AN H_∞/LTR METHOD FOR ROBUST CONTROLLER DESIGN.

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Abstract.

This paper describes a Loop Transfer Recovery (LTR) method for the design of controllers based on specifications for robust stability and performance. Optimality for the controllers are expressed as an $\mathcal{H}_\infty$ norm constraint. The $\mathcal{H}_\infty$ problem emerging from various LTR specifications always have the form of a weighted $\mathcal{H}_\infty$ optimization of a certain matrix valued function, the Recovery Matrix. The solution to the involved $\mathcal{H}_\infty$/LTR problem proceeds by two steps. First the static part of the controller is designed, which can be done independently of the recovery optimization. Subsequently, the $\mathcal{H}_\infty$ optimization is performed by the design of a Luenberger observer. The paper describes the complete procedure from specifications to final controller design.

1. Introduction.

The Loop Transfer Recovery (LTR) method originated [Doyle and Stein 1979, Athans 1986, Stein and Athans 1987] as an approach to overcome the robustness problems which arise in classical observer based controller design. It became evident that plant dynamics and observer dynamics can not be treated separately in a design proces for an uncertain system. However, a two step design procedure is conceptually appealing and observers offer optional information for use e.g. in supervision applications.

The LTR design method involves two design steps for observer based controllers. First, a state feedback design is constructed which satisfies the performance and robustness specifications. Second, a dynamical controller is designed such that the properties of the state feedback are reobtained.

Classically, LQG methods were applied to implement the LTR procedure. Inherently, however, to the structure of the LQG criterion no a *priori* guarantees for robust stability can be given for LQG controllers. Consequently, any LQG/LTR method has to be of an intrinsically iterative nature.

As an alternative $\mathcal{H}_\infty$ theory offers straightforward means of designing for robust stability. Therefore, it is natural to embed the LTR methodology in the $\mathcal{H}_\infty$ framework. The first attempt to do this was done by [Moore and Tay 1989]. They applied a frequency domain $\mathcal{H}_\infty$ method combined with perturbation techniques. The same $\mathcal{H}_\infty$/LTR problem was addressed in [Stoustrup 1990, Stoustrup and Niemann 1990] where time domain techniques were applied, resulting in closed form state space formulae. An $\mathcal{H}_\infty$/LTR method for the design of lower order controllers were described in [Niemann, Søgaard-Andersen and Stoustrup 1991b].

In this paper we address the $\mathcal{H}_\infty$/LTR problem directly in terms of robust control specifications. Embarking from the original performance and robust stability constraints, we arrive at a general $\mathcal{H}_\infty$/LTR formulation, which include all previous approaches as special cases.

2. The Loop Transfer Recovery Principle.

In this section we shall briefly introduce the Loop Transfer Recovery (LTR) design problem.

Let us consider a finite dimensional, linear, time invariant plant model, represented by a minimal state space realization (A, B, C, 0):

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y = Cx & y \in \mathbb{R}^p \end{cases} \qquad (2.1)$$

with open loop transfer function:

$$H(s) = C(sI-A)^{-1}B \qquad (2.2)$$

where A, B and C are matrices of appropriate dimensions. The system is assumed to be stabilizable, detectable and left invertible. Moreover, we shall make the technical assumption, that $\Lambda(A) \cap \mathbb{C}^0 = \emptyset$. Note, however, that this can always be achieved by applying a preliminary static output feedback. Furthermore, this preliminary static output feedback can be chosen arbitrarily small.

It is assumed throughout this paper that Σ does not have a direct feedthrough term. However, the results derived can with minor changes be generalized to handle systems with direct feedthrough terms.

To design a controller for the system Σ by the LTR methodology, we first determine a (static) state feedback, the target design, which satisfies our design specifications. The design specifications, such as robustness and performance, are assumed to be reflected to the input node [Athans 1986, Stein and Athans 1987]. The target sensitivity function $S_F(s)$, the target complementary sensitivity function $T_F(s)$ and the target closed loop transfer function $G_F(s)$ resulting when formally applying the target state feedback are respectively:

$$\begin{aligned} S_F(s) &= (I-F(sI-A)^{-1}B)^{-1} \\ T_F(s) &= I-S_F(s) \\ G_F(s) &= C(sI-A-BF)^{-1}B \end{aligned} \tag{2.3}$$

Second, the LTR step must be performed, where the target design is recovered in terms of the above three transfer functions by a dynamic compensator $C(s)$ giving the following (full loop) sensitivity function $S(s)$, the (full loop) complementary sensitivity function $T(s)$ and the (full) closed loop transfer function $G(s)$, respectively:

$$\begin{aligned} S(s) &= (I-C(s)H(s))^{-1} \\ T(s) &= I-S(s) \\ G(s) &= H(s)(I-C(s)H(s))^{-1} \end{aligned} \tag{2.4}$$

We need a measure of the quality of the recovery step. To that end we define the sensitivity, the complementary sensitivity and the closed loop recovery errors [Niemann, Søgaard-Andersen and Stoustrup 1991a].

Definition 2.1. *The sensitivity recovery error* $E_S(s)$, *the complementary sensitivity recovery error* $E_T(s)$ *and the closed loop recovery error* $E_G(s)$ *are defined by:*

$$\begin{aligned} E_S(s) &= S_F(s) - S(s) \\ E_T(s) &= T_F(s) - T(s) = -E_S(s) \\ E_G(s) &= G_F(s) - G(s) \end{aligned} \tag{2.5}$$

Note, that in Definition 2.1, $E_S(s)$, $E_T(s)$ and $E_G(s)$ are defined independent of the applied controller type. In the sequel we shall consider specific observer based controller types for which the recovery errors can be written in more convenient forms.

2.1. The Luenberger Observer.

In the following we shall consider Luenberger observers, which are selected because of their generality. Indeed, any stabilizing controller can be implemented as a Luenberger observer based controller. Important special cases are full order observers and minimal order observers. Another important example of a Luenberger observer is the Q–observer [Niemann, Søgaard-Andersen and Stoustrup 1991a] which will be studied further in Section 4.

A Luenberger observer based controller [Luenberger 1971] for the system Σ is given by a state space realization of the following form:

$$\Sigma_L : \begin{cases} \dot{\xi} = D\xi + Gu + Ey & \xi \in \mathbb{R}^r,\ u \in \mathbb{R}^m \\ u = P\xi \qquad\quad + Vy & y \in \mathbb{R}^p \end{cases} \tag{2.6}$$

where D, G, E, P and V are the Luenberger matrices, which have to satisfy

$$\lambda(D) \in \mathbb{C}^-,\ TA - DT = EC,\ G = TB,\ F = PT + VC \tag{2.7}$$

for some T of appropriate dimensions. Then the output signal u is related to the state feedback signal as $u = F\hat{x}$, where $\hat{x}$ is an estimate of the state vector x in Σ.

The Luenberger observer described by (2.6) has the following transfer function:

$$C(s) = V + P(sI-D+GP)^{-1}(E-GV) \tag{2.8}$$

In the sequel it will prove useful to rewrite $C(s)$ in the so called *recovery form* [Niemann, Søgaard-Andersen and Stoustrup 1991a], i.e. as:

$$C(s) = (I+M(s))^{-1}N(s) \tag{2.9}$$

where

$$\begin{aligned} M(s) &= P(sI-D)^{-1}G \\ N(s) &= P(sI-D)^{-1}E + V \end{aligned} \tag{2.10}$$

The matrix valued function $M(\cdot)$ is called the *recovery matrix* and turns out to be central in recovery design of observer based controllers.

2.2. Recovery Errors.

Using the recovery matrix $M(s)$ it can be shown [Niemann, Søgaard-Andersen and Stoustrup 1991a] that the three recovery errors in Definition 2.1 attain the following forms.

Lemma 2.2. *Assume that a Luenberger observer based controller of the form (2.6) has been applied. Then the sensitivity recovery error $E_S(s)$, the complementary sensitivity recovery error $E_T(s)$ and the closed loop recovery error $E_G(s)$ become:*

$$E_S(s) = S_F(s)M(s) = -E_T(s)$$
$$E_G(s) = G_F(s)M(s) \qquad (2.11)$$

Note, that each recovery error is given as a target transfer function times the recovery matrix $M(s)$. The target functions are constant during the recovery step of the design procedure. This means that they can simply be regarded as weighting functions in an $\mathcal{H}_\infty$ optimization of the recovery matrix. Hence, the recovery matrix can in this sense be considered as a measure of the quality of the recovery design step.

Based on these observations, we are motivated to introduce a more general type of recovery error for observer based controllers.

To that end, consider a finite dimensional, linear time invariant system given by the state space representation (A_w, B_w, C_w, D_w):

$$\Sigma_W : \begin{cases} \dot{x}_w = A_w x_w + B_w u & x \in \mathbb{R}^{n_w}, u \in \mathbb{R}^{m_w} \\ y_w = C_w x_w + D_w u & y \in \mathbb{R}^{p_w} \end{cases} \qquad (2.12)$$

with the transfer function

$$W(s) = D_w + C_w(sI-A_w)^{-1}B_w \qquad (2.13)$$

(A_w, B_w, C_w and D_w are matrices of appropriate dimensions.) The system Σ_w is assumed to be stabilizable, detectable and without zeros on the imaginary axis. Now we apply $W(s)$ as a weighting function for $M(s)$.

Definition 2.3. *Let $M(s)$ be the recovery matrix corresponding to a given Luenberger observer based controller, and let $W(s)$ be a transfer function as above. Then we define by the* (general) recovery error $E(\cdot)$ *the following:*

$$E(s) = W(s)M(s) \qquad (2.14)$$

By comparing terms with the three errors of Lemma 2.2 we see that $E(s) = E_S(s)$ for $W(s) = S_F(s)$, that $E(s) = E_T(s)$ for $W(s) = -S_F(s)$ and that $E(s) = E_G(s)$ for $W(s) = G_F(s)$.

3. The Robust H_∞/LTR Design Problem.

Robust control design deals with the design of dynamic feedback compensators applied to uncertain dynamical systems. Typically, such a controller has to optimally track a reference signal in the face of various kinds of uncertainties which are usually thought of as e.g. disturbances, measurement noise, unmodeled system dynamics or unknown dynamics of future reference signals to be applied.

The closed loop effect of a large class of uncertainties relate to the magnitude of the sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$. $S(s)$ has to be small to suppress disturbances and obtain good tracking. On the other hand, to achieve robust stability subject to unmodeled dynamics and to eliminate the influence of measurement noise, $T(s)$ has to be small, which is a contrary design objective. By introducing weighting functions expressing the *a priori* knowledge of the frequency contents of disturbances and reference signals, $W_1(\cdot)$, and the frequency contents of measurement noise and unmodeled dynamics, $W_2(\cdot)$, we are led to the following problem formulation.

Problem 3.1. *Let $\gamma > 0$ be given, and let $W_1(\cdot)$ and $W_2(\cdot)$ be weight functions for the sensitivity function and the complementary sensitivity function, respectively. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller $C(s)$ such that the resulting sensitivity and complementary sensitivity functions satisfy:*

$$\| W_1(\cdot)S(\cdot) \|_\infty < \gamma \text{ and } \| W_2(\cdot)T(\cdot) \|_\infty < \gamma \qquad (3.1)$$

Problem 3.1 is an optimization problem for which there is no direct translation to an $\mathcal{H}_\infty$ standard problem (see [Francis and Doyle 1987] for a description of the $\mathcal{H}_\infty$ standard problem). Hence, to obtain feasible solutions by means of an $\mathcal{H}_\infty$ standard model, we have to introduce some amount of conservatism. In this section we shall discuss several approaches to perform this $\mathcal{H}_\infty$ standard problem modeling in a Loop Transfer Recovery (LTR) setting.

To design a controller for a given system by the LTR methodology, we first determine a formal (static) state feedback $u = Fx$, the target design, which satisfies our design specifications, which in this case are:

$$\| W_1(\cdot)S_F(\cdot) \|_\infty < \gamma_F$$
and
$$\| W_2(\cdot)T_F(\cdot) \|_\infty < \gamma_F \tag{3.2}$$

where $S_F(s)$ and $T_F(s)$ are the sensitivity function, resp. the complementary sensitivity function, obtained when applying a state feedback, and γ_F is the *target sensitivity specification*, $0 < \gamma_F < \gamma$. The $\mathcal{H}_\infty$ target design can be performed in a number of ways, which will not be discussed further in the present paper. Subsequently, the recovery step of the LTR design procedure has to recover the target design within the *recovery gap:*

$$\varepsilon := \gamma - \gamma_F \tag{3.3}$$

by means of a dynamic measurement feedback controller.

The first approach to formulate Problem 3.1 as an $\mathcal{H}_\infty$ standard problem in the LTR setting was taken by Moore and Tay [1989]. They applied a frequency domain method to an $\mathcal{H}_\infty$ optimization problem based on the sensitivity recovery error $E_S(s) = -E_T(s)$, i.e. a general recovery error where the target sensitivity $S_F(s)$ is used as weight matrix for $M(s)$, $E(s) = E_S(s) = S_F(s)M(s)$ (see Section 2). Hence, the idea is that when the actual sensitivity functions $S(s)$ and $T(s)$ are sufficiently close to the target sensitivity functions $S_F(s)$ and $T_F(s)$, respectively, so will (3.1) be satisfied when (3.2) is. To be more precise, we wish

$$\| W_1(\cdot)S(\cdot) \|_\infty < \gamma, \ \| W_2(\cdot)T(\cdot) \|_\infty < \gamma \tag{3.4}$$

which is satisfied if

$$\| W_1(\cdot)(E_S(\cdot)+S_F(\cdot)) \|_\infty < \gamma,$$
and
$$\| W_2(\cdot)(T_F(\cdot)-E_S(\cdot)) \|_\infty < \gamma \tag{3.5}$$

By using the triangular inequality we obtain the following sufficient condition for (3.5):

$$\|W_1(\cdot)\|_\infty\|E_S(\cdot)\|_\infty + \|W_1(\cdot)S_F(\cdot)\|_\infty < \gamma$$
and
$$\|W_2(\cdot)\|_\infty\|E_S(\cdot)\|_\infty + \|W_2(\cdot)T_F(\cdot)\|_\infty < \gamma \tag{3.6}$$

This in turn is implied by:

$$\|W_1(\cdot)\|_\infty\|E_S(\cdot)\|_\infty < \varepsilon$$
and
$$\|W_2(\cdot)\|_\infty\|E_S(\cdot)\|_\infty < \varepsilon \tag{3.7}$$

Hence, (3.1) is guaranteed if $E_S(s)$ satisfies the bound in the following problem:

PROBLEM 3.2. *Let $\varepsilon > 0$ be given. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller $C(s)$ such that the resulting recovery error $E_S(s)$ satisfies:*

$$\|E_S(\cdot)\|_\infty < \gamma_{LTR} \tag{3.8}$$

where

$$\gamma_{LTR} := \varepsilon \cdot \min\left\{ \frac{1}{\|W_1(\cdot)\|_\infty}, \frac{1}{\|W_2(\cdot)\|_\infty} \right\}$$

It should be noted that (inevitably) the formulation in Problem 3.2 provides only a sufficient condition for solvability of Problem 3.1. To be more precise we have:

LEMMA 3.3. *Assume that $C(s)$ is a solution to Problem 3.2. Then $C(s)$ also solves Problem 3.1.*

In [Moore and Tay 1989] an $\mathcal{H}_\infty$ problem with an optimization constraint on $E_S(s)$ was studied in frequency domain, giving rise to controllers of order $3n-1$ or $2n$. State space formulae for the solution to Problem 3.2 were given in [Stoustrup and Niemann 1990] with controller orders of at most $2n$.

In [Niemann, Søgaard-Andersen and Stoustrup 1991b] it is shown that the bound given by (3.8) is rather conservative. The reason is that an unweighted optimization of $E_S(s)$ is considered which means that we have to accept a low frequency sensitivity error of the same magnitude as the unavoidable error caused by non-minimum phase zeros. This makes the performance specifications conservative.

Surprisingly, better results can in general be achieved even with a n'th order controller. To obtain an n'th order controller, we must use a zero'th order weight matrix for $M(s)$, which after a change of coordinates basically means $W(s) = I$, $E(s) = W(s)M(s) = I \cdot M(s)$. This means that the recovery step involves an $\mathcal{H}_\infty$ optimization of $M(s)$, directly. This leads us to the following $\mathcal{H}_\infty$/LTR formulation.

PROBLEM 3.4. *Let $\varepsilon > 0$ be given. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller $C(s)$ such that the resulting recovery matrix $M(\cdot)$ satisfies:*

$$\|M(\cdot)\|_\infty < \gamma_{LTR} \tag{3.9}$$

where

$$\gamma_{LTR} := \varepsilon \cdot \min\left\{ \frac{1}{\|W_1(\cdot)S_F(\cdot)\|_\infty}, \frac{1}{\|W_2(\cdot)S_F(\cdot)\|_\infty} \right\}$$

Lemma 3.5. *Assume that* C(s) *is a solution to Problem 3.4. Then* C(s) *also solves Problem 3.1.*

The proof of Lemma 3.5 is equivalent to the proof of Lemma 3.3.

In [Niemann, Søgaard-Andersen and Stoustrup 1990b] it is shown that Problem 3.4 has a solution in some cases where Problem 3.2 does not, meaning that Problem 3.2 has a more conservative bound than Problem 3.4. The reason is that for typical designs the 'min' of (3.8) and of (3.9) in both cases are determined by the first operand. This means that if $\Delta = 20\cdot(\log \|W_1(\cdot)S_F(\cdot)\|_\infty - \log \|W_1(\cdot)\|_\infty)$ is a positive number, which it will always be for reasonable target designs, then an $\mathcal{H}_\infty$/LTR method based on Problem 3.4 will be Δ dB less conservative than a method based on Problem 3.2. Moreover, since the dynamic order of M(s) is n, the solution to Problem 3.4. will be a controller of order at most n, whereas $\mathcal{H}_\infty$/LTR design based on Problem 3.2 results in $2n$'th order controllers.

In some cases, though, more than n controller states are required. In that case we have to introduce the weights $W_1(s)$ and $W_2(s)$, which are associated with S(s) and T(s), respectively, for $E_S(s)$. This gives rise to the following problem.

Problem 3.6. *Let* ε > 0 *be given. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller* C(s) *such that the resulting (weighted) recovery error* $E_S(\cdot)$ *satisfies:*

$$\left\| \begin{bmatrix} W_1(\cdot) \\ W_2(\cdot) \end{bmatrix} E_S(\cdot) \right\|_\infty < \gamma_{LTR} \tag{3.10}$$

where

$$\gamma_{LTR} = \frac{\varepsilon}{\sqrt{2}}$$

As in Problem 3.2 and Problem 3.4 the right hand side of (3.10) is an upper bound for the $\mathcal{H}_\infty$ optimization of the weighted sensitivity recovery error, which guarantee that a controller solving Problem 3.6 also is a solution of Problem 3.1:

Lemma 3.7. *Assume that* C(s) *is a solution to Problem 3.6. Then* C(s) *also solves Problem 3.1.*

Again, the result is verified by arguments related to those preceding Problem 3.2.

Problem 3.6 is an $\mathcal{H}_\infty$ optimization of a general recovery error E(s) with:

$$E(s) = \begin{bmatrix} W_1(s) \\ W_2(s) \end{bmatrix} E_S(s) = \begin{bmatrix} W_1(s)S_F(s) \\ W_2(s)S_F(s) \end{bmatrix} M(s) = W(s)M(s) \tag{3.11}$$

Problem 3.6 is, in a certain sense, the best possible approach to a robust $\mathcal{H}_\infty$ design method in the LTR methodology with regard to conservatism. Provided the target feedback has been carefully selected, the design will be no more conservative than 6 dB. On the other hand, a conservatism of 6 dB is inevitable when modeling Problem 3.1 as an $\mathcal{H}_\infty$ standard problem.

The drawback of applying an $\mathcal{H}_\infty$/LTR method based on Problem 3.6, is the associated controller order. Since the dynamic order of $E_S(s)$ is $2n$, a controller for Problem 3.6 will typically be of order $2n+n_{w_1}+n_{w_2}$ where n_{w_1} and n_{w_2} are the dynamic orders of $W_1(s)$ and $W_2(s)$, respectively.

All the above $\mathcal{H}_\infty$/LTR formulations involves an $\mathcal{H}_\infty$ optimization of an expression of the form E(·) = W(·)M(·). Thus motivated, we are led to the following *General $\mathcal{H}_\infty$/LTR Design Problem:*

Problem 3.8. *Let* $\gamma_{LTR} > 0$ *be given. Let* W(·) *be any stable weight matrix. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller* C(s) *such that the resulting general recovery error* E(·) *satisfies:*

$$\|E(\cdot)\|_\infty < \gamma_{LTR} \quad \text{or} \quad \|W(\cdot)M(\cdot)\|_\infty < \gamma_{LTR} \tag{3.12}$$

Problem 3.8 includes Problem 3.2, 3.4 and 3.6 as special cases. Note, however, that as a cost of generality an *a priori* value of γ_{LTR} can not be estimated. Hence, γ_{LTR} has to be regarded as a design parameter if W(s) is chosen completely arbitrary.

In the remaining sections of this paper we shall provide the solution to Problem 3.8.

4. The Q-Observer Based Controller.

In this section we shall consider a special case of the Luenberger observer, the Q-observer. The Q-observer is a parameterized observer based controller architecture which implements the class of all stabilizing controllers by means of the Youla (or Q-) parameterization. Briefly, the principle in the Youla parameterization is to pick any stabilizing controller which is thereafter fixed, and then make a certain interconnection structure for which the original controller is retained if nothing is attached at the interconnection nodes. The class of all stabilizing controllers is then parameterized by applying the class of all $\mathcal{H}_\infty$ systems at the interconnection terminals.

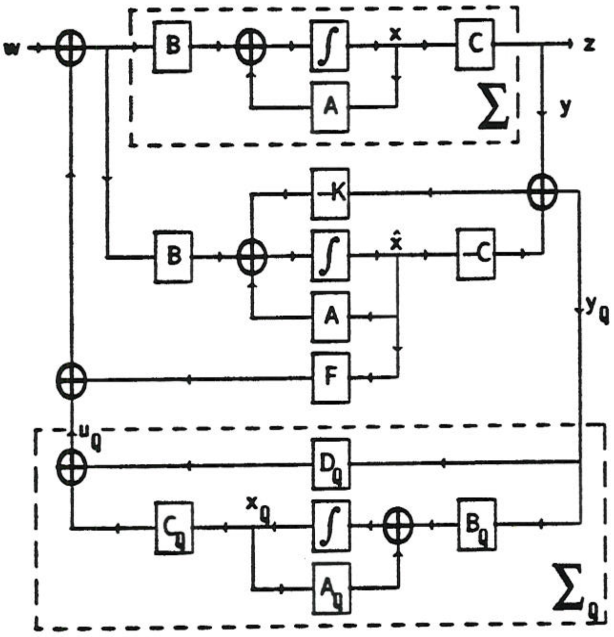

Fig. 4.1. The Q–Observer.

In [Boyd et al. 1988] it has been shown, that the construction shown in fig 4.1 is an implementation of the Youla parameterization. Fig. 4.1 is a full order observer based controller with (stabilizing) observer gain K, interconnected with a parameter part. We shall refer to this construction as the *Q–observer* in the sequel, motivated by the following result.

LEMMA 4.1. *Assume that* $Q \in \mathcal{RH}_\infty$ *is given by a state space representation:*

$$\Sigma_Q : \begin{cases} \dot{x}_Q = A_Q x_Q + B_Q y_Q \\ u_Q = C_Q x_Q + D_Q y_Q \end{cases} \tag{4.1}$$

where $x_Q \in \mathbb{R}^q$, *and* q *is the order of* Q. *Then the corresponding Q–observer (as shown in Fig. 4.1) is a Luenberger observer with the following parameters:*

$$D = \begin{bmatrix} A+KC & 0 \\ B_Q C & A_Q \end{bmatrix},\ G = \begin{bmatrix} B \\ 0 \end{bmatrix},\ P = \begin{bmatrix} F+D_Q C & C_Q \end{bmatrix} \tag{4.2}$$

$$E = \begin{bmatrix} -K \\ -B_Q \end{bmatrix},\ V = -D_Q \text{ and } T = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

PROOF. The expressions of Lemma 4.1 are obtained simply by comparing terms with (2.8). Since both $A+KC$ and A_Q are stable by assumption so are D. The remaining three Luenberger conditions, namely $TA-DT=EC$, $G=TB$ and $F=PT+VC$ are verified directly by substitution of the above parameters.

In [Niemann, Søgaard-Andersen and Stoustrup 1991a] it has been shown that the poles of the state feedback part, the full order observer part and of $Q(s)$ can be assigned separately.

When imposing the Q–observer structure to the general $\mathcal{H}_\infty$/LTR problem (3.8) we get the following formulation.

PROBLEM 4.2. The general $\mathcal{H}_\infty$/LTR Q–observer design problem. *Let a weight matrix* $W(s)$ *as in (2.13), let a stabilizing full order observer with observer gain* K *and let* $\gamma > 0$ *be given. Find if possible,* $Q \in \mathcal{RH}_\infty$ *such that the weighted recovery matrix for the corresponding Q–observer satisfies:*

$$\|W(s)M(s)\|_\infty < \gamma \tag{4.3}$$

or equivalently

$$\|W(s)[F(sI-A-KC)^{-1}B + Q(s)C(sI-A-KC)^{-1}B]\|_\infty < \gamma \tag{4.4}$$

In the rest of this paper we shall discuss the general $\mathcal{H}_\infty$/LTR design problem in the formulation of Problem 4.2. In the next section we shall derive state space formulae for a controller $Q \in \mathcal{RH}_\infty$ solving Problem 4.2.

5. Controller Design.

A step by step controller design procedure for the $\mathcal{H}_\infty$/LTR design problem as stated in Problem 4.2 will be given in this section. A complete discussion every single step will not be given here, please see instead [Niemann and Stoustrup 1991] for a detailed discussion. The purpose of the exposition here is to give design procedures in a form, which do not presume detailed knowledge of the underlying results from $\mathcal{H}_\infty$ theory. Consequently, involved algorithms are included to the necessary extent.

The weighted recovery matrix corresponding to the Q–observer structure given by

$$\|W(s)[F(sI-A-KC)^{-1}B + Q(s)C(sI-A-KC)^{-1}B]\|_\infty < \gamma \quad (5.1)$$

has the following standard state space $\mathcal{H}_\infty$ representation (see e.g. [Francis and Doyle 1987] for a description of the $\mathcal{H}_\infty$ standard problem):

$$\Sigma_G: \begin{cases} \dot{x} = \begin{bmatrix} A+KC & 0 \\ B_w F & A_w \end{bmatrix} x + \begin{bmatrix} 0 \\ B_w \end{bmatrix} u + \begin{bmatrix} B \\ 0 \end{bmatrix} w \\ y = \begin{bmatrix} C & 0 \end{bmatrix} x \qquad\qquad + \; 0 \cdot w \\ z = \begin{bmatrix} D_w F & C_w \end{bmatrix} x + \; D_w \cdot u \end{cases} \quad (5.2)$$

or short:

$$\Sigma_G: \begin{cases} \dot{x} = \bar{A}\cdot x + \bar{B}\cdot u + \bar{E}\cdot w \\ y = \bar{C}_1 x \qquad\quad + \bar{D}_1 w \\ z = \bar{C}_2 x + \bar{D}_2 u \end{cases} \quad (5.3)$$

The $\mathcal{H}_\infty$ standard problem associated with the state space representation (5.2) is a so called singular problem, because the direct feedthrough term $\bar{D}_1$ of the $w \mapsto y$ transfer function does not have full row rank (it is zero) as it is required in order to apply the standard regular $\mathcal{H}_\infty$ theory as in e.g. [Doyle et al. 1989]. Instead the approach of [Stoorvogel 1990, 1991, Stoorvogel and Trentelman 1991] will be taken, which is a generalization of the approach in [Doyle et al. 1989]. As a main difference, the singular $\mathcal{H}_\infty$ approach of [Stoorvogel 1990, 1991, Stoorvogel and Trentelman 1991] involves two certain quadratic matrix inequalities with some associated rank conditions rather than the two matrix Riccati equations known from [Doyle et al. 1989]. In our case, however, it is possible to recover one of the DGKF type Riccati equations if the direct feedthrough term $\bar{D}_2$ of the $w \mapsto z$ transfer function has full column rank.

5.1. The Singular $\mathcal{H}_\infty$ Design Procedure.

To find a solution to a singular $\mathcal{H}_\infty$ problem is slightly more complicated than to find the solution of a regular $\mathcal{H}_\infty$ problem. The single steps of the solution are described in this section in brief outline. More details will appear from the subsequent sections.

<u>Design Procedure.</u>

Step 1. Find the positive semidefinite solutions P and Q which satisfy two certain quadratic matrix inequalities along with for each two associated rank conditions. Further P and Q have to satisfy $\rho(PQ) < \gamma^2$.

Step 2. Based on the two solutions P and Q of Step 1, a new system $\Sigma_{PQ,G}$ is determined as a transformation of Σ_G :

$$\Sigma_G: \qquad\qquad\qquad \Sigma_{PQ,G}:$$

$$\begin{bmatrix} \bar{A} & \bar{B} & \bar{E} \\ \bar{C}_1 & 0 & \bar{D}_1 \\ \bar{C}_2 & \bar{D}_2 & 0 \end{bmatrix} \xrightarrow{T_{PQ}} \begin{bmatrix} \bar{A}_{PQ} & \bar{B}_{PQ} & \bar{E}_{PQ} \\ \bar{C}_{PQ,1} & 0 & \bar{D}_{PQ,1} \\ \bar{C}_{PQ,2} & \bar{D}_{PQ,2} & 0 \end{bmatrix}$$

The transformed system $\Sigma_{PQ,G}$ has two important properties: (1) a controller $Q(s)$ is a solution to the posed $\mathcal{H}_\infty$ problem for Σ_G if and only if it also solves the $\mathcal{H}_\infty$ problem for $\Sigma_{PQ,G}$, and (2) the transformed system $\Sigma_{PQ,G}$ is minimum phase.

Step 3. A state feedback gain L and an observer gain M are computed by solving two almost disturbance decoupling problems related to the transformed system $\Sigma_{PQ,G}$.

Step 4. An admissible $\mathcal{H}_\infty$ controller is given as:

$$Q(s) = -L(sI-\bar{A}_{PQ}-\bar{B}_{PQ}L-M\bar{C}_{PQ,1})^{-1}M .$$

For an $\mathcal{H}_\infty$ standard problem, where the direct feedthrough matrices $\bar{D}_1$ and $\bar{D}_2$ satisfy the usual regularity assumptions, the first step of the design procedure becomes equivalent to the two matrix Riccati equations known from [Doyle et al. 1989]. The coupling condition $\rho(PQ) < \gamma^2$ is common to the singular and the regular approaches.

The transformation in Step 2 is implicit in the usual design algorithm for a regular $\mathcal{H}_\infty$ problem, but it is important for the theory in both cases.

Step 3 involves two almost disturbance decoupling problems, which can be solved explicitly by a variety of methods. In the regular case they are replaced by two exact disturbance decoupling problems.

Both for the regular and the singular $\mathcal{H}_\infty$ problem, the above method provides observer based controller implementation.

An algorithm for computing the unique solutions to the two quadratic matrix inequalities with associated rank conditions is given in Appendix A.

5.2. The $\mathcal{H}_\infty$/LTR Controller.

The four step design procedure outlined in the previous subsection is now applied to the $\mathcal{H}_\infty$ standard problem described by (5.2) to derive expressions for the Q–observer implementation of the $\mathcal{H}_\infty$/LTR controller. Only equations required in the design process are given here. A more detailed description of each step in the design procedure applied to $\Sigma_{PQ,G}$ can be found in [Niemann and Stoustrup 1991].

<u>Step 1.</u>

In this design step we first determine a positive semidefinite matrix P which makes a certain matrix function $F_\gamma(P)$ positive semidefinite whence satisfying two rank type side constraints (an algorithm can be found in Appendix A). In this special case it turns out that the function $F_\gamma(P)$ becomes affine in P. P is uniquely determined by the following three conditions:

$$F_\gamma(P) = \begin{bmatrix} A_w^T P + P A_w + C_w^T C_w & P B_w + C_w^T D_w \\ B_w^T P + D_w^T C_w & D_w^T D_w \end{bmatrix} \geq 0 \quad (5.4)$$

$$\text{rank } F_\gamma(P) = \text{normrank } W(\cdot) \quad (5.5)$$

$$\text{rank} \begin{bmatrix} sI - A_w & -B_w \\ \multicolumn{2}{c}{F_\gamma(P)} \end{bmatrix} = n_w + \text{normrank } W(\cdot)\,,\ \forall s \in \bar{\mathbb{C}}^+ \quad (5.6)$$

where

$$\text{normrank } W(\cdot) = \max_s \text{rank } W(s) \quad (5.7)$$

Effectively, (5.4)–(5.7) reduces to a reduced order algebraic Riccati equation, see Appendix A.

Note, that P does depend on neither γ nor the control system Σ but only on Σ_w

Next, by similar methods we have to find a positive semidefinite matrix Q satisfying a Riccati inequality, a linear (homogeneous) matrix equation and two rank conditions:

$$G_\gamma(Q) = \bar{A}Q + Q\bar{A}^T + \bar{E}\bar{E}^T + \gamma^{-2} Q \bar{C}_2^T \bar{C}_2 Q \geq 0 \quad (5.8)$$

$$\begin{bmatrix} C & 0 \end{bmatrix} Q = 0 \quad (5.9)$$

$$\text{rank } G_\gamma(Q) = n_g \quad (5.10)$$

$$\text{rank} \begin{bmatrix} sI - \bar{A} - \gamma^{-2} Q \bar{C}_2^T \bar{C}_2 & G_\gamma(Q) \\ \bar{C}_1 & 0 \end{bmatrix} = n + n_w + n_g\,,\ \forall s \in \bar{\mathbb{C}}^+ \quad (5.11)$$

where

$$n_g = \text{normrank } H(\cdot) = \max_s \text{rank } C(sI-A)^{-1}B \quad (5.12)$$

Like above the calculation of Q essentially involves solving a reduced order Riccati equation, see Appendix A.

Finally, P and Q have to satisfy the coupling condition:

$$\rho(PQ) < \gamma^2 \quad (5.13)$$

If, eventually, after going through the above calculations, (5.13) is not satisfied, or if any of the two involved Riccati equations (see Appendix A) fails to be solvable, the design specifications can not be met by any admissible controller. Hence, Step 1 has to be repeated with a relaxed value of γ.

<u>Step 2.</u>

The transformed system $\Sigma_{PQ,G}$ is determined as:

$$\Sigma_{PQ,G} : \begin{cases} \dot{x} = \bar{A}_{PQ} \cdot x + \bar{B}_{PQ} \cdot u + \bar{E}_{PQ} \cdot w \\ y = \bar{C}_{PQ,1} x + \bar{D}_{PQ,1} w \\ z = \bar{C}_{PQ,2} x + \bar{D}_{PQ,2} u \end{cases} \quad (5.14)$$

where the parameters are defined by the following sequence:

$$Y := (I - \gamma^{-2} QP)^{-1} Q =: \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \quad (5.15)$$

$$\bar{A}_{PQ} = \begin{bmatrix} A_{PQ}^{11} & A_{PQ}^{12} \\ A_{PQ}^{21} & A_{PQ}^{22} \end{bmatrix} \quad (5.16)$$

$$\bar{B}_{PQ} = \begin{bmatrix} B_{PQ}^{11} \\ B_{PQ}^{21} \end{bmatrix} \quad (5.17)$$

with:

$$A^{11}_{PQ} := A+KC+\gamma^2 Y_{11}F^TD_w^TD_wF+\gamma^2 Y_{12}(PB_wF+C_w^TD_wF)$$

$$A^{12}_{PQ} := \gamma^2 Y_{11}(F^TB_wP+F^TD_wC_w)+\gamma^2 Y_{12}C^T_{P,2}C_{P,2} \tag{5.18}$$

$$A^{21}_{PQ} := B_wF+\gamma^2 Y^T_{12}F^TD_w^TD_wF+\gamma^2 Y_{22}(PB_wF+C_w^TD_wF)$$

$$A^{22}_{PQ} := A_w+\gamma^2 Y^T_{12}(F^TB_wP+F^TD_wC_w)+\gamma^2 Y_{22}C^T_{P,2}C_{P,2}$$

and

$$B^{11}_{PQ} := \gamma^2 Y^T_{11}F^TD_w^TD_w+\gamma^2 Y_{12}C^T_{P,2}D_w \tag{5.19}$$

$$B^{21}_{PQ} := B_w+\gamma^2 Y^T_{12}F^TD_w^TD_w+\gamma^2 Y_{22}C^T_{P,2}D_w$$

where $C_{P,2}$ is defined as a row minimal square root:

$$C^T_{P,2}C_{P,2} := A_w^TP+PA_w+C_w^TC_w \tag{5.20}$$

Now, $\tilde{C}_{PQ,2}$ is given by

$$\tilde{C}_{PQ,2} := \begin{bmatrix} D_wF & C_{P,2} \end{bmatrix} \tag{5.21}$$

Likewise, we define $\tilde{E}_{PQ}$ as a column minimal square root:

$$\tilde{E}_{PQ}\tilde{E}^T_{PQ} := \tilde{A}_{PQ}Y+Y\tilde{A}^T_{PQ}+\tilde{E}\tilde{E}^T+\gamma^2 Y\tilde{C}^T_{PQ,2}\tilde{C}_{PQ,2}Y \tag{5.22}$$

The remaining two parameters of $\Sigma_{PQ,G}$ are unchanged:

$$\tilde{C}_{PQ,1} = \begin{bmatrix} C & 0 \end{bmatrix} \tag{5.23}$$

$$\tilde{D}_{PQ,1} = 0 \tag{5.24}$$

Step 3.

Next we have to compute the two gains L and M as the solutions to two almost disturbance decoupling problems. The state feedback gain L is given by:

$$L = \begin{bmatrix} -F & L_2 \end{bmatrix} \tag{5.25}$$

where L_2 satisfies:

$$\|(C_{P,2}+D_wL_2)(sI-A^{22}_{PQ}-B^{21}_{PQ}L_2)\|_\infty < \gamma/(3\|\tilde{E}_{PQ}\|) \tag{5.26}$$

The state feedback problem (5.26) is very simple, since $(A^{22}_{PQ}, B^{21}_{PQ}, C_{P,2}, D_w)$ is a minimum phase system.

The observer gain $M = \begin{bmatrix} M_1^T & M_2^T \end{bmatrix}^T$ has to satisfy:

$$\| (sI-\tilde{A}_{PQ}-M\tilde{C}_{PQ,1})^{-1}\tilde{E}_{PQ} \|_\infty < \tilde{\gamma}$$

or

$$\left\| \begin{bmatrix} sI-A^{11}_{PQ}-M_1C & -A^{12}_{PQ} \\ -A^{21}_{PQ}-M_2C & sI-A^{22}_{PQ} \end{bmatrix}^{-1} \tilde{E}_{PQ} \right\|_\infty < \tilde{\gamma} \tag{5.27}$$

where

$$\tilde{\gamma} = \min \{ \gamma/\|D_wL\| , \|\tilde{E}_{PQ}\|/\|\tilde{B}_{PQ}L\| \} \tag{5.28}$$

Also (5.28) is a minimum phase problem.

Step 4.

Finally, we are ready to provide a formulae for an admissible $Q \in \mathbb{R}\mathcal{H}_\infty$ i.e. an internally stabilizing controller such that when we apply the control law $u = Qy$ for the system (5.2), the closed loop transfer function form w to z has $\mathcal{H}_\infty$ norm smaller than γ.

$$Q(s) = -L(sI-\tilde{A}_{PQ}-\tilde{B}_{PQ}L-M\tilde{C}_{PQ,1})^{-1}M \tag{5.29}$$

$$= \begin{bmatrix} F & -L_2 \end{bmatrix} \begin{bmatrix} sI-A^{11}_{PQ}+B^{11}_{PQ}F-M_1C & -A^{12}_{PQ}-B^{11}_{PQ}L_2 \\ -A^{21}_{PQ}+B^{21}_{PQ}F-M_2C & sI-A^{22}_{PQ}-B^{21}_{PQ}L_2 \end{bmatrix}^{-1} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

To summarize we have the following result, which is proved by verifying each of the four design steps.

THEOREM 5.1. *When the controller Q given by (5.29) is applied as the parameter in a Q–observer based controller with observer gain K, the transfer function from input to output in the closed loop control system has $\mathcal{H}_\infty$ norm smaller than γ.*

The controller given by (5.29) has dynamic order $n+n_w$ which means that the complete controller will be of order $2n+n_w$ if no model order reduction is carried out. However, if we allow also the observer gain K to be a design parameter, it turns out that a structural reduction can be performed which in no way affects the obtainable $\mathcal{H}_\infty$ norm.

The basic idea is to first design the controller of order $2n+n_w$ as explained above, and second to simultaneously modify the observer gain to $K^* = K+M_1$ and the parameter part Σ_Q to Σ_Q^* (see Fig. 5.1). The interconnection structure will in general be more complicated for the transformed controller. We only for directly reobtain the Q–observer structure for D_w injective. But the controller can always be reduced to order $n+n_w$ as a Luenberger observer. The Luenberger parameters are listed in Theorem 5.2.

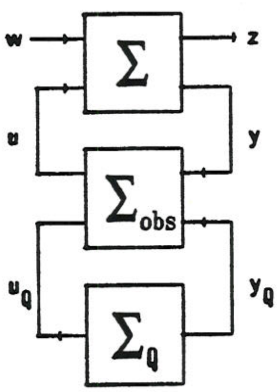

Fig. 5.1(a). Original system with controller of order $2n+n_w$.

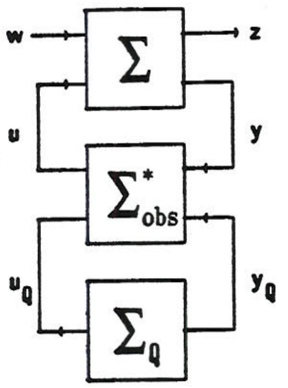

Fig. 5.1(b). Transformed system with controller of order $n+n_w$.

Theorem 5.2. *The cascade of Σ^*_{obs} and Σ^*_Q as explained above (see Fig. 5.1) is a Luenberger observer based controller with the following choice of parameters:*

$$D = \begin{bmatrix} A+KC+M_1C & A^{12}_{PQ}+B^{11}_{PQ}L_2 \\ M_2C & A^{22}_{PQ}+B^{21}_{PQ}L_2 \end{bmatrix}, \quad G = \begin{bmatrix} B \\ 0 \end{bmatrix}, \tag{5.30}$$

$$P = \begin{bmatrix} F & -L_2 \end{bmatrix}, \quad E = \begin{bmatrix} K+M_1 \\ M_2 \end{bmatrix}, \quad V = 0, \quad T = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

This Luenberger observer based controller, when applied to the weighted recovery matrix as described by (2.14) makes the closed loop transfer function from w to z smaller than γ.

Proof. By computing the transfer function of the Luenberger observer based controller given in Theorem 5.2 the same transfer function is obtained as when applying the Q-parameter given by (5.29) in a Q–observer based controller.

It is seen from the Luenberger parameters in Theorem 5.1, that the dynamic order of the controller is $n+n_w$.

6. Concluding Remarks.

A general $\mathcal{H}_\infty$/LTR method has been proposed in this paper which includes previous methods as special cases.

Based on robust design specifications, sufficient conditions have been derived for the design of observer based $\mathcal{H}_\infty$ controllers in the LTR framework.

First, a general recovery design problem was formulated in terms of the specifications for robust stability and performance. Second, this recovery problem was restated as an $\mathcal{H}_\infty$ optimization problem. Third, state space formulae were provided for the solution to the $\mathcal{H}_\infty$ problem both in terms of a Luenberger observer based controller and, alternatively, in terms of a Q–observer based controller.

Inherent to the structure of control problems with simultaneous specifications for robust stability and performance, conservatism is introduced when the problem is restated as an $\mathcal{H}_\infty$ standard problem. The nature of this conservatism has been analyzed, and a quantitative analysis given.

The dynamic order of the $\mathcal{H}_\infty$/LTR controllers is in general $n+n_w$ ($2n+n_w$ for the Q–observer). When using a weight matrix W(·) with higher order dynamics, better robustness can be obtained at the cost of additional controller states.

Finally, a step by step algorithm has been provided for the $\mathcal{H}_\infty$/LTR controller design scheme.

Appendix A.

In this appendix we shall describe an algorithm for the solution to the *quadratic matrix inequality* which is a basic ingredient of the design algorithm of Section 5.

To be more precise: given a constant $\gamma > 0$ and five matrices (A, B, E, C, D), which can be interpreted as the following dynamical system:

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew & x \in \mathbb{C}^n \;\; u \in \mathbb{C}^m \;\; w \in \mathbb{C}^q \\ z = Cx + Du & z \in \mathbb{C}^p \end{cases}$$

we wish to find a positive semidefinite matrix $P \geq 0$ such that the following three are all satisfied:

(a)
$$\begin{bmatrix} A^TP+PA+C^TC+\gamma^{-2}PEE^TP & PB+C^TD \\ B^TP+D^TC & D^TD \end{bmatrix} =: \begin{bmatrix} C_P^T \\ D_P^T \end{bmatrix} \begin{bmatrix} C_P & D_P \end{bmatrix} \geq 0$$

(b) $\operatorname{rank} C_P = p$

(c) $(A+\gamma^2 EE^TP, B, C_P, D_P)$ is a minimum phase system.

The algorithm below is based on [Stoorvogel 1990]. We shall use the following notation:

$Y := \mathrm{RAN}(X)$, where X is a matrix, gives a matrix Y whose columns are a basis for the column space of X.

$Y := \mathrm{NULL}(X)$, where X is a matrix, gives a matrix Y whose columns are a basis for the kernel of X, $XY = 0$.

Step 1 – Calculation of the Almost Controllability Subspace.

(1) *Initialize:* $R_a := R_b := 0$.

(2) $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} := \mathrm{NULL}([CR_b\ ,\ D])$

(3) $R_a := R_b S_1$

(4) $R_b := \mathrm{RAN}([AR_a + BS_2\ ,\ B*\mathrm{NULL}(D)])$

(5) *Stop, if* R_b *did not increase its number of columns in* (4), *otherwise go to* (2).

Step 2 – State Space Transformation.

(6) $T_2 := R_a$

(7) *Determine* T_3 *such that* $\mathrm{RAN}([T_2, T_3]) = \mathrm{RAN}(R_b)$ *and the columns of* $[T_2, T_3]$ *are linearly independent.*

(8) *Determine* T_1 *such that the columns of* $[T_1, T_2, T_3]$ *form a basis of* $\mathbb{R}^n$.

(9) $[S_1^T, S_2^T, S_3^T]^T := [T_1, T_2, T_3]^{-1}$

(10) *Determine* F *as any solution to* $D^T DF = -D^T C$

(11) $A_{11} := S_1(A+BF)T_1$

(12) $A_{13} := S_1(A+BF)T_3$

(13) $B_1 := S_1 B$

(14) $C_{21} := (C+DF)T_1$

(15) $C_{23} := (C+DF)T_3$

(16) $E_1 := S_1 E$

Step 3 – Reduced Order Riccati Equation.

(17) $\bar{A} := A_{11} - A_{13}(C_{23}^T C_{23})^{-1} C_{23}^T C_{21}$

(18) $\bar{R} := \gamma^{-2} E_1 E_1^T - B_1(\mathrm{RAN}(D)^T \mathrm{RAN}(D))^{-1} B_1^T$

(19) $\bar{Q} := C_{21}^T C_{21} - C_{21}^T C_{23}(C_{23}^T C_{23})^{-1} C_{23}^T C_{21}$

(20) *Solve the algebraic Riccati equation:*
$\bar{P}\bar{A} + \bar{P}\bar{A} + \bar{P}\bar{R}\bar{P} + \bar{Q} = 0$

Step 4 – Solution to Quadratic Matrix Inequality.

(21) $P := S_1^T \bar{P} S_1$

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