Loop Transfer Recovery with an H_∞ Optimality Criterion.

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Abstract.

A formulation of the Loop Transfer Recovery (LTR) design problem using an $\mathcal{H}_\infty$ (sub-) optimality criterion is presented in this paper and state space formulae are given as its solution. The LTR problem is formulated as an $\mathcal{H}_\infty$ norm constraint for a recovery error which can be defined in two ways: either as a recovery error of the sensitivity function or as a recovery error of the input-output transfer function. Applying output feedback controllers we proceed from the recovery errors to an $\mathcal{H}_\infty$ state space formulation. The control problems corresponding to the two recovery error types are given as two different $\mathcal{H}_\infty$ state space problems. Each $\mathcal{H}_\infty$ problem is decomposed into the well known regular problem and a totally singular problem. The dynamics of the obtained controllers are at most of order $2n$. Moreover, the $\mathcal{H}_\infty$/LTR method handles both minimum phase as well as non minimum phase systems in a common framework.

1. Introduction.

In the original setting, LTR was intimately related to LQG design methods of full order observers [Doyle and Stein 1981] for the design of robust observer based control systems. Later, however, other design methods such as eigenstructure assignment techniques for full order observer based controllers [Søgaard-Andersen 1989] etc. have also proved to be efficient LTR design methods.

LTR design is the last step in a three step procedure for the design of robust observer based controllers. In the first step, the design specifications, i.e. robust stability and performance specifications, are formulated. The second step is a state feedback (target) design, which has to satisfy the design specifications, followed by the LTR-step where the target loop is recovered by using a dynamic measurement based controller [Athans 1986].

Recently, [Moore and Tay 1989] pioneered a new approach to the LTR problem. Their approach is based on an $\mathcal{H}_\infty$ optimization of a suitably chosen recovery function for a controller structure, the Q–observer, consisting of a standard full order observer with an additional dynamic feedback structure attached at the estimation error node. The approach presented in [Moore and Tay 1989] suffers from a number of drawbacks. First, the approach handles only the minimum phase part of a system, and for systems with RHP zeros no guaranteed norm bounds can be given for the overall system. Moreover. the resulting controller orders turn out to be at least $2n$, which is unnecessarily large and due to the fact that the authors use frequency domain methods rather than the state space methods, which has meanwhile proven more powerful.

In this paper we present an alternative approach to the $\mathcal{H}_\infty$/LTR design problem based on the standard $\mathcal{H}_\infty$ setup. The design method is based on the main results in the Ph.D. thesis by Stoustrup [1990] also reported in [Stoustrup and Niemann 1990] where a more thorough treatment of the method can be found. Further, an equivalent $\mathcal{H}_\infty$/LTR approach based on observer based controllers can be found in [Stoustrup 1990 and Stoustrup and Niemann 1990]. The $\mathcal{H}_\infty$/LTR problem is formulated in Section 2 based on the recovery error concept [Niemann et al. 1990]. Two $\mathcal{H}_\infty$/LTR design problems are formulated based on optimizations of the sensitivity recovery error and the input-output recovery error. The two problems are solved in Secs. 3 and 4, respectively by using the socalled singular $\mathcal{H}_\infty$ approach [Stoorvogel 1989, Stoorvogel and Trentelman 1990], which make it possible to calculate the $2n$'th order controllers (n'th order for minimum phase systems) in a straightforward manner. A discussion is made in Section 5.

2. The $\mathcal{H}_\infty$/LTR Problem Formulation.

In this section we shall shortly introduce the Loop Transfer Recovery (LTR) design method. Further, the $\mathcal{H}_\infty$/LTR design problems will be formulated as standard $\mathcal{H}_\infty$ problems.

2.1. Loop Transfer Recovery (LTR).

Let us consider a finite dimensional, linear, time invariant (FDLTI) plant model, represented by a state space realization $(A, B, C, 0)$:

$$\sum : \begin{cases} \dot{x} = Ax + Bu \\ z = Cx \end{cases} \tag{2.1}$$

with transfer function $G(s) = C(sI-A)^{-1}B$. where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $z \in \mathbb{R}^p$, and A, B and C are matrices of appropriate dimensions. The system is assumed to be stabilizable, detectable and left invertible. Moreover, we shall make the technical assumption, that $\Lambda(A) \cap \mathbb{C}^0 = \emptyset$. Note, however, that this can always be achieved by applying a preliminary static output feedback. Furthermore, this preliminary static output feedback can be chosen arbitrarily small.

To design a controller for the system Σ by the LTR methodology, we first determine a (static) state feedback, the target design, which satisfies our design specifications. The design specifications, such as robustness and performance, are assumed to be reflected to the input node [Athans 1986, Stein and Athans 1987]. The resulting target loop transfer function becomes $G_{TFL} = F(sI-A)^{-1}B$, where F is the target (state feedback) design. Second, the LTR step is performed, where the target design is recovered over the range of frequencies by a dynamic compensator $C(s)$, giving a full loop transfer of the form $G_I(s) = C(s)G(s)$.

The associated sensitivity and input output transfer functions are given by:

$$\begin{aligned} &S_{TFL}(s) = (I - G_{TFL}(s))^{-1},\ S_I(s) = (I - G_I(s))^{-1} \\ &G_{IO,TFL}(s) = G(s)S_{TFL}(s),\ G_{IO}(s) = G_I(s)S_I(s) \end{aligned} \tag{2.2}$$

Using these transfer functions, we can define the sensitivity recovery error and the input-output recovery error.

DEFINITION 2.1. *The sensitivity recovery error E_S is defined by:*

$$\begin{aligned} E_S(s) &= S_{TFL}(s) - S_I(s) \\ E_{IO}(s) &= G_{IO,TFL}(s) - G_{IO}(s) \end{aligned} \tag{2.3}$$

Other types of recovery errors are considered in [Niemann et al. 1990].

2.2. The $\mathcal{H}_\infty$/LTR Setup.

The $\mathcal{H}_\infty$ standard philosophy is to define a fictitious plant Σ^* which is a realization of the compound transfer function on which the $\mathcal{H}_\infty$ constraint is posed, rather than of the plant itself (see e.g. [Francis and Doyle 1987]). Consider the closed loop system in Fig. 2.1.

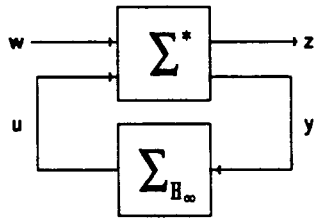

Fig 2.1. The $\mathcal{H}_\infty$ Standard Problem.

Denote the transfer function of the controller Σ_{H_∞} by $Q(s) \in \mathcal{RH}_\infty$. Then the closed loop transfer function from w to z becomes:

$$G_{zw}(s) = T_{zw}(s) + T_{zu}(s)Q(s)(I-T_{yu}(s)Q(s))^{-1}T_{yw}(s) \quad (2.4)$$

where $T_{zw}(s)$, $T_{zu}(s)$, $T_{yu}(s)$ and $T_{yw}(s)$ are the open loop transfer functions from $w \mapsto z$, $u \mapsto z$, $u \mapsto y$ and $w \mapsto y$, respectively.

Now, with $G_{zw}(s)$ being the sensitivity recovery error $E_S(s)$ or the input-output recovery error $E_{IO}(s)$ introduced in Definition 2.1., we have:

LEMMA 2.2. *A linear fractional transformation of $E_S(s)$ and $E_{IO}(s)$ in the form (2.4) are given by the following transfer functions:*

E_S:	E_{IO}:
$T_{zw}(s) = (I-F(sI-A)^{-1}B)^{-1}-I$	$T_{zw}(s) = C(sI-A-BF)^{-1}B-C(sI-A)^{-1}B$
$T_{zu}(s) = -I$	$T_{zu}(s) = -C(sI-A)^{-1}B$
$T_{yu}(s) = C(sI-A)^{-1}B$	$T_{yu}(s) = C(sI-A)^{-1}B$ (2.5)
$T_{yw}(s) = C(sI-A)^{-1}B$	$T_{yw}(s) = C(sI-A)^{-1}B$

Using the expressions in Lemma 2.2 for $E_S(s)$ and $E_{IO}(s)$ we get the following two $\mathcal{H}_\infty$/LTR problem formulations:

PROBLEM 1. *Let $\gamma > 0$ be given. Find, if possible, a* FDLTI *controller $Q(s)$ such that when applied as a dynamic measurement feedback controller we achieve:*

$$\| E_S(s) \|_\infty < \gamma \quad (2.6)$$

and the closed loop system is internally stable.

PROBLEM 2. *Let $\gamma > 0$ be given. Find, if possible, a* FDLTI *controller $Q(s)$ such that when applied as a dynamic measurement feedback controller we achieve:*

$$\| E_{IO}(s) \|_\infty < \gamma \quad (2.7)$$

and the closed loop system is internally stable.

3. Sensitivity Recovery in the $\mathcal{H}_\infty$ Standard Formulation.

In this section we shall provide state space solutions to the sensitivity recovery problem, applying the so-called singular $\mathcal{H}_\infty$ approach which is briefly summarized in Appendix A.

When applying a general controller Q, $Q \in \mathcal{RH}_\infty$, the sensitivity recovery error introduced in Section 2.1 has the form (Section 2.2):

$$E_S(s) = F(sI-A-BF)^{-1}B-Q(s)(I-C(sI-A)^{-1}BQ(s))^{-1}C(sI-A)^{-1}B \quad (3.1)$$

which is a linear fractional transformation in $Q(s)$.

The state space formulation equivalent to (3.1) is:

$$\sum_S : \begin{cases} \dot{x} = \begin{bmatrix} A & 0 \\ 0 & A+BF \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} B \\ B \end{bmatrix} w \\ y = \begin{bmatrix} C & 0 \end{bmatrix} x + 0w \\ z = \begin{bmatrix} 0 & F \end{bmatrix} x - Iu \end{cases} \quad (3.2)$$

or, short

$$\sum_S : \begin{cases} \dot{x} = \bar{A}x + \bar{B}u + \bar{E}w \\ y = \bar{C}_1 x + \bar{D}_1 w \\ z = \bar{C}_2 x + \bar{D}_2 u \end{cases}$$

In this case Assumption A.1. amounts to the requirement that (A, B, C, 0) has neither zeros nor poles on the imaginary axis. This is assumed throughout this section.

In the sequel we shall study the solutions of two certain matrix inequalities introduced in Appendix A: The Quadratic Matrix Inequality (QMI) and the Dual Quadratic Matrix Inequality (DQMI) for which the solutions are crucial to the controller expressions below.

We see that $\bar{D}_2 = I$ is injective and $\bar{D}_1 = 0$, which means that the Quadratic Matrix Inequality is regular, and that the Dual Quadratic Matrix Inequality is totally singular - see Appendix A.

In the sequel the structure of the solutions to the QMI and the DQMI will be described. Based on these solutions we shall describe the so-called QM- and DQM-transformations, which transform the feedback and estimation subproblems into equivalent minimum phase problems, for which the controllers can be derived.

For the QMI we have:

THEOREM 3.1. *For the system Σ_S described by (3.2), the solution $\bar{P}$ of the QMI is:*

$$\bar{P} = \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} \quad (3.3)$$

where P is the unique solution to the algebraic Riccati equation:

$$A^TP + PA - PBB^TP = 0 \quad (3.4)$$

P is given by:

$$P = -\Pi^T(\Pi G_c \Pi^T)^{-1}\Pi \quad (3.5)$$

Here, G_c is the controllability gramian, and Π is the orthogonal projection onto $X_-(A)^\perp$ along $X_-(A)$, the generalized stable eigenspace of A.
The associated quadratic matrix becomes:

$$\bar{F}_\gamma(\bar{P}) = \begin{bmatrix} -PB \\ PB+F^T \\ -I \end{bmatrix} \times \begin{bmatrix} -B^TP & B^TP+F & -I \end{bmatrix} \quad (3.6)$$

PROOF. See [Stoustrup and Niemann 1990].

Note that solvability of, and the solution to, the QMI does not depend on γ. Hence solvability of the $\mathcal{H}_\infty$ problem is equivalent to solvability of the transformed DQMI below.

From Appendix A we achieve the following QM-transformed matrices associated with the $\mathcal{H}_\infty$ problem (3.2):

$$\bar{A}_P = \bar{A}, \quad \bar{C}_P = \bar{C}_1, \quad \bar{C}_{2,P} = \begin{bmatrix} -B^TP & B^TP+F \end{bmatrix}, \quad \bar{D}_P = \bar{D}_2 \quad (3.7)$$

Now, the dual version of Corollary A.4 can be applied to the QM-transformed system to obtain the solution of the DQMI.

LEMMA 3.2. *For the DQMI associated with the system* Σ_{IO} *with* $\bar{C}_2$ *replaced by* $\bar{C}_{2,P}$, *the solution* $\bar{Y}$:

$$\bar{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \tag{3.8}$$

is the unique solution to:

(i) $\bar{A}\bar{Y}+\bar{Y}\bar{A}^T+\bar{E}\bar{E}^T+\bar{Y}\bar{C}_{2,P}^T\bar{C}_{2,P}\bar{Y} =: \bar{E}_{P,Q}\bar{E}_{P,Q}^T \geq 0$

(ii) $CY_{11} = 0$ *and* $CY_{12} = 0$

(iii) $\text{rank}(\bar{E}_{P,Q}) = \text{normrank } H(s)$

(iv) $\text{rank}\begin{bmatrix} sI-\bar{A} & -\bar{E}_{P,Q} \\ \bar{C}_1 & 0 \end{bmatrix} = n + \text{normrank } H(s), \ \forall s \in \bar{\mathbb{C}}^+$

with $H(s) = C(sI-A)^{-1}B$.

In [Stoustrup and Niemann 1990] an algorithm is provided to construct $\bar{Y}$ satisfying conditions (i–iv) of Lemma 3.2. It turns out that the algorithm involves only the solution of a reduced order Riccati equation.

The DQM-transformation proceeds as follows:

$$\bar{A}_{P,Q} = \bar{A} + \gamma^{-2}\bar{Y}\bar{C}_{2,P}^T\bar{C}_{2,P} = \begin{bmatrix} A_{P,Q}^{11} & A_{P,Q}^{12} \\ A_{P,Q}^{21} & A_{P,Q}^{22} \end{bmatrix} \tag{3.10}$$

with:

$$A_{P,Q}^{11} = A+\gamma^{-2}(Y_{11}-Y_{12})PBB^TP-\gamma^{-2}Y_{12}F^TB^TP$$

$$A_{P,Q}^{21} = \gamma^{-2}(Y_{12}^T-Y_{22})PBB^TP-\gamma^{-2}Y_{22}F^TB^TP$$

$$A_{P,Q}^{12} = \gamma^{-2}(Y_{12}-Y_{11})(PBB^TP+PBF)+\gamma^{-2}Y_{12}(F^TB^TP+F^TF)$$

$$A_{P,Q}^{22} = A_F+\gamma^{-2}(Y_{22}-Y_{12}^T)(PBB^TP+PBF)+\gamma^{-2}Y_{22}(F^TB^TP+F^TF)$$

$$\bar{B}_{P,Q} = \bar{B} + \gamma^{-2}\bar{Y}\bar{C}_{2,P}^T\bar{D}_P = \begin{bmatrix} B+\gamma^{-2}(Y_{11}-Y_{12})PB-\gamma^{-2}Y_{12}F^T \\ -\gamma^{-2}(Y_{12}^T-Y_{22})PB-\gamma^{-2}Y_{22}F^T \end{bmatrix} \tag{3.11}$$

After these two transformations, we have a transformed system which is minimum phase, and the final controller $Q(s)$ can be designed directly, by means of the solutions to the two norm inequalities given by (A.11) and (A.12). It is readily seen that (A.11) is trivially satisfied for:

$$L = \begin{bmatrix} -B^TP & B^TP+F \end{bmatrix} \tag{3.12}$$

since this choice solve an (exact) disturbance decoupling problem.

LEMMA 3.3. *Let* P *be as above and let* $M = \begin{bmatrix} M_1^T & M_2^T \end{bmatrix}^T$ *be an output injection satisfying:*

$$\| (sI-\bar{A}_{P,Q}-M\bar{C}_1)^{-1}\bar{E}_{P,Q} \|_\infty < \gamma/\|\bar{C}_{2,P}\| \tag{3.13}$$

with $\bar{A}_{P,Q}+M\bar{C}_1$ *stable. Then an admissible controller for the above* $\mathcal{H}_\infty$ *problem is given by:*

$$Q(s) = -\begin{bmatrix} -B^TP & B^TP+F \end{bmatrix} \times \begin{bmatrix} sI-A+BB^TP-M_1C & -BB^TP-BF \\ -M_2C & sI-A-BF \end{bmatrix}^{-1} \times \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \tag{3.14}$$

PROOF. Lemma 3.3. follows by substituting the above matrices in the expression of Theorem A.5. The relaxed norm bound in (3.13) (compared to (A.12)) is achieved by exploiting that L solves an exact disturbance decoupling problem.

The selection of M in Lemma 3.3. is always possible, since the transformed system $(\bar{A}_{P,Q}, \bar{E}_{P,Q}, \bar{C}_1)$ is minimum phase. Note that the controller depends on $\bar{Y}$ only indirectly (via M).

If the system is invertible and minimum phase, the following controller results by substituting $\bar{Y} = 0$ in Theorem A.5.

LEMMA 3.4. *Assume that* $\bar{Y} = 0$ *is the solution of the DQMI. Then an admissible controller for the above* $\mathcal{H}_\infty$ *problem is given by:*

$$Q(s) = F(sI-A-BF-NC)^{-1}N \tag{3.15}$$

where N *satisfies:*

$$\| (sI-A-NC)^{-1}B \|_\infty < \tilde{\gamma} \tag{3.16}$$

with $A+NC$ *stable, and:*

$$\tilde{\gamma} = \gamma/\|B^TP(sI-A-BF)^{-1}BF+F(sI-A-BF)^{-1}(sI-A)\|_\infty \tag{3.17}$$

PROOF. Assume that N is any (stabilizing) matrix satisfying (3.16). Then by substitution it can be shown that $M = [N^T\ N^T]^T$ satisfies (3.13) and that $\bar{A}_{P,Q}+M\bar{C}_1$ is stable. (3.15) is obtained directly by reduction of (3.14).

4. Input-Output Recovery in the $\mathcal{H}_\infty$ Standard Formulation.

In this section we shall consider the input-output recovery problem with an $\mathcal{H}_\infty$ optimality criterion (Problem 2).

When applying a general controller Q, $Q \in \mathcal{H}_\infty$, the input-output recovery error introduced in Section 2.2 has the form:

$$E_{IO}(s) = C(sI-A-BF)^{-1}B - C(sI-A)^{-1}B - C(sI-A)^{-1}BQ(s)(I-C(sI-A)^{-1}BQ(s))^{-1}C(sI-A)^{-1}B \tag{4.1}$$

which is again a linear fractional transformation in $Q(s)$. The state space formulation equivalent to (4.1) is:

$$\sum_{IO}: \begin{cases} \dot{x} = \begin{bmatrix} A & 0 \\ 0 & A+BF \end{bmatrix}x + \begin{bmatrix} B \\ 0 \end{bmatrix}u + \begin{bmatrix} B \\ B \end{bmatrix}w \\ y = \begin{bmatrix} C & 0 \end{bmatrix}x + 0w \\ z = \begin{bmatrix} -C & C \end{bmatrix}x + 0u \end{cases} \tag{4.2}$$

or, short

$$\sum_{IO}: \begin{cases} \dot{x} = \bar{A}x + \bar{B}u + \bar{E}w \\ y = \bar{C}_1x + 0w \\ z = \bar{C}_2x + 0u \end{cases}$$

The $\mathcal{H}_\infty$ problem to be considered in the following is totally singular (see Appendix A) since both D_1 and D_2 are equal to zero. Consequently Corollary A.4 can be applied to solve the associated QMI and DQMI. This time Assumption A.1. amounts to the requirement that (A, B, C, 0) has no purely imaginary zeros.

THEOREM 4.1. *Consider the system* Σ_{IO} *given by (4.2). The solution* $\bar{P}$ *to the associated QMI has the following form:*

$$\bar{P} = \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} \tag{4.3}$$

where P *is the unique matrix satisfying:*

$$(i)\ A^TP+PA+C^TC =: C_{2,P}^T C_{2,P} \geq 0 \qquad (4.4)$$

(ii) $PB = 0$

(iii) $\text{rank}(C_{2,P}) = \text{normrank } G(s)$

$$(iv)\ \text{rank}\begin{bmatrix} sI-A & -B \\ C_{2,P} & 0 \end{bmatrix} = n + \text{normrank } G(s),\ \forall s \in \bar{\mathbb{C}}^+$$

with $G(s) = C(sI-A-BF)^{-1}B$.
The associated quadratic matrix (see Appendix A) becomes:

$$F_\gamma(\bar{P}) = \begin{bmatrix} C_{2,P}^T \\ -C_{2,P}^T \\ 0 \end{bmatrix} \times \begin{bmatrix} C_{2,P} & -C_{2,P} & 0 \end{bmatrix}. \qquad (4.5)$$

PROOF. Follows by Corollary A.4., see [Stoustrup and Niemann 1990].

Note, that the QMI in this case reduces to a dissipation inequality (known from classical LQ-theory) of n'th order. This inequality is normally solved by transformation to a reduced order Riccati equation.

The QM-transformation becomes:

$$\bar{A}_P = \bar{A},\ \bar{C}_{1,P} = \bar{C}_1,\ \bar{C}_{2,P} = \begin{bmatrix} C_{2,P} & -C_{2,P} \end{bmatrix},\ D_P = 0 \qquad (4.6)$$

On the QM-transformed system, the dual version of Corollary A.4. can now be applied to derive the solution of the DQMI.

LEMMA 4.2. *For the DQMI associated with the system* Σ_{IO} *with* $\bar{C}_2$ *replaced by* $\bar{C}_{2,P}$, *the solution* $\bar{Y}$:

$$\bar{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \qquad (4.7)$$

is the unique solution to:

(i) $\bar{A}\bar{Y}+\bar{Y}\bar{A}^T+\bar{E}\bar{E}^T+\bar{Y}\bar{C}_{2,P}^T\bar{C}_{2,P}\bar{Y} =: \bar{E}_{P,Q}\bar{E}_{P,Q}^T \geq 0$

(ii) $CY_{11} = 0$ *and* $CY_{12} = 0$

(iii) $\text{rank}(\bar{E}_{P,Q}) = \text{normrank } H(s)$

$$(iv)\ \text{rank}\begin{bmatrix} sI-\bar{A} & -\bar{E}_{P,Q} \\ \bar{C}_1 & 0 \end{bmatrix} = n + \text{normrank } H(s),\ \forall s \in \bar{\mathbb{C}}^+$$

with $H(s) = C(sI-A)^{-1}B$.

Again, conditions (i–iv) can be reformulated as a reduced order Riccati equation.

If the system (A, B, C, 0) is minimum phase, $\bar{P} = 0$ is the unique solution to the QMI satisfying the involved rank conditions (iii–iv). Further if (A, B, C, 0) is also invertible, then $\bar{Y} = 0$ is the unique solution to the DQMI satisfying the two rank conditions.

For non-trivial transformations we obtain the following matrices for the transformed system:

$$\bar{A}_{P,Q} = \begin{bmatrix} A_{P,Q}^{11} & A_{P,Q}^{12} \\ A_{P,Q}^{21} & A_{P,Q}^{22} \end{bmatrix}$$

where:

$$A_{P,Q}^{11} = A + \gamma^{-2}(Y_{11}-Y_{12})C_{2,P}^T C_{2,P}$$

$$A_{P,Q}^{12} = \gamma^{-2}(Y_{12}-Y_{11})C_{2,P}^T C_{2,P}$$

$$A_{P,Q}^{21} = \gamma^{-2}(Y_{12}^T-Y_{22})C_{2,P}^T C_{2,P}$$

$$A_{P,Q}^{22} = A + BF + \gamma^{-2}(Y_{22}-Y_{12}^T)C_{2,P}^T C_{2,P}$$

$$\bar{B}_{P,Q} = \bar{B} \qquad (4.8)$$

$$\bar{E}_{P,Q} = \begin{bmatrix} E_{P,Q}^1 \\ E_{P,Q}^2 \end{bmatrix}$$

$$\bar{C}_{1,P} = \bar{C}_1$$

$$\bar{C}_{2,P} = \begin{bmatrix} C_{2,P} & -C_{2,P} \end{bmatrix}$$

Eventually, an admissible controller, solving the $\mathcal{H}_\infty$ problem is obtained in terms of these transformed matrices.

LEMMA 4.3. *Let* $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$ *be a state feedback satisfying (A.11), and let* $M = \begin{bmatrix} M_1^T & M_2^T \end{bmatrix}^T$ *be an output injection satisfying (A.12). Then, an internally stabilizing controller, making the* $\mathcal{H}_\infty$ *norm of the closed loop transfer function from* w *to* z *smaller than* γ *is given by:*

$$Q(s) = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \times \begin{bmatrix} sI-A_{P,Q}^{11}-BL_1-M_1C & -A_{P,Q}^{12}-BL_2 \\ -A_{P,Q}^{21}-M_2C & sI-A_{P,Q}^{22} \end{bmatrix}^{-1} \times \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \qquad (4.9)$$

Again, in the minimum phase case, we only need an n'th order controller:

LEMMA 4.4. *If* $\bar{P} = 0$ *and* $\bar{Y} = 0$ *are solutions to QMI and DQMI, resp., the controller is given by:*

$$Q(s) = F(sI-A-BF-NC)^{-1}N \qquad (4.10)$$

where N *is any matrix satisfying:*

$$\| (sI-A-NC)^{-1}B \|_\infty < \gamma/\|F\| \qquad (4.11)$$

with $A+NC$ *stable.*

PROOF. By substitution, it can be verified that $L = [\ F\ 0\ ]$ and $M = [\ N^T\ 0\ ]^T$, with N as above, satisfies (A.11) and (A.12) in the minimum phase case.

5. Discussion.

An alternative approach to the LTR design philosophy is introduced in this paper where the LTR design problem is restated as an $\mathcal{H}_\infty$ standard problem. The actual $\mathcal{H}_\infty$ standard problem, however, does not satisfy the normal regularity assumptions in $\mathcal{H}_\infty$ theory, and therefore we invoke the so called singular $\mathcal{H}_\infty$ theory for the calculation of the controllers. Programs implementing the obtained controller formulae are included in the MATLAB toolbox [Niemann and Stoustrup 1991].

This $\mathcal{H}_\infty$/LTR design method provides directly controllers of the same dynamic order as the order of the $\mathcal{H}_\infty$ standard problem formulation, meaning $2n$ for general systems and n for minimum phase systems. This is a pay-off of the state space approach, since frequency domain methods give controllers of dynamic order $3n-1$ [Moore and Tay 1989].

Two problems are considered: The sensitivity recovery problem and the input-output recovery problem. The two resulting controllers are given in terms of the unique solutions to two Dual Quadratic Matrix Inequalities (DQMI's) or order $2n$, and additionally by the solution to an n'th order singular Riccati equation (sensitivity recovery) or, respectively, by an n'th order dissipation inequality (input-output recovery). It turns out that the solution of the two DQMI's with additional rank constraints involves only the solution of two reduced order Riccati equations.

Comparing the $\mathcal{H}_\infty$/LTR design methods proposed above to traditional LTR methods, a major advantage is that non-minimum phase systems can be treated by exactly the same techniques as minimum phase systems after a preliminary transformation has been performed. This preliminary transformation involves a state space transformation and the solution to a reduced order Riccati equation. The preliminary transformation is a one-shot process requiring no iterations.

The main limitation of the suggested methods is due to the fact that $\mathcal{H}_\infty$ methods generally try to average out the errors over the whole frequency range. This situation is not desirable for LTR problems, since the acceptable errors might be low for instance at low frequencies (performance specs.). Hence, to overcome this limitation, it would sometimes be reasonable to incorporate weighting functions in the problem formulation. This can easily be done, but only at the cost of more controller states.

As an alternative to the method proposed in this paper, one might restrict the attention to consider only observer based controllers, motivated by the traditional LTR setup. In fact, this can be done in a similar way to the above. The observer based $\mathcal{H}_\infty$/LTR methods include a direct and an indirect method. The direct method [Stoustrup 1990, Stoustrup and Niemann 1990] involves an observer based solution to the problems considered in the present paper. The indirect method [Stoustrup 1990, Stoustrup and Niemann 1991] is based on sufficient conditions only for the solution to the above problems, but provides always an n'th order controller.

APPENDIX A.

The necessary preliminaries for the $\mathcal{H}_\infty$ methods used in this paper will be introduced in this appendix. The approach taken is based on the results in [Stoorvogel 1989, Stoorvogel and Trentelman 1990], the so called singular approach. This is a very general approach which includes the well known approach by Doyle et al. [1989] as a special case.

In the state space approach to $\mathcal{H}_\infty$ the standard problem is as follows:

Consider a finite dimensional, linear, time invariant system:

$$\sum \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1 x + D_1 w \\ z = C_2 x + D_2 u \end{cases} \qquad (A.1)$$

We assume that $\gamma > 0$ has been given. We wish to design, if possible, an internally stabilizing FDLTI compensator $u = Q(s)y$ such that the $\mathcal{H}_\infty$ norm of the resulting closed-loop transfer function from w to z is smaller than γ.

ASSUMPTION A.1. It is assumed that the systems (A,B,C_2,D_2) and (A,E,C_1,D_1) have no invariant zeros in $\mathbb{C}^0$.

The main result is:

THEOREM A.2. *Consider the system Σ above satisfying Assumption A.1. Let $\gamma > 0$ be given. Then, there exists a FDLTI compensator $u = Q(s)y$ for which the $\mathcal{H}_\infty$ norm of the resulting closed-loop transfer function from w to z is smaller than γ,* **if and only if** *there exist $P \geq 0$ and $Q \geq 0$ for which:*

(1) $F_\gamma(P) \geq 0$

(2) $G_\gamma(Q) \geq 0$

(3) rank $F_\gamma(P)$ = normrank G

(4) rank $G_\gamma(Q)$ = normrank H

(5) $$\text{rank} \begin{bmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{bmatrix} = n + \text{normrank } G, \quad \forall s \in \mathbb{C}^+ \cup \mathbb{C}^0$$

(6) $$\text{rank} \begin{bmatrix} M_\gamma(Q,s) & G_\gamma(Q) \end{bmatrix} = n + \text{normrank } H, \quad \forall s \in \mathbb{C}^+ \cup \mathbb{C}^0$$

(7) $\rho(PQ) < \gamma^2$

where the notation used is as follows:

$$F_\gamma(P) = \begin{bmatrix} A^T P + PA + C_2^T C_2 + \gamma^{-2} PEE^T P & PB + C_2^T D_2 \\ B^T P + D_2^T C_2 & D_2^T D_2 \end{bmatrix} \qquad (A.2)$$

$$G_\gamma(Q) = \begin{bmatrix} AQ + QA^T + EE^T + \gamma^{-2} QC_2^T C_2 Q & QC_1^T + ED_1^T \\ C_1 Q + D_1 E^T & D_1 D_1^T \end{bmatrix} \qquad (A.3)$$

$$L_\gamma(P,s) = \begin{bmatrix} sI - A - \gamma^{-2} EE^T P & -B \end{bmatrix}$$

$$M_\gamma(Q,s) = \begin{bmatrix} sI - A - \gamma^{-2} QC_2^T C_2 \\ -C_1 \end{bmatrix} \qquad (A.4)$$

$$G(s) = C_2(sI - A)^{-1}B + D_2, \quad H(s) = C_1(sI - A)^{-1}E + D_1 \qquad (A.5)$$

The proof of Theorem A.2 can be found in [Stoorvogel 1989]. We shall refer to condition (1) as the *Quadratic Matrix Inequality* (QMI), and any $P \geq 0$ satisfying (1) will be called a solution to QMI. Analogously we shall call (2) the *Dual Quadratic Matrix Inequality* (DQMI), and refer to solutions of DQMI any $Q \geq 0$ satisfying (2). Conditions (3) and (5) guarantees that a solution to QMI is unique and of minimal rank (and dually for DQMI with (4) and (6)). (7) is a typical $\mathcal{H}_\infty$ coupling condition, which also appears in [Doyle et al. 1989].

Further, we shall need a couple of corollaries.

COROLLARY A.3. The Regular Case. *Assume that D_2 is injective. Then (1), (3) and (5) is satisfied if and only if*

$$A^T P + PA + C_2^T C_2 + \gamma^{-2} PEE^T P - (PB + C_2^T D_2)(D_2^T D_2)^{-1}(B^T P + D_2^T C_2) = 0$$

and

$$\Lambda(A + \gamma^{-2} EE^T P - B(D_2^T D_2)^{-1}(B^T P + D_2^T C_2)) \subset \mathbb{C}^-$$

COROLLARY A.4. The Totally Singular Case. *Assume $D_2 = 0$. Then (1) is equivalent to:*

$$A^T P + PA + C_2^T C_2 + \gamma^{-2} PEE^T P \geq 0$$

where P satisfies $PB = 0$.

The two corollaries have straightforward duals, which we shall also utilize in the sequel.

Expressions for admissible controllers will be given in the following in terms of the matrices for certain transformations of Σ. First we define $C_{2,P}$ and D_P by the following factorization:

$$F_\gamma(P) = \begin{bmatrix} C_{2,P} & D_P \end{bmatrix}^T \times \begin{bmatrix} C_{2,P} & D_P \end{bmatrix}, \qquad (A.6)$$

Moreover, we will need the following matrices:

$$A_P = A+\gamma^{-2}EE^TP,\ C_{1,P} = C_1+\gamma^{-2}D_1E^TP \qquad (A.7)$$

$$Y = (I-\gamma^{-2}QP)^{-1}Q \qquad (A.8)$$

$$A_{P,Q} = A_P+\gamma^{-2}YC_{2,P}^TC_{2,P},\ B_{P,Q} = B+\gamma^{-2}YC_{2,P}^TD_P \qquad (A.9)$$

We shall refer to the system where A_P, $C_{1,P}$, $C_{2,P}$ and D_P substitute A, C_1, C_2 and D_2 as the *QM-transform* of the system Σ. The DQMI for the QM-transformed system becomes:

$$\tilde{G}_\gamma(Y) = \begin{bmatrix} A_PY+YA_P^T+EE^T+\gamma^{-2}YC_{2,P}^TC_{2,P}Y & YC_{1,P}^T+ED_1^T \\ C_{1,P}Y+D_1E^T & D_1D_1^T \end{bmatrix} =$$

$$\begin{bmatrix} E_{P,Q}^T & D_{P,Q}^T \end{bmatrix}^T\times\begin{bmatrix} E_{P,Q}^T & D_{P,Q}^T \end{bmatrix} \geq 0 \qquad (A.10)$$

Substituting $A_{P,Q}$, $B_{P,Q}$, $E_{P,Q}$ and $D_{P,Q}$ for the corresponding variables in the previous system will be referred to as the DQM-transformation.

In terms of these transformed system matrices we can compute the desired $\mathcal{H}_\infty$ controller:

THEOREM A.5. *Let $A_{P,Q}$, $B_{P,Q}$ and $C_{1,P}$ be as above. Let L be a state feedback L, such that $A_{P,Q}+B_{P,Q}L$ is stable, and such that:*

$$\|(C_{2,P}+D_PL)(sI-A_{P,Q}-B_{P,Q}L)^{-1}\|_\infty < \gamma/(3\cdot\|E_{P,Q}\|) \qquad (A.11)$$

Let M be an output injection, such that $A_{P,Q}+MC_{1,P}$ is stable and further:

$$\| (sI-A_{P,Q}-MC_{1,P})^{-1}(E_{P,Q}+MD_{P,Q}) \|_\infty < \tilde{\gamma} \qquad (A.12)$$

where

$$\tilde{\gamma} = \min\{\ \gamma/(3\cdot\|D_PL\|),\ \|E_{P,Q}\|/\|B_{P,Q}L\|\ \}$$

Then the controller:

$$u = -L(sI-A_{P,Q}-B_{P,Q}L-MC_{1,P})^{-1}My \qquad (A.13)$$

makes the $\mathcal{H}_\infty$ norm of the resulting closed loop transfer function from w to z in Σ smaller than γ.

The significance of Theorem A.5 is to transform the original $\mathcal{H}_\infty$ problem to two disturbance attenuation problems, which can be solved by well known methods, see e.g. [Stoorvogel 1989, Trentelman 1986, Willems 1981].

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