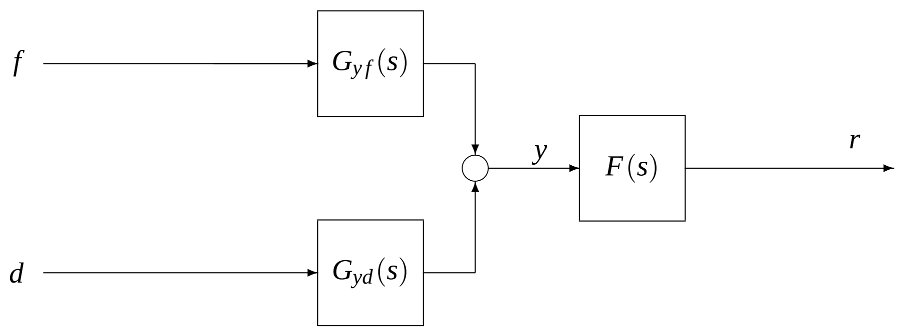


Figure 7.1: Block diagram for the fault residual signal r

7.3 The Uncertain FDI Setup

The nominal FDI setup described in section 7.2 will be extended in this section to uncertain systems. Customarily system uncertainty is transformed into an external disturbance input and considered as disturbance [PC96]. Where it is shown that the disturbance and the uncertainties effect the residual vector in the same way and it is therefore difficult to seperate these two effects. In a number of cases, this description is valid for the fault detector design. However, if we truly want to optimise the fault detector with respect to the uncertainties, the description of uncertainties as external disturbance is inadequate. One reason is that the external disturbance is assumed to be uncorrelated with the system. This is not the case when uncertainty is described as external disturbance. Including an explicit discription of the uncertainties in connection with fault detection has been done in [MAVV95, SGN97]. It is shown in [SGN97], that the design of fault detectors and feedback controllers are coupled in the uncertain case whereas they are uncoupled in the nominal case, see table 9.2 on page 92. It is therefore well motivated to look at the fault detection problem for uncertain systems.

Consider the following uncertain system G_{unc} given by:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{zw} & G_{zf} & G_{zd} \\ G_{yw} & G_{yf} & G_{yd} \end{bmatrix} \begin{bmatrix} w \\ f \\ d \end{bmatrix} \tag{7.4}$$

where z and y are the external output signal and the measurement output signal, respectively. The input signals are the external input signal w, the fault signal f and

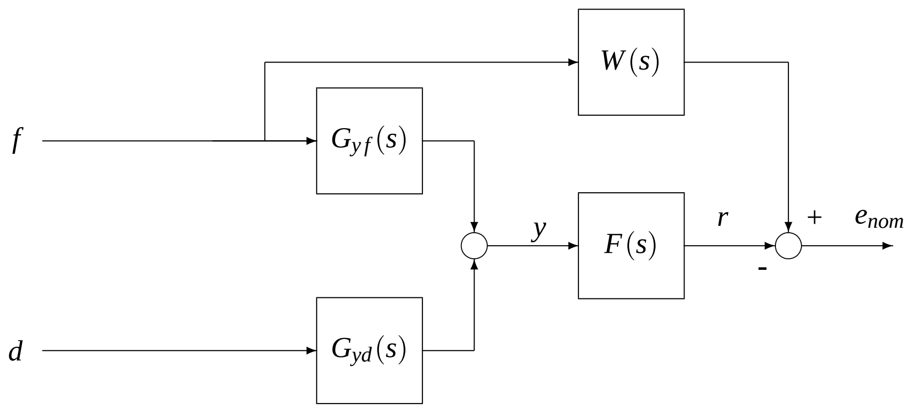


Figure 7.2: Block diagram for the residual error e_{nom}

the disturbance input signal d.

The external input signal w and the external output signal z are closed through the uncertain perturbation block $\Delta(s)$, i.e.

$$w = \Delta(s)z \tag{7.5}$$

It is assumed that the perturbation block $\Delta(s)$ is scaled such that $\|\Delta\| \leq 1, \forall\omega$ and the scaling function is included in G_{unc}. There is no assumption on the structure of Δ.

Based on the results in Section 8.1, we consider the transfer function for the residual error e_{unc}. In the uncertain case, the residual error is given by:

$$e_{unc} = W(s)f - F(s)(G_{yw}w + G_{yf}(s)f + G_{yd}(s)d) \tag{7.6}$$

in the open loop case, i.e. the uncertain feedback loop defined by (7.5) is not closed. When this loop is closed, the residual error given by (7.6) takes the following form:

$$\begin{aligned} e_{unc} = {} & (W(s) - F(s)G_{yw}\Delta S_\Delta G_{zf} - F(s)G_{yf})\, f \\ & - (F(s)G_{yw}\Delta S_\Delta G_{zd} + F(s)G_{yd}(s))\, d \end{aligned} \tag{7.7}$$

or by using (7.3):

$$e_{unc} = e_{nom} - F(s)G_{yw}\Delta S_\Delta (G_{zf}f + G_{zd}d) \tag{7.8}$$

where $S_\Delta = (I - G_{zw}\Delta)^{-1}$.

The fault estimation error given by (7.7) is shown in fig. 7.3.

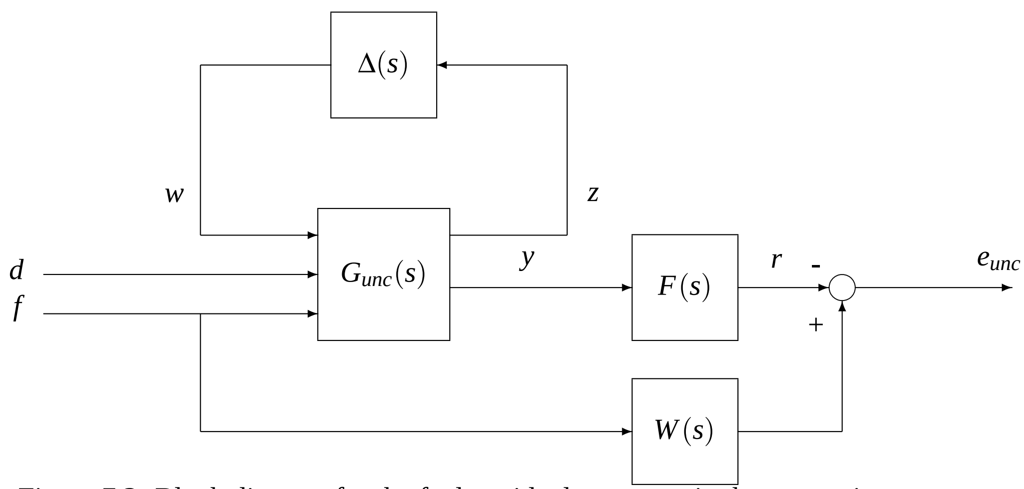


Figure 7.3: Block diagram for the fault residual error e_{unc} in the uncertain case

Chapter 8

Norm Based Design of Fault Detectors

It is apparent from the citations in chapter 7 that the FDI problem is well studied. However, the design of a fault detector is only half of a complete FDI design. The other part is the selection/calculation of a threshold for the fault detector. This part may be the most important part as, generally, it is more difficult to find a threshold value than designing the corresponding fault detector. One of the reasons is that there does not exist any systematic way to obtain a threshold value. However, the threshold selection problem has been considered in a number of papers see [ENAR88, Pat94, DG96, DF91, Hor88, Wei93, DGF93, QG93, FK96].

The selection of the threshold is quite closely related to the ability to reject disturbances and model uncertainties and at the same time to be able to get a reasonably detection of fault signals. Based on this fact, different performance indices for the fault detection problem can be found in the literature which reflects the above properties. Examples of these indices can be found in [DG96, QG93]. The index derived in [DG96] results in an optimisation problem which can not be solved in a systematic way. The reason is that the index include a maximisation of the smallest gain of a transfer function. The index applied in [QG93] includes only a optimisation of the maximal gains of some transfer functions, which can be done in an optimal way. In this case an $\mathcal{H}_\infty$ method has been applied. The problem with this index is that it only focuses on the largest gain of the transfer function from fault signal to estimated fault signal. However, this is not so important, the smallest gain is more important. Comparing this gain with the maximal gain from disturbance to estimated fault signal gives an indication of how small a fault signal that can be detected. This is precisely what the index derived in [DG96] takes care of.

The main topic for this chapter is to investigate the selection of thresholds and performance indices in connection with FDI and estimation i.e. FDE. The investigation of threshold selection will be carried out using norms on the related transfer functions. The results presented can be applied in connection with all types of fault detectors.

Based on an analysis of the fault detection problem, an index is derived. The index reflects the design constraint on the transfer functions from disturbance and fault signals to the estimation signals. Further, it is only based on the maximal gains of the involved transfer functions. This makes it easier to derive an optimal design method for the FDI problem.

8.1 An Analysis of the Nominal Case

First an analysis based on the estimate of the residual signal given by (7.2) is considered followed by an analysis based on the residual error given by (7.3).

8.1.1 Applying the Residual Signal

Consider the fault detection signal given by (7.2). Now let the norms of the two transfer functions FG_{yf} and FG_{yd} be as follows:

$$\|FG_{yf}\| \leq \alpha, \;\; \|FG_{yd}\| \leq \beta \tag{8.1}$$

where $\alpha \geq \beta$ otherwise it is generally not possible to distinguish fault from disturbance.

It is clear that the ratio between β and α is very important in connection with the quality of the fault detector. A large $\frac{\beta}{\alpha}$ will in general make fault detection very difficult because the disturbance effect on the residual vector is quite large. For obtaining a good fault detection, we need to have the ratio $\frac{\beta}{\alpha}$ small such that the effect from the disturbance on the residual vector is minimised. An obvious design condition is therefore to minimise this ratio. However, a minimisation of the ratio $\frac{\beta}{\alpha}$ will not necessarily result in a good fault detection. The reason is that we are only looking at the largest gain from fault signal to residual signal. Further, the direction for this gain might not even be a realistic situation, i.e. the related fault vector might not appear at any time.

To overcome this problem, the smallest gain of the FG_{yf} can be applied instead, i.e. $\|FG_{yf}\|_- = \alpha$, see e.g. [DG96] and [SPC97]. Wherein an evaluation function $\|\cdot\|_e$, which may not be a norm, is used to define $\|M\|_- = \inf_{\|x\|_e=1}\|Mx\|_e$, this is neither a norm. The problem with using the smallest gain of FG_{yf} is that $\|\cdot\|_-$ is not a norm. This will make the following optimisation more difficult than necessary. Further, standard optimisation methods known from optimal and robust control cannot be applied in a straightforward manner.

Instead of using the residual signal r directly we will use the residual error (7.3).

8.1.2 Applying the Residual Error

Let the norms of the two transfer functions $W - FG_{yf}$ and FG_{yd} be as follows:

$$\|W - FG_{yf}\| \leq \alpha, \quad \|FG_{yd}\| \leq \beta \tag{8.2}$$

In the matching problem G_{yf} may be strictly proper and/or have zeros which restricts W. Moreover, we may only have a certain frequency region of interest. The combined frequency region where matching is wanted is denoted Ω and given by

$$\Omega = ([\omega_l^1, \omega_u^1], \dots, [\omega_l^n, \omega_u^n])$$

Let the smallest desired gain W be

$$\delta := \inf_{\omega \in \Omega} \underline{\sigma}(W(j\omega))$$

With a proper choice of W one can find F with *maximal estimation error* $\alpha \leq \delta$.

We will use the lemmas in section 2.7.1 to relate the smallest gain from fault to residual with the residual error. Assuming the induced norm in (8.2) to be the $\mathcal{H}_\infty$-norm and using lemma 2.4 on page 20 we have

$$\delta - \alpha \leq \underline{\sigma}(FG_{yf}(j\omega)) \quad \forall \omega \in \Omega \tag{8.3}$$

Likewise in the case without frequency restrictions where any induced norm applies. Let $\delta = \inf_{\|x\|=1} \|Wx\|$. In which case lemma 2.2 gives

$$\delta - \alpha \leq \inf_{\|x\|=1} \|FG_{yf}x\| \tag{8.4}$$

From the triangle-inequality we always have an upper bound by $\|FG_{yf}\| \leq \|W\| + \alpha$.

By studying the residual error instead of the residual signal directly, it has been possible to give both an upper bound and, more important, a lower bound for the largest and smallest gain of FG_{yf}. Based on the norm bound we can setup under which condition it is possible to detect a fault signal. We will use (8.4) since the application of (8.3) is parallel.

It will in the following be assumed that the matrix $W(s)$ is given as a diagonal matrix. When the matrix $W(s)$ is selected as a diagonal matrix, the analysis is derived with respect to how sensitive the ith residual signal r_i is to the ith fault signal f_i and how insensitive it is to the other fault signals.

First we consider the residual vector, when only disturbance appear in the system. Then the maximal norm of r is the given as:

$$max_{d,f=0}\|r\| = max_d\|FG_{yd}d\| = \|FG_{yd}\| =: \beta$$

since d is scaled such that $\|d\| \leq 1$. Let the threshold value be denoted Γ. Hence, it is clear that we will not get false alarms for $\Gamma \geq \beta$.

We have the following four cases for fault detection depending on the selection of the threshold value:

1. fault signals $\|f\|_\Omega$ larger than $\frac{\Gamma+\beta}{\delta-\alpha}$ are detected independent of the noise.

2. fault signals $\frac{\beta}{\delta-\alpha} \leq \|f\|_\Omega \leq \frac{\Gamma+\beta}{\delta-\alpha}$ may be detected. The disturbance signal may obstruct a detection.

3. fault signals detected in the range $\frac{\Gamma}{\delta-\alpha} \leq \|f\|_\Omega \leq \frac{\beta}{\delta-\alpha}$ may not be fault signals for $\Gamma < \beta$.

4. fault signals $\|f\|_\Omega \leq \frac{\Gamma}{\delta-\alpha}$ cannot be detected in general unless the disturbance signal “helps”.

This also holds in the case without frequency restriction; for ease of notation we simply write $\|\cdot\|_\Omega$ in any case. The above analysis based on norms of the two transfer functions FG_{yf} and FG_{yd} is quite superficial. It is e.g. not taken into account that only a limited number of faults are assumed to appear at the same time. In the analysis given above, all faults can appear at the same time.

An important thing to note with respect to the above analysis is that the norm of the disturbance transfer function FG_{yd}, β, is very important in connection with the selection of the threshold value. If we want to avoid false alarms, the threshold value must be selected to β.

A new performance index will be derived in the next section based on the analysis results given in this section.

8.2 Performance Index

As pointed out in section 8.1, it is quite obvious to select the threshold value equal to or larger than β. A way to design the FDI filter is to reduce the norm of the transfer function from d to r as much as possible; to reduce the threshold without getting any false alarms. However, this will in general not result in a minimisation of the smallest fault signal that can be detected. There is a trade off between good fault detection and good disturbance rejection. This limitation in filtering has been considered in [SBG97].

Instead, we need to consider a performance index which directly takes care of a minimisation of the disturbance effect on the estimated fault signal and at the same time maximises the estimated fault signal. Such a performance index has been formulated in e.g. [DG96] and [ENAR88]. The performance index in [DG96] is given by:

$$J = 2\inf \frac{\|FG_{yd}\|}{\|FG_{yf}\|_-} \tag{8.5}$$

The performance index in (8.5) gives the smallest fault signal that is guaranteed to be detected. This can be seen by using the results from section 8.1. Therefore, a design method which will minimise the index J in (8.5) needs to be applied. However, the optimisation of the performance index is difficult as pointed out in [DG96], because the denominator is not a norm.

Instead of applying the above index, we can formulate an index based on the residual error. Motivated by section 8.1 we introduce the following index:

$$\|f\|_\Omega \geq \eta = 2\inf_F \frac{\|FG_{yd}\|}{\delta - \|W - FG_{yf}\|} \tag{8.6}$$

Faults $\|f\|_\Omega \geq \eta$ are guaranteed to be detected, however, $\eta \geq J$ since the bound may be conservative. It is important to note that the index in (8.6) only includes norms of transfer functions. This will make an optimisation of the index much more simple than an optimisation of the index given by (8.5). It should be pointed out that the index in (8.6) is based on no false alarms.

In the SISO case, where $|W| \approx 1$ for $\omega \in \Omega$, we get that it is required that $|G_{yf}| > 2|G_{yd}|$ to guarantee for any fault detection, i.e. $\eta < 1$. Another case is when the disturbance is measurement noise, i.e. $G_{yd} = k = constant$. To obtain fault detection, it is required that $|G_{yf}| > 2k$ in the frequency range where fault detection is wanted.

By using the index in (8.6), we do not only have the possibility to optimise the size of the fault signal that is guaranteed to be detected, it is also possible to give a structure of the residual vector r. This is important to improve the performance of the FDI filter. By structure of the residual vector, we mean how the residual signals depend of the fault signals. If W is selected as a diagonal matrix as above, r_i will be sensitive to f_i and insensitive to the other fault signals. This will in general not be the optimal selection due to performance limitations. A more useful selection of W would be to use e.g. a triangle matrix in the index. There is no systematic way to select a good W. One possibility is to optimise W in an iterative design. Such a design can be done in the following way. Select a suitable matrix W, followed be a design of a fault detector for this matrix. Then optimise W for fixed fault detector etc. It should be noted that we need to require that the smallest gain of W is larger than a specified level for for $\omega \in \Omega$. If this is not required, the design method will give $F(s) = 0$ and $W(s) = 0$ as the result.

8.2.1 Performance Index with Two System Norms

The performance index (8.6) makes sense whenever maximal estimation error is bounded by an induced norm, therefore, it makes sense to consider

$$\eta = 2 \inf_F \frac{\|FG_{yd}\|_a}{\delta - \|W - FG_{yf}\|} \tag{8.7}$$

where $\|\cdot\|_a$ is any system norm. Hence, the disturbance attenuation may be with respect to white noise, i.e. $\mathcal{H}_2$, whereas the estimation error minimisation may be over energy or amplitude induced norm, i.e. $\mathcal{H}_\infty, \ell_1$.

An optimisation method for (8.6) is given in section 8.4, however the setup may also be applied for the multiobjective design, see [SGC97, ED97].

8.2.2 Performance Index for a Fixed Number of Faults

The performance indices given above are all based on the assumption that all fault signals can appear at the same time. However, in practice there will generally be an assumption on the number of faults that can appear at the same time. This means that the performance index given in (8.6) can be conservative. We might optimise the performance index for some fault vectors f which cannot appear.

Consider a FDI problem where m faults can appear. It is assumed that only $p, (p < m)$ faults can appear at the same time. To take care of this in the performance index, we can rewrite the index. For doing this, let the transfer function $W - FG_{yf}$ be partitioned as:

$$W - FG_{yf} = [g_{w1}, \cdots, g_{wm}] \tag{8.8}$$

The worst gain of $W - FG_{yf}$ when only p fault signals can appear at the same time is then given by:

$$\|W - FG_{yf}\| = max\{\sum_{i_1=1}^{i_1=m} \cdots \sum_{i_p=p}^{i_p=m} \|g_{wi_1} \ldots g_{wi_p}\|\},\ i_1 \neq \cdots \neq i_p \tag{8.9}$$

The performance index in (8.6) is now given by:

$$\|f\|_\Omega \geq \eta = 2 \inf_F \frac{\|FG_{yd}\|}{\delta - max\{\sum_{i_1=1}^{i_1=m} \cdots \sum_{i_p=p}^{i_p=m} \|g_{wi_1} \ldots g_{wi_p}\|\},\ i_1 \neq \cdots \neq i_p} \tag{8.10}$$

One way to transform the new performance index given in (8.10) into a setup which can be solved is to stack the problem, see [SN97]. Then the same method, as will be described in section 8.4, can be applied.

An important case is when only one fault signal appears; this will in general be the case. The performance index in (8.10) is then given by:

$$\|f\|_\Omega \geq \eta = 2 \inf_F \frac{\|FG_{yd}\|}{\delta - max\{\sum_{i_1=1}^{i_1=m} \|g_{wi_1}\|\}} \tag{8.11}$$

8.2.3 Performance Index for Individual Residual Signals

Until now, the fault detection design problem has only been considered as one single design problem. However, it is also possible to split the design problem into m separate designs. This is done by considering the design of fault detectors for each residual signal separately. Let $W(s)$, $F(s)$ and G_{yf} be partitioned as follows:

$$\begin{aligned} W(s) &= \begin{bmatrix} w_1(s) \\ \vdots \\ w_m(s) \end{bmatrix} \\ F(s) &= \begin{bmatrix} F_1(s) \\ \vdots \\ f_m(s) \end{bmatrix} \\ G_{yf}(s) &= \begin{bmatrix} G_{yf,1} & \cdots & G_{yf,m} \end{bmatrix} \end{aligned} \tag{8.12}$$

Based on (8.12), the design problem for the design of a fault detector for the ith residual error signal is then given by:

$$e_i = (w_i(s) - F_i(s)G_{yf})f - F_i(s)G_{yd}d \tag{8.13}$$

The performance index given in (8.6) can not be applied directly for this design problem, because δ will in general be equal to zero in this case. First, we need to rewrite the equation for the ith residual signal. Assume that we want to have the ith residual signal sensitive to the first j fault signal and insensitive to the other $m-j$

fault signals. Then (8.13) can be written as:

$$
\begin{aligned}
e_i &= \left(\begin{bmatrix} w_{i,1} & \cdots & w_{i,j} \end{bmatrix} - F_i(s)\begin{bmatrix} G_{yf,1} & \cdots & G_{yf,j} \end{bmatrix}\right)\begin{bmatrix} f_1 \\ \vdots \\ f_j \end{bmatrix} \\
&\quad - F_i\left(\begin{bmatrix} G_{yf,j+1} & \cdots & G_{yf,m} \end{bmatrix}\begin{bmatrix} f_{j+1} \\ \vdots \\ f_m \end{bmatrix} + G_{yd}(s)d\right) \\
&= (\bar{W}_1(s) - F_i(s)\bar{G}_{yf}(s))\bar{f} - F_i(s)\bar{G}_{yd}\bar{d}
\end{aligned}
\tag{8.14}
$$

where

$$
\begin{aligned}
w_i(s) &= \begin{bmatrix} w_{i,1} & \cdots & w_{i,m} \end{bmatrix} \\
&= \begin{bmatrix} \bar{W}_1 & \bar{W}_2 \end{bmatrix} \\
\bar{G}_{yf}(s) &= \begin{bmatrix} G_{yf,1} & \cdots & G_{yf,j} \end{bmatrix} \\
\bar{G}_{yd}(s) &= \begin{bmatrix} G_{yf,j+1} & \cdots & G_{yf,m} & G_{yd} \end{bmatrix} \\
\bar{f} &= \begin{bmatrix} f_1 \\ \vdots \\ f_j \end{bmatrix} \\
\bar{d} &= \begin{bmatrix} f_{j+1} \\ \vdots \\ f_m \\ d \end{bmatrix}
\end{aligned}
$$

By this rewriting of e_i the performance index given by (8.6) can now be applied for the residual error given by (8.14).

8.3 An Analysis of the Uncertain FDI Case

Based on the results derived in section 8.1 and 8.2 for the nominal case, equivalent results will be given in this section for the uncertain case. One of the key results from section 8.1 was that if the fault residual error is applied, the performance index given by (8.6) is only based on norms. Consequently we only consider the fault residual error in this section.

Let the norm of the two transfer functions from (7.7) be given by:

$$
\|W - F(G_{yw}\Delta S_\Delta G_{zf} - G_{yf})\| \leq \alpha, \;\; \|F(G_{yw}\Delta S_\Delta G_{zd} + G_{yd})\| \leq \beta \tag{8.15}
$$

The above norms are not easy to calculate in general, especially not if the uncertainty block has a structure. One method is to apply the μ analysis for the calculation of the norms, see [ZDG95].

Based on the norms of these two transfer functions, we can give an equation for the threshold value which needs to be optimised. In the case when we do not want any false alarms, the threshold value needs to be selected as:

$$\Gamma \geq \beta$$

and the smallest fault signal guaranteed to be detected is given by the performance index:

$$\|f\|_\Omega \geq \eta = 2 \inf_F \frac{\|F(G_{yw}\Delta S_\Delta G_{zd} + G_{yd})\|}{\delta - \|F(G_{yw}\Delta S_\Delta G_{zf} + G_{yf})\|} \tag{8.16}$$

The optimisation problem in the uncertain case is therefore exactly the same as in the nominal case given by (8.6).

In connection with this optimisation problem, it should be mentioned that the problem of selection of thresholds for the uncertain FDI problem has also been considered in [DG96]. But the results derived in [DG96] are based on the fault estimation signal. Further, only systems with open loop uncertainties are considered. Instead of using the description by feeding back the uncertain block as described by (7.5), it is instead assumed that the input to the uncertain block is bounded. This will in general give an inadequate description of the uncertain part of the system. On the other side, the calculation of the norms is less complex than calculation of the norms in (8.15).

8.4 Design of Threshold

This section is denoted the study of (8.6) which includes an optimisation of the norms of two transfer functions. The design method for the design of filters for fault detection which will be presented in the following will not directly give an optimal value of the performance index in (8.6). To optimise the index in (8.6) an iteration process needs to be applied.

In this section we will apply a standard $\mathcal{H}_\infty$ design method for the design of the filter. Consider the nominal case shown in fig. 8.1. For obtaining a reasonable design, weight functions at both the input and at the output must be included. This is shown in fig. 8.1.

The weight matrices at the input signals should reflect the frequency contents of the two signals d and f.

Assume that the two weight matrices W_d and W_f at the input shown in fig. 8.1 are included in the state space description for the system given by (7.1). Further,

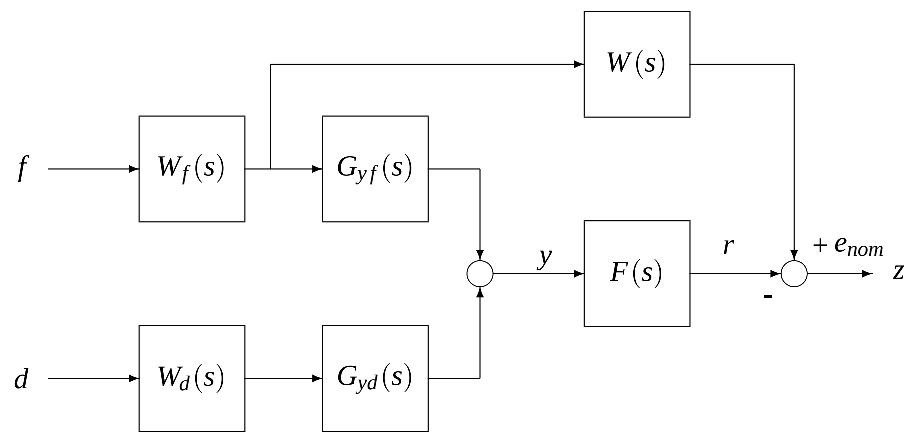


Figure 8.1: The nominal fault detection setup with weight matrices

let the weight matrix W have the state space description (A_w, B_w, C_w). It is without loss of generality to assume that the weight matrix is strictly proper. It will in general be at low frequencies that we want to detect the fault signals. A complete state space description is given by:

$$
\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{x}_w \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x \\ x_w \end{bmatrix} + \kappa \begin{bmatrix} B_d \\ 0 \end{bmatrix} d + \begin{bmatrix} B_f \\ B_w \end{bmatrix} f \\
z &= \begin{bmatrix} 0 & C_w \end{bmatrix} \begin{bmatrix} x \\ x_w \end{bmatrix} - Iu \\
y &= \begin{bmatrix} C_y & 0 \end{bmatrix} \begin{bmatrix} x \\ x_w \end{bmatrix} + \kappa D_{yd} d + D_{yf} f
\end{aligned}
\tag{8.17}
$$

or in a compact form:

$$
\begin{aligned}
\dot{\bar{x}} &= \bar{A}\bar{x} + \kappa \bar{B}_d d + \bar{B}_f f \\
z &= \bar{C}_z \bar{x} - Iu \\
y &= \bar{C}_y \bar{x} + \kappa D_{yd} d + D_{yf} f
\end{aligned}
$$

where the standard setup has been applied. A block diagram of the standard setup is shown in fig. 8.2.

Note that a scalar parameter $\kappa \in \mathbb{R}^+$ has been included in (8.17) to weight the two transfer functions from f and d to z against each other. κ gives the trade-off between good fault detection and good disturbance rejection. κ needs to be selected such that the performance index is minimised.

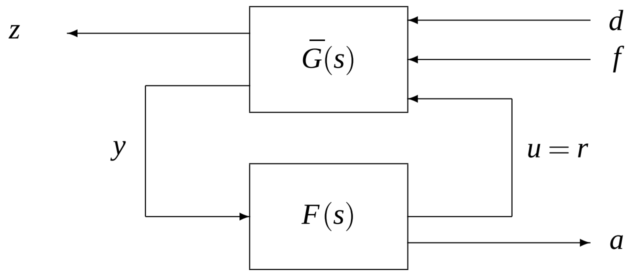


Figure 8.2: The fault detection problem in the standard formulation

The solution of (8.17) gives

$$\|\kappa F G_{yd}\; W - F G_{yf}\| \leq \gamma(\kappa) \tag{8.18}$$

which clearly implies

$$\eta = 2 \inf_F \frac{\|F G_{yd}\|}{\delta - \|W - F G_{yf}\|} \leq \frac{2\gamma(\kappa)}{\kappa(\delta - \gamma(\kappa))} \tag{8.19}$$

with $\delta = \inf_{\|x\|=1} \|Wx\|$, which is an upper bound on the smallest fault signal f that is guaranteed to be detected for small γ. Using the upper bound given by (8.18) together with the fact that the fault signal is scaled such that $\|f\| \leq 1$ gives the following condition on $\gamma(\kappa)$ for fixed κ to guarantee η to be less than 1:

$$\gamma < \frac{\kappa\delta}{2+\kappa} \tag{8.20}$$

If W_f and W_d are not included in (7.1), then (8.18) and (8.19) are given by:

$$\|\kappa F G_{yd} W_d \;\; (W - F G_{yf}) W_f\| \leq \gamma(\kappa)$$

and

$$\eta = 2 \inf_F \frac{\|F G_{yd} W_d\|}{\delta - \|(W - F G_{yf}) W_f\|}$$

Note that δ needs to be calculated from WW_f instead of W for $W_f \neq I$.

Based on the above setup, it is possible to design an $\mathcal{H}_\infty$ filter for the fault detection problem. Changing the scalar parameter κ, it is possible to optimise the design index given by (8.6), i.e. an iterative optimisation of the index.

It is also possible in an equivalent way to set up the fault detection problem in the uncertain case and apply an $\mathcal{H}_\infty$ optimisation method. If the design method is applied directly, the result can be conservative. The reason is that the uncertain block together with the performance specifications has a structure, which in general results in conservative controllers/filter, see [ZDG95]. Instead, an μ optimisation needs to be applied for removing the conservatism from the filter, see [NS97, NS96].

8.5 Example

The fault detection problem which will be considered in this example is based on a model of a jet engine. Here we do not comply with the standing assumption that the faults allowing for a zero or almost zero threshold are handled as such; since we only want to emphasis the new part.

The state space description of the jet engine is given by, [VTL87]:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where the four matrices are given by:

$$\begin{aligned} A &= \begin{bmatrix} -3.370 & 1.636 \\ -0.325 & -1.896 \end{bmatrix} \\ B &= \begin{bmatrix} 0.586 & -1.419 & 1.252 \\ 0.410 & 1.118 & 0.139 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.731 & 0.786 \end{bmatrix} \\ D &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.267 & -0.025 & -0.146 \end{bmatrix} \end{aligned}$$

Both actuator fault as well as sensor fault can appear. This means that there are 6 possible fault signals, 3 actuator faults and 3 sensor faults. It is not possible to detect and isolate all 6 fault signals at the same time, because we have only 3 measurement signals. We will therefore consider two FDI problems in the following, one setup where the actuator faults are considered and one setup where the sensor faults are considered. The setup for the two cases are given by (7.1). With disturbance signal given by:

$$
\begin{aligned}
B_d &= \begin{bmatrix} 0.100 \\ 0.100 \end{bmatrix} \\
D_{yd} &= \begin{bmatrix} 0.200 \\ 0.200 \\ 0.200 \end{bmatrix}
\end{aligned}
$$

and the two matrices B_f and D_{yf} are given by:

- Actuator fault problem

$$
\begin{aligned}
B_f &= B \\
D_{yd} &= D
\end{aligned}
$$

- Sensor fault problem

$$
\begin{aligned}
B_f &= 0 \\
D_{yd} &= I
\end{aligned}
$$

Applying the setup given in section 8.4; with W_d as a weight function on the disturbance signal and W as the desired transfer function from fault signal f to residual signal r, shown in fig. 8.3. The frequency range where we want fault detection and isolation is given by $\Omega = [0,\ 2] rad/sec$.

Figure 8.3: W_d solid line and W dashed line.

The standard $\mathcal{H}_\infty$ design method will be applied for the design of fault detectors. The direct matrix D_{21} has not full rank in the actuator fault case. D_{21} is perturbed to get full rank so the standard $\mathcal{H}_\infty$ design method can be applied. This is without loss of generality in this example, because we will here focus on the performance of the FDI filters. If we instead want to implement the FDI filters, design methods as LMI based $\mathcal{H}_\infty$ or the singular $\mathcal{H}_\infty$ approach need to be applied. These methods do not require full rank of the two direct matrices D_{12} and D_{21}.

The FDI filters for the two problems will be designed by minimising the $\mathcal{H}_\infty$ norm constraint given by (8.18) for a given κ.

The value of κ will be selected such that a given performance index is minimised. The FDI filters will be designed with respect to that all three fault signal can appear at the same time. Evaluation of the FDI filters will be done by using the performance index given by (8.6) and the index given by (8.10) or (8.11). We will consider the three different performance indices, the case where all 3 fault signal can appear simultaneously as well as the cases where 2 and 1 fault signals can happen at a time.

First, consider the actuator fault detection problem. An FDI filter is calculated for different values of the scalar constant κ and the resulting filters are evaluated with respect to 3 different performance indices with respect to the number of faults that is assumed to appear at the same time. The values of the 3 performance indices as functions of κ are shown in fig. 8.4.

A performance index equal to 1 in the figure indicates that the performance index is either larger than 1 or that the performance index gives a negative value. Filters which give these results can not be applied in practice. It will either not be possible to detect any fault signal with guarantee or the estimation error will be more than 100%.

If we consider the FDI sensor problem, we will see the same again. The sensor FDI design problem is shown in fig. 8.5.

From fig. 8.4 and 8.5, we can see that the decision about how many faults that are assumed to appear simultaneously is very important. A good choice for κ when we assume that only 1 or 2 faults can appear simultaneously may not be a good choice for the case where 3 faults appear simultaneously. To use the information given in fig. 8.4 and fig. 8.5 in a constructive way in connection with the selection of the scalar parameter κ is to setup a new index given by:

$$J = \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3$$

where $J_i, i = 1,2,3$ are the three performance indices applied and $\alpha_i, i = 1,2,3$ are scalar parameters that must reflect the relative relationship between the cases where 1, 2 and 3 fault appears. This means that we will have $\alpha_1 \geq \alpha_2 \geq \alpha_3$. In

Figure 8.4: The actuator FDI problem. J_3 (3 faults) dash dotted line, J_2 dashed line and J_1 solid line.

Figure 8.5: The sensor FDI problem. J_3 (3 faults) dash dotted line, J_2 dashed line and J_1 solid line.

this example, it is quite clear that κ must be selected around 100 in the actuator case and around 200 – 500 in the sensor case. In general, it is not so easy to select κ. It is not enough to look at the performance indices, we also need to look at the derived fault detector. Conditions about gains etc. in the fault detector must also be taken into consideration for the selection of κ.

8.6 Notes and References

In the example, it has been shown that the choice of performance index is very crucial for the selection of the final FDI filter.

We have only discussed the selection of the threshold equal to or larger than the norm of the transfer function from disturbance to residual error. This selection will avoid false alarms. If the threshold is reduced then false alarms may occur. An analysis needs to be made in every single case to give the level of the threshold if a number of false alarms can be accepted. Such a reduction of the threshold level will not change the performance index.

Chapter 9

Design of Controller and Fault Detector

The integrated FDI problem was studied in [NJM88] using the four-parameter controller and the control/diagnostic objectives were captured by transfer matrices. In a direct line therefrom follows [AK93] which gives a discrete time-domain closed-loop approach.

The integrated approach has the advantage compared to model-based methods with independent FDI modules that the interactions between the control system and the diagnostic module are taken into account as well as the overall control/diagnostic system has an order which is at most the order of the plant plus the order of weigthings in the $\mathcal{H}_\infty$ and the $\mathcal{H}_2$ cases.

Here we state the problem in a standard 2×2 setup form, see section 3.2, and give a state-space description essentially having the same objectives as in [NJM88], i.e. closed-loop stability and minimisation of certain transfer matrices, and also have the option to include uncertainties.

9.1 State-Space Setup for FDI and Control

We consider the following FDLTI system in the compact notation

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

It is assumed that (A, B_2) is stabilisable and (C_2, A) is detectable.

The setup is depicted in figure 9.1 as a standard rejection problem, where G is the plant and K is the controller. Sensor and actuator faults have been added as

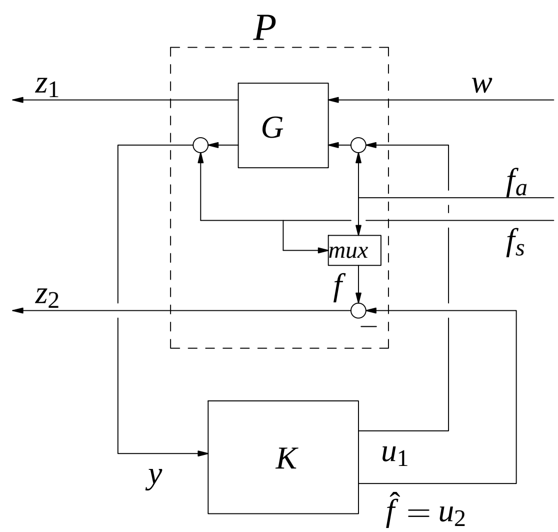


Figure 9.1: Control system with Diagnostics

inputs and the diagnostic output (estimate) $\hat{f} \triangleq u_2$ is obtained by introducing an extra to-be-controlled output with

$$z_2 = I \begin{bmatrix} f_a \\ f_s \end{bmatrix} + \begin{bmatrix} 0 & -I \end{bmatrix} u \, .$$

The following stacked signals are used,

$$f \triangleq \begin{bmatrix} f_a \\ f_s \end{bmatrix}, \; u \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, z \triangleq \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, v \triangleq \begin{bmatrix} w \\ \begin{bmatrix} f_a \\ f_s \end{bmatrix} \end{bmatrix}, u = Ky$$

A state-space describtion for the augmented system reordered into standard form is given by

$$\begin{array}{rclclclcl} \dot{x} & = & Ax & + & B_1 w & + & \begin{bmatrix} B_2 & 0 \end{bmatrix} \begin{bmatrix} f_a \\ f_s \end{bmatrix} & + & \begin{bmatrix} B_2 & 0 \end{bmatrix} u \\ z_1 & = & C_1 x & + & D_{11} w & & & + & \begin{bmatrix} D_{12} & 0 \end{bmatrix} u \\ z_2 & = & & & & & I \begin{bmatrix} f_a \\ f_s \end{bmatrix} & + & \begin{bmatrix} 0 & -I \end{bmatrix} u \\ y & = & C_2 x & + & D_{21} w & + & \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} f_a \\ f_s \end{bmatrix} & & \end{array} \tag{9.1}$$

However, the faults are only expected in a certain frequency region [1], hence we filter the signal z_2 through

$$W(s) = \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array}\right]$$

and obtain

$$P = \left[\begin{array}{c|cc} \begin{bmatrix} A & 0 \\ 0 & A_W \end{bmatrix} & \begin{bmatrix} B_1 & \begin{bmatrix} B_2 & 0 \end{bmatrix} \\ 0 & B_W \end{bmatrix} & \begin{bmatrix} B_2 & 0 \\ 0 & -B_W \end{bmatrix} \\ \hline \begin{bmatrix} C_1 & 0 \\ 0 & C_W \end{bmatrix} & \begin{bmatrix} D_{11} & 0 \\ 0 & D_W \end{bmatrix} & \begin{bmatrix} D_{12} & 0 \\ 0 & -D_W \end{bmatrix} \\ \begin{bmatrix} C_2 & 0 \end{bmatrix} & \begin{bmatrix} D_{21} & \begin{bmatrix} 0 & I \end{bmatrix} \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right]. \tag{9.2}$$

9.2 Discussion of Design Methods

Now consider the transfer matrix from v to z of the augmented system,

$$T_{zv} = P \star K = \begin{bmatrix} T_{z_1 w} & T_{z_1 f} \\ T_{z_2 w} & T_{z_2 f} \end{bmatrix}$$

where we note for some suitable norm that

- $\|T_{z_1 w}\|$ small implies disturbance rejection.
- $\|T_{z_1 f}\|$ small means that undetected failures are not disastrous.
- $\|T_{z_2 w}\|$ small secures no false alarms.
- $\|T_{z_2 f}\| \to 0 \;\Rightarrow\; u_2 \to f$ i.e a good estimate.

This means we want to solve the straightforward and simple problem of minimising,

$$\|T_{zv}\|. \tag{9.3}$$

In the case where undetected failures are not disastrous the following multiobjective problem may be considered

$$\|T_{zw}\| < \gamma_1 \text{ and } \|T_{z_2 f}\| < \gamma_2 \tag{9.4}$$

[1] In fact, the same holds for disturbances and model uncertainty. For simplicity we did not include these weightings, but the extension is straightforward, (or regard them already absorbed into G)

which will lead to a less conservative design than (9.3). One method is to use [KRS93] where the system needs to comply with additional constraints, the system should be left invertible and a less restrictive rank condition on the B-matrix.

However, in both cases the problem is inheritly singular which somewhat limits the number of optimisation methods of choice or at least a (somewhat tricky) regularisation is needed.

Next we discuss methods for the problem in 9.3, depending on the signals (energy bounded, amplitude bounded, white noise) one performs ($\mathcal{H}_\infty$, ℓ_1, $\mathcal{H}_2$) a design for which various known approaches can be used (e.g. [ZDG95, DDB95]).

Using LMIs [SIG97] for solving the $\mathcal{H}_\infty$ disturbance rejection problem $\|T_{zv}\|_\infty$, give at most a design with order $n-1$ and the singular structure is handled smoothly. LMIs offer in addition a possibility to compute low order controllers.

In the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ cases[2] the simultaneous design of fault detection and control has the advantage that the resulting control/estimator has order n whereas a controller design followed by adding a fault estimator using an output estimator and a filter gives order $3n$, and does not take into account that control and FDI might be contrary objectives.

Actuelly, the latter has been clearfied in the line of work [SGN97, SG96, NS97] see table 9.2, however, some order reduction should be applied when performed separate. In other words the same observer may serve both the controller and the filter in a coupled design. Read $\mathcal{H}_\infty$ for any unstructured and μ for structured optimisation.

Filter	Nominal Perf.		Robust Perf.	
Controller	Design	Optim.	Design	Optim.
N. Performance	Separate	$\mathcal{H}_\infty$	Coupled	$\mathcal{H}_\infty,\ \mu$
R. Stability	Separate	$\mathcal{H}_\infty$	Coupled	μ
R. Performance	Separate	$\mu,\ \mathcal{H}_\infty$	Coupled	μ

Table 9.1: FDI seperation

9.3 Example

We give an example illustrating the simultaneous design of $\mathcal{H}_\infty$-control and estimation of faults in the actuators and the sensors. It should be noted that the design

[2] In the ℓ_1 case this order inflation tendency will in general be even worse since there is no bound on the controller order

could be done seperately see table 9.2 but for the order. The system considered is stable, 4th order, and has 1 disturbance, 1 actuator, 1 output, and 2 sensors. The state-space data is given by the compact system matrix

$$G(s) = \left[\begin{array}{cccc|cc} -4.873 & 0.758 & 9.541 & 8.763 & 0.483 & 0.003854 \\ 4.673 & -5.385 & 10.94 & -435.2 & 1.498 & 9.072\cdot 10^{-4} \\ -0.3003 & -0.6001 & -5.334 & -20.23 & 9.889 & -2.865\cdot 10^{-5} \\ 0.03852 & 0.7446 & 0.5703 & -83.28 & 0.1282 & -13.96 \\ \hline 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 \end{array}\right]$$

The purpose is to design a combined controller and estimator such that good disturbance attenuation from w to z together with good estimate of f_a and f_s are obtained. The output is filtered by: $W(s) = \frac{10}{s+10} I_{2\times 2}$.

The derived $\mathcal{H}_\infty$ minimisation problem is solved by the LMI-method and resulted in a controller of order 1 with $\|T_{zv}\|_\infty = 0.57$.

The $\mathcal{H}_\infty$ norm of the elements in T_{zv} was calculated to $\|T_{z_1w}\|_\infty = 0.29$, $\|T_{z_1f}\|_\infty = 0.001$, $\|T_{z_2w}\|_\infty = 0.33$, and $\|T_{z_2f}\|_\infty = 0.49$. The norm of T_{z_1w} and T_{z_2f} indicates that good disturbance attenuation and fault estimation could be expected. However, since T_{z_2f} is a 2×2 transfer matrix more could be said about the fault estimation when calculating the $\mathcal{H}_\infty$ norm of each element in T_{z_2f}. $\|(T_{z_2f})_{11}\|_\infty = 0.49$, $\|(T_{z_2f})_{22}\|_\infty = 0.02$, and the off-diagonal elements are close to zero. This means that the estimates of f_a and f_s are nearly isolated, i.e. does not influence each other, and that a very fine estimate of f_s could be expected. The $\mathcal{H}_\infty$ norm of $(T_{z_2f})_{11}$ was reached near DC and goes to zero beside DC, so the steady-state estimate of f_a is expected to be approximately 50% of the actual value.

Fig. 9.2 shows a simulation of the system with a actuator failure and fig. 9.3 shows a simulation with a sensor failure. In both cases the disturbance, w, was coloured noise with an amplitude about 0.15V. The figures are in good accordance with the expectation.

Figure 9.2: Actuator and sensor estimate for actuator failure

Figure 9.3: Actuator and sensor estimate for sensor failure

Chapter 10

Conclusion

The robust control framework has been applied and extended in this thesis to obtain new results on sampled-data systems and fault detection and isolation. We will briefly discuss these and suggest some further research directions; a list of contributions is given on page 6.

In the lifting technique framework for SDS taking the intersample behaviour into account the robust $\mathcal{H}_2$ problem has been given a thorough study. Encompassing extentions to the well-known LTR procedure and the recent results on robust $\mathcal{H}_2$ performance to the SD setting. Thereby adding to the merits of the lifting technique.

For completeness related multiobjective sampled-data designs are sketched and discussed. These are mainly casted in the fast discretisation setting.

From a practitioner's point of view fast discretisation offers a simple (approximative) way to handle the SD problems in general. However, this is not without cavaets, most notably by higher computional burden.

In other words the direct SD (exact) discretisations is prefered in the theoretical sense as well as for computional schemes.

Using the notion of a smallest gain a new (possible conservative) norm based index quantifying the smallest fault guaranteed to be detected is deducted. It opens for optimisation based design methods and is given in various forms. The integrated FDI and controller design problem is setup to exploit their interaction.

Thus a full FDI design is facilitated by optimisation methods.

Further Directions:

These are mostly given were appropriate within the text, but we stress a few below.

- An overview study with comparison between the various $\mathcal{H}_2$ generalisations and likewise robust $\mathcal{H}_2$ performance conditions.

SDS:

- Find state-space conditions as in [Pag96d] for the robust $\mathcal{H}_2$ performance SD LTV case.
- For mixed SD $\mathcal{H}_\infty/\ell_1$ and $\mathcal{H}_2/\mathcal{H}_\infty$ derive a bound, see remark 12 on page 103.

FDI:

- Optimisation design for W to exploit the freedom see page 76.
- A design method as in section 8.4 where LMIs for both smallest and largest gain are combined.

Finally, we like to point out, remark 4 on page 97, that there is no a priori check condition available for when a direct SD design is needed or not.

A Redbook

A.1 Sampled-Data Design

This brief redbook is meant as a summary on how and when to do a sampled-data design. It is based on results given in the comprehensive literature[1] most notably [CF95]. A similar discussion is partly given in [DD95]. The notation follows chapter 4 and the redbook should be read thereafter.

Generally the intersample behaviour is of interest when the sampling rate is limited which to some extent is always the case. Even if the rate is free and therefore can be chosen corresponding to a frequency much higher than the bandwidth of the closed-loop, say a hundred times, there may be complications.

We will assume that an appropriate lowpass prefilter is absorbed in the plant. Given specifications herein a choise of norm and a plant.

Problem A.1 *Solve the SD controller problem in fig. 4.1 i.e. find a* $K = HK_dS$ *which assures nominal stability and performance.*

The SD problem can be approached indirectly or directly. Stability is always implicit with a direct design, but requires a check in the indirect case.

Remark 4 *As there is no a priori check condition available for when a direct SD design is needed or not; simply apply it every time*[2].

When we write SD design we implicitly assume a continuous-time measure hence taking the intersample behaviour into account. For the indirect designs the SD norm has to be analysed, i.e. recalculated in SD sense, hence we introduce a little notation.

- $\gamma_c = \|G(j\omega) \star K^c(j\omega)\|$. Continuous-time optimal controller.

  Continuous SD design

- $\gamma_c^{si} = \|G(j\omega) \star HK_d{}^{si}S\|$. Step Invariant transformation of K^c to $K_d{}^{si}$.
- $\gamma_c^{bi} = \|G(j\omega) \star HK_d{}^{bi}S\|$. BIlinear transformation of K^c to $K_d{}^{bi}$.

  Discrete SD design

- $\gamma_d^{si} = \|G(j\omega) \star HK_d{}^{fd_1}S\|$. Discretise G by the step invariant transformation, find discrete-time controller $K_d{}^{fd_1}$.

[1] in the author's interpretation

[2] Some direct design methods are implemented in Matlab TM

- $\gamma_d^{fd} = \|G(j\omega)\star HK_d{}^{fd_n}S\|$. Discretise G by fast discretisation, find discrete-time controller $K_d{}^{fd_n}$.

  Direct SD design

- $\gamma_{sd} = \|G\star HK_dS\|$.

Assume the sampling rate is fixed

Then there is a certain (restriction) gap between γ_c and γ_{sd}. In fig. A.1 the performance indices for continuous SD design using the step invariant and bilinear transformation[3] are compared with the direct design index.

Figure A.1: Continuous SD design

In fig. A.2 the performance indices for discrete SD design using step invariant[4] and fast discretisation are compared with the direct design index.

Figure A.2: Discrete SD design

Remark 5 *Empirically, the continuous SD design, especially using the bilinear transformation, is better than the discrete SD design. However, as the number of oversamples[5], n, increases using fast discretisation this index converges to the direct design index.*

That is fast discretisation approximates the direct SD design, $\gamma_d^{fd} \to \gamma_{sd}$ as $n \to \infty$, but this convergens is very slow and the computational burden grows as the

[3] the bilinear transformation is often better than the step invariant
[4] same as fast discretisation with $n = 1$
[5] the computational burden is soon larger than the direct design

number of input and output grows linearly with n for the optimisation problem to be solved.

An advantage of fast discretisation is that it is very simple to implement, further it is the only known way to solve the SD problem for the ℓ_1-norm (4.26) and mixed problems with it.

Remark 6 *If no direct SD design method is known*[6] *fast discretisation is then the better choice.*

Next we will assume that the sampling rate is free to design

As the sampling rate is increased (fast sampling) then the performance of the continuous-time optimal (analog) controller is recovered by the continuous SD design, $\gamma_c^{si}, \gamma_c^{bi} \to \gamma_c$ as $h \to 0$. Furthermore, there exists a h_0 so that for $h < h_0$ stability is assured.

Fast sampling is not a "free lunch", it is costly and gives rise to FWL (Finite Word Length) problems etc. Furthermore,

Remark 7 *Rules of thumb are inadequate. Say that the rate is chosen corresponding to a frequency even a hundred times higher than the bandwidth of the closed-loop, there may anyway be complications, (arbitrary) difference in norm.*

E.g. due to resonance peaks [DD95].

Remark 8 *A simple analysis is better. Calculate the SD norm* $\|G \star H{K_d}^i S\|$ *with the given controller* ${K_d}^i$ *(or approximate it by fast discretisation) and compare it to the continuous-time optimal controller* γ_c*. Judge if the archived performance is acceptable; remember there is a restriction gap.*

Note that the gap between γ_c and γ_{sd} also vanish when the sampling is fast.

A direct SD design often needed when the sample rate is low or the specifications are against the underlaying continuous-time nature, ill-posed (e.g. Dead-beat or no cost on actuators) or the system has resonance peaks. But there is no safe clue to when they are not needed. Hence comply with the synthesis and analysis remarks[7] 4, 6 and 8 which are also good for the uncertain case.

Finally, note that the problem of given a performance level find an appropriate h is also superiorly solved by the direct SD methods.

[6] implemented

[7] Remark 4 is an open problem, but it seems it will remain so

A.2 Fast Discretisation

Fast discretisation, section 4.5.1, due to [KA92], formulas from [CF95], in Matlab:

```
function [dsys]=sdfast(sdsys,nmeas,ncon,h,N)

%[dsys]=sdfast(sdsys,nmeas,ncon,h,N)
%
%   Inputs:
%      SDSYS    -   interconnection matrix for control design
%                    (continuous time) see Matlab mu-tools TM
%      NMEAS    -   # controller inputs (np2)
%      NCON     -   # controller outputs (nm2)
%      h        -   sampling period of the controller to be designed
%      N        -   number of over samples
%
%      From 8.3 SD [CF95]
%      MLR 1995

if nargin~=5,
   error('usage:%[dsys]=sdfast(sdsys,nmeas,ncon,h,N)');
   return
end

[typ,p,m,n]=minfo(sdsys);

if typ~='syst',
   error(' Plant is not a system')
   return
end

[a,b,c,d]=unpck(sdsys);

b1=b(1:n,1:m-ncon);              b2=b(1:n,m-ncon+1:m);
c1=c(1:p-nmeas,1:n);             c2=c(p-nmeas+1:p,1:n);
d11=d(1:p-nmeas,1:m-ncon);       d12=d(1:p-nmeas,m-ncon+1:m);
d21=d(p-nmeas+1:p,1:m-ncon);     d22=d(p-nmeas+1:p,m-ncon+1:m);

% check data
if rank( diag( exp(eig(a)*h) + ones(n,1) ) ) < n
   error(' System is pathalogically sampled')
   return
   end

% Fast discretisation, first discretise at the slow rate
[ad,b2d]=c2d(a,b2,h);
% Discretise at the fast rate
[af,bt]=c2d(a,[b1,b2],h/N);

b1f=bt(:,1:m-ncon);            b2f=bt(:,m-ncon+1:m);
at=eye(n);
bv1=b1f;                       cv1=c1;
dv11=[d11 zeros(p-nmeas,(N-1)*(m-ncon))];
dv21=[d21 zeros(nmeas,(N-1)*(m-ncon))];
dv12=d12;
for i=1:N-1
   dv11=[dv11;[c1*at*b1f dv11((i-1)*(p-nmeas)+1:i*(p-nmeas), ...
                                   (m-ncon)+1:N*(m-ncon))]];
   dv12=[dv12;c1*at*b2f+dv12((i-1)*(p-nmeas)+1:i*(p-nmeas),:)];
   at=at*af;
   bv1=[at*b1f bv1];
   cv1=[cv1;c1*at];
end
dsys=pck(ad,[bv1 b2d],[cv1;c2],[dv11 dv12; dv21 d22]);
```

B Multiobjective Sampled-Data Design

The aim here is a discussion of $\mathcal{H}_2/\mathcal{H}_\infty$ mixed control and related multiobjective control in the SD setting. This serves to find good starting points for D-K iteration like iteration for robust SD $\mathcal{H}_2$ performance and

Remark 9 *It is hard quantify a good controller into a single number/norm.*

In other words to capture the multiobjective nature of many control problems. This may to some extent be handled by "the art" of choosing weighting matrices, some clues are given in [SP96].

Remark 10 *The optimal solution generally has an extreme behavior, hence choose a solution only close to the optimal solution.*

This remark is most notable for $\mathcal{H}_\infty$ design; anyhow it is general. However, the methods mentioned below are often suboptimal in nature.

In the line of work pointing towards robust $\mathcal{H}_2$ performance [BH89, ZGBD94, DZGB94, KR91] the controller provides nominal performance and robust stability. Furthermore, the approach is restricted to either one input ($p = w$) or one output ($q = z$) in fig. 3.3.

A connection to a similar $\mathcal{H}_2/\mathcal{H}_\infty$ problem, see [Pag96a], and the analysis condition 2 on page 59 is,

Proposition B.1 *Partition the input* $w = \begin{bmatrix} w_\infty & w_2 \end{bmatrix}'$ *for the system* $M = \begin{bmatrix} M_\infty & M_2 \end{bmatrix}$ *then the following are equivalent:*

1. *condition 2 holds for* $X = I$ *and some* $Y(\theta)$
2. *For* $B > 0, \exists \eta > 0 : \sup\left(\|Mw\|^2_{\mathcal{L}_2} : \|w_\infty\|^2_{\mathcal{L}_2} + \frac{1}{m}\|w_2\|^2_{\mathcal{L}_2} \leq 1, w_2 \in W_{\eta,B} \right)$

Proof. See proof of 6.20 and [Pag96a]. ■

Here we prefer to study the 2×2 RP problem. This may be attacked using multiobjective control $\mathcal{H}_2/\mathcal{H}_\infty$ as in [SGC97] and $\ell_1/\mathcal{H}_\infty$ [SB98] and $\mathcal{H}_2/\ell_1$ with more in [ED97].

B.1 Multiobjective Sampled-Data Design

The $\mathcal{H}_2/\ell_1$ problem is similar to the $\mathcal{H}_2/\mathcal{H}_\infty$ problem, but the uncertainty is here bounded in the ℓ_1-norm. The "dual" problem is also solved [ED97].

The $\mathcal{H}_\infty/\ell_1$ problem is similer to RP $\mathcal{H}_\infty$ (μ), but the uncertainty is here bounded in the ℓ_1-norm. Combines time and frequency requirements [SB98], it is a convex problem.

For the multiobjective sampled-data setup we use the notation in chapter 4. The system is given by

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{B.1}$$

Consider a and b channels in w and z

$$\hat{G}(s) = \left[\begin{array}{c|ccc} A & B_1^a & B_1^b & B_2 \\ \hline C_1^a & D_{11}^{aa} & D_{11}^{ab} & D_{12}^a \\ C_1^b & D_{11}^{ba} & D_{11}^{bb} & D_{12}^b \\ C_2 & 0 & 0 & 0 \end{array}\right] \tag{B.2}$$

and

$$K(\lambda) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{B.3}$$

This setup gives 4 combinations of closed-loops. However, we prefer the somewhat more flexibel setup where the closed-loop is

$$T = G \star HK_dS \tag{B.4}$$

and the appropriate input/output channels are choosen by the matrices L_j, R_j as

$$T_j = L_j T R_j \tag{B.5}$$

i.e. $w = R_j w_j$ and $z_j = L_j z$, see [BB91]. Let T_a, T_b be two closed-loops as above with the same dynamics. Then a generic (two) multiobjective problem with two norms is given by

Problem B.1 *Find controller K,*

$$\inf_{K \text{ internally stabilising}} \|T_a\|_a \tag{B.6}$$

subject to:

$$\|T_b\|_b \leq \gamma \tag{B.7}$$

This can be restated using the result of parameterisations of all stabilising controllers [YJB76, Fra87]. This is a main idea[1] in [SB98, ED97].

For the multiobjective sampled-data problems the following is apparent

[1] more details are out of the scope

Remark 11 *A solution based on fast discretisation, section 4.5.1, is straightforward, the cavaet is finding bounds like (4.26) from [BDP93].*

Moreover, one do not expect any better for the cases with ℓ_1 included.

Remark 12 *Some motivation is found in that the result like (4.26) from [BDP93] holds for general $\mathcal{L}_p$-induced spaces. The $\mathcal{L}_2$ version is proved in [KA92] which a better convergence rate than $\frac{1}{n}$ was obtained.*

Though it seems worth to conjecture[2] that these bounds exist, it will require elaborated versions of proof depending on case.

For the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem we expect an "exact" discretisation as this is the case in the pure forms the problems. We will outline[3] a such line of attack in below section with some cavaets.

B.2 $\mathcal{H}_2/\mathcal{H}_\infty$ Multiobjective Sampled-Data Design

The synthesis (and analysis) method in [SGC97] (may handle even more objectives) requires

1. Common dynamics A matrix and state.

2. Single Lyapunov matrix.

The conservatism introduced by 2. is justified by the tracable solution see reference. Further, the optimal $\mathcal{H}_2/\mathcal{H}_\infty$ controller is infinite dimensional, but the single Lyapunov matrix gives one with the same order as the plant.

Whereas in the SD setting 1. complicates things a bit, since usual $\mathcal{H}_2$ and $\mathcal{H}_\infty$ discretisation give different A matrices. But 1. can be accomplished in to ways

- fast discretisation (i.e. approximation), leaves a question of convergence rate and the problem becomes computational heavier as n is increased. However, it is a well motivated approach, see remark 12.

- $\mathcal{H}_\infty$ SD without Loop-Shifting.

Pick $\mathcal{H}_2$ and $\mathcal{H}_\infty$ channels in w and z. Assume D_{11} is zero for two reasons; to have a finite $\mathcal{H}_2$ norm and have a simple way to express $\mathcal{H}_\infty$-norm.

For the $\mathcal{H}_2$ channels the usual $\mathcal{H}_2$ discretisation is applicable,

[2] At least M. A. Dahleh with his deep ℓ_1 insight expect so for $\mathcal{H}_\infty/\ell_1$

[3] this is from work under preparation

$$\bar{G}(\lambda) = \left[\begin{array}{c|cc} A_d & \bar{B}_1 & B_{2d} \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right] \tag{B.8}$$

However, for the $\mathcal{H}_\infty$ channels we need a representation with the same dynamics A_d, this is the remaining task in next section.

Besides, one must find

- Linearising change of variable discrete version (similar [SGC97] (33)).
- Obtain discrete versions of ([SGC97] (v),(42)).

B.2.1 $\mathcal{H}_\infty$ SD without Loop-Shifting

Recall that the lifting step gives a system of the form

$$\tilde{G} = \left[\begin{array}{c|cc} A_d & \tilde{B}_1 & B_{2d} \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{B.9}$$

the operators are given in (4.6) and (A_d, B_{2d}) is the usual discretisation (step invariant) of (A, B_2).

$$T = \left[\begin{array}{c|c} A_{cl} & \tilde{B}_{cl} \\ \hline \tilde{C}_{cl} & \tilde{D}_{11} \end{array}\right] \triangleq \tilde{G} \star K_d \tag{B.10}$$

$\tilde{D}_{11}$ is compact as $D_{11} = 0$. So is T therefore its norm equals its largest singular value.

We will next use the "$\mathcal{H}_\infty$ norms, Riccati equations, symplectic matrix, LMIs" lemma in [PD93].

A starting point for finding the representation is [CF95], but the pitfall here is the implicit use of loop-shifting.

- We need to find the representation.

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