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A GENERAL H_∞/LTR DESIGN PROBLEM

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ABSTRACT.

The emphasis of this paper is on an alternative approach to the Loop Transfer Recovery (LTR) design problem based on H_∞ optimization. An H_∞/LTR design problem is formulated as an H_∞ optimization of the weighted Recovery Matrix.

This general recovery formulation includes the indirect H_∞/LTR design problem (equivalent to LQG/LTR), the H_∞/LTR sensitivity and the input-output recovery problem as special cases. Moreover, the weight matrix is also used for obtaining robustness in the final design. The control problem corresponding to the general H_∞/LTR design problem is formulated as a standard H_∞ state space problem. The state-space solution to the H_∞ problem is derived and the corresponding H_∞/LTR controller is implemented as a Luenberger observer of order at most $n + n_w$ (n_w is the order of the weight on the Recovery Matrix).

The proposed H_∞/LTR method handles both minimum phase as well as non-minimum phase systems in the same framework.

1. INTRODUCTION.

Since the paper by Doyle and Stein [4,5] has introduced the concept of Loop Transfer Recovery (LTR) design of observer-based controllers, a lot of methods based on this concept has been derived, see e.g. [1,8,11,12,14,17,24,27] for both continuous-time and discrete-time systems.

The first recovery methods was based on the LQG method, LQG/LTR, and only related to minimum phase systems, where the difference between the desired and the obtained transfer functions can be made arbitrary small, i.e. asymptotic recovery can be obtained, [14,19]. The applied norm on the difference between the desired and the obtained transfer functions, which will be called the recovery error [15], is of less importance when asymptotic recovery can be obtained. The most common way for analyzing the recovery design is by using singular value plots of the recovery errors. Such a design scheme has an iterative nature where no guarantee is given for obtaining a satisfactory recovery design except in the asymptotic case. However, the iterative part in the LTR design process can be removed by using H_∞ methods in combination with LTR. The LTR step in the controller design process becomes a systematic design step when H_∞ methods are applied. The LTR step can then be separated into three parts: First, specification of a weight matrix on the recovery error, which reflect the upper bounds on the recovery error and a recovery bound. Second, testing (by a one shot method) if the specified recovery bounds can be satisfied by an H_∞ controller. At last, an H_∞ controller is determined which satisfies the specified recovery conditions.

This LTR-concept has first been introduced by Moore and Tay [12] where the sensitivity recovery error, i.e. the difference between the desired and the obtained sensitivity transfer functions, has been minimized by using an H_∞ method. However, the H_∞/LTR method derived in [12] has three drawbacks: First, only approximative H_∞ solutions are derived, due to the used frequency-domain method. Secondly, the order of the final observer based controllers are 2n for square systems and 3n-1 otherwise in the minimum phase case. However, it is always possible to reduce the H_∞ norm of the recovery errors by nth order controllers in the minimum phase case, [22]. At last, for non-minimum phase systems only the minimum phase part is considered and no norm bounds are guaranteed for the over-all system. But as a matter of fact, the main importance of direct design methods, are their application to non-minimum phase systems, as it will appear in the course of this paper.

The key contribution of this paper is to formulate a general and concise H_∞/LTR design problem based on the Luenberger observer and derive the associated state-space solutions. By using recovery errors, a certain matrix named the Recovery Matrix, has a very central role for observer-based controllers, [15]. Our general H_∞/LTR design problem is formulated as a minimization of the H_∞ norm of the weighted recovery matrix. The direct and indirect H_∞/LTR methods for observer based controllers [22,23] are all included in this formulation as special cases.

The freedom in selecting the weight matrix in the criterium gives a very flexible LTR design method. This freedom can be used to obtain H_∞/LTR control feedback systems satisfying robustness specifications, minimizing RHP zeros' influence on the final feedback loop etc. It is also possible to make LQG/LTR-like controllers by this method.

Derivation of the state space solution to the H_∞/LTR problem is based on the requirement that the applied observer is a Q-parameterized observer. This requirement does never prevent solvability, since all Luenberger observers can be realized as Q-observers, [15].

The solution is derived based on the singular H_∞ approach [18,19,20], which is a generalization of the regular DGKF approach [3]. As a main difference, the singular H_∞ approach includes two certain quadratic matrix inequalities with some associated rank constraints, rather than the two matrix Riccati equations known from [3]. When the singular H_∞ theory is applied on the H_∞/LTR problem, we can derive the associated H_∞/LTR observer-based controllers and reduce the order to $n + n_w$ (n_w = order of the weight matrix) without any approximations. Further, only one of the applied quadratic matrix inequalities depend on the selected H_∞ constraint. Both minimum phase as well as non-minimum phase systems are handled in a common framework.

The paper is organized as follows. In section 2 the background will briefly be summarized and the general H_∞/LTR design problem will be formulated. Further, an analysis of the selection of the weight matrix for obtaining robust feedback design are included. In section 3 the state-space solutions are derived based on the singular H_∞ theory followed by a conclusion in section 4.

2. A GENERAL LTR PROBLEM FORMULATION.

A general H_∞/LTR design problem will be formulated in this section when the general dynamical measurement feedback controller, the Luenberger observer is used.

2.1 The Luenberger observer based controller.

Let's consider the FDLTI system represented by the state space realization (A,B,C):

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{2.1}$$

with the transfer function:

$$G(s) = C(sI - A)^{-1}B = C\Phi(s)B \tag{2.2}$$

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$ and A,B and C are matrices of appropriate dimensions. The systems Σ is assumed to be stabilizable, detectable and left invertible. Moreover, we shall make the technical assumption that (A,B,C) has no transmission zero on the imaginary axis.

A Luenberger observer [10] is used for the feedback control of the system Σ. The Luenberger observer is described by the following state-space realization:

$$\Sigma_L: \begin{cases} \dot{z} = Dz + Gu + Ey \\ v = Pz + Vy \end{cases} \tag{2.3}$$

where the matrices in (2.3) satisfies [10]:

$$\text{i)}\quad \Lambda(D) \subset \mathbb{C}^- \qquad \text{ii)}\quad TA - DT = EC$$
$$\text{iii)}\quad G = TB \qquad \text{iiii)}\quad F = PT + VC \tag{2.4}$$

The output signal w from the Luenberger observer is related to the state feedback signal as:

$$u = v + r = F\hat{x} + r \tag{2.5}$$

where $\hat{x}$ is an estimate of the plant state and F is the state-feedback gain.
Due to the following recovery design, the implementation of the Luenberger observer is given in the following form, also called the recovery form [15]:

$$C(s) = (I + M_l(s))^{-1}N_l(s) \tag{2.6}$$

where

$$M_l(s) = P(sI - D)^{-1}G$$
$$N_l(s) = P(sI - D)^{-1}E + V \tag{2.7}$$

The same observer form has been described in [9] in connection with a coprime factorization of full order observers.

$M_l(s)$ in (2.7) will in the following be called the Recovery Matrix, because M_l has a very central role in the recovery design.

2.2 The General H_∞/LTR Design Formulation.

The general H_∞/LTR design problem will firstly be defined when a Luenberger observer are used as the feedback controller, followed by a description of the generality in the H_∞/LTR design problem.

Consider the following FDLTI system represented by the state space realization (A_w,B_w,C_w,D_w):

$$\Sigma_w: \begin{cases} \dot{x} = A_w x + B_w u \\ y = C_w x + D_w u \end{cases} \tag{2.8}$$

with the transfer function:

$$W(s) = C_w(sI - A_w)^{-1}B_w + D_w \tag{2.9}$$

Here $x \in \mathbb{R}^{nw}$, $u \in \mathbb{R}^r$, $y \in \mathbb{R}^p$ and A_w, B_w, C_w and D_w are matrices of appropriate dimensions. The system Σ_w is assumed to be stabilizable, detectable and without transmissions zeros on the imaginary axis.

With W(s) as the weight matrix in the H_∞/LTR design problem, we are now able to formulate the general H_∞/LTR design problem:

Problem 2.1. *The general H_∞/LTR design problem.*
Let the weight matrix W(s) be as in (2.9) and let $\gamma > 0$ be given. Find, if possible, a FDLTI Luenberger controller such that the weighted transfer function from u to v, M_l, satisfies:

$$\|W(s)M_l(s)\|_\infty < \gamma \tag{2.10}$$

Here $\|\cdot\|_\infty$ is the H_∞ norm.

2.3 Selection of W(s) for the recovery design.

Some special selections of the weight matrix W(s) in Problem 2.1 are studied in this section. Further, the connection between Problem 2.1 and other H_∞/LTR design methods [16,22,23] are also derived. A more detailed description of the results given in this section can be found in [15,22,23].

Let the target open-loop, the target sensitivity and the input-output (closed-loop) transfer functions be given by:

$$G_{TFL}(s) = F(sI - A)^{-1}B$$
$$S_{TFL}(s) = (I - G_{TFL}(s))^{-1} \tag{2.11}$$
$$G_{TFL,IO}(s) = C(sI - A - BF)^{-1}B$$

and equivalently for the full-loop transfer functions:

$$G_l(s) = C(s)G(s)$$
$$S_l(s) = (I - G_l(s))^{-1} \tag{2.12}$$
$$G_{l,IO}(s) = G(s)(I - G_l(s))^{-1}$$

where C(s) is the Luenberger observer given by (2.6). Further, let the open-loop recovery error $E_O(s)$, the sensitivity recovery error $E_S(s)$ and the input-output recovery error $E_{IO}(s)$ be defined as the difference between the respective target and full-loop transfer functions, see [8,15,22]. These three recovery errors can be rewritten in more convenient forms:

Lemma 2.2.
Let the recovery matrix $M_l(s)$ be given by (2.7). Then

$$E_O(s) = M_l(s)(I - M_l(s))^{-1}(I - G_{TFL}(s))$$
$$E_S(s) = S_{TFL}(s)M_l(s) \tag{2.13}$$
$$E_{IO}(s) = G_{TFL,IO}(s)M_l(s)$$

Proof. The proof of Lemma 2.2 can be found in [15].

The recovery matrix $M_l(s)$ introduced in (2.6) as an open-loop transfer function from u to v in the Luenberger observer is very strongly related to the recovery errors as shown in Lemma 2.2.
By selecting the weight matrix W(s) in the H_∞/LTR problem as the target sensitivity or the target input-output transfer function, $S_{TFL}(s)$ or $G_{TFL,IO}(s)$ resp., the obtained design problem is a minimization of the associated sensitivity or input-output recovery error as given in Lemma 2.2 [22,23]. Another special H_∞/LTR problem is the indirect design case, which is obtained by using W(s) = I in Problem 2.1.

Robust controller design using LTR can also be handled by the selection of the weight matrix W(s).

Suppose the design specifications for the final feedback system are given as bounds on the sensitivity transfer function S(·) and the complementary sensitivity transfer function T(·):

$$\|W_p(s)S(s)\|_\infty < \gamma$$
$$\|W_\pi(s)T(s)\|_\infty < \gamma \tag{2.14}$$

The performance specification (e.g. asymptotic tracking, bandwidth) are expressed by the weight function $W_p(s)$ on the sensitivity function [5]. The weight $W_\pi(s)$ on T(·) reflects the systems uncertainties such as disturbance, noise and modelling errors, i.e. robust stability specification.
It is assumed that the target design satisfies the design specifications given by (2.14), i.e.

$$\|W_p(s)S_{TFL}(s)\|_\infty < \gamma_{TFL}, \quad \|W_n(s)T_{TFL}(s)\|_\infty < \gamma_{TFL} \quad \gamma_{TFL} < \gamma \qquad (2.15)$$

The full-loop transfer function will then satisfies the design specification in (2.14) if, [23]:

$$\left\| \begin{bmatrix} W_p(\bullet) \\ W_n(\bullet) \end{bmatrix} E_r(\bullet)\right\|_\infty = \left\| \begin{bmatrix} W_p(\bullet) \\ W_n(\bullet) \end{bmatrix} S_{TFL}(\bullet)M_I(\bullet)\right\|_\infty < \gamma_{LTR} \qquad (2.16)$$

where

$$\gamma_{LTR} = \frac{\gamma - \gamma_{TFL}}{\sqrt{2}} \qquad (2.17)$$

The weight matrix W(s) must therefore be selected as:

$$W(s) = \begin{bmatrix} W_p(s) \\ W_n(s) \end{bmatrix} S_{TFL}(s) \qquad (2.18)$$

Note, that an upper bound on the recovery level is given in this case.

2.4 The Q-parameterized Observer.

In this section we shall consider a special architecture for the Luenberger observer, the Q-observer. The Q-observer is a parameterized implementation of the Luenberger observer, which realize the class of all internally stabilizing controllers in an observer based form by means of the Youla (or Q-) parameterization [2]. Briefly, the principle in the Youla parameterization is to take any stabilizing controller which is thereafter fixed, and then make a certain interconnection structure. The class of all stabilizing controllers are parameterized by applying the class of all H_∞ systems at the interconnection nodes. In [2] it has been shown, that the construction shown in fig. 2.1 is an implementation of the Youla parameterization, which we shall refer to in the sequel as the Q-observer.

We shall need the following result:

Lemma 2.3.
Assume that $Q \in RH_\infty$, with a state-space representation, say,

$$\Sigma_Q : \begin{cases} \dot{\xi} = A_Q\xi + B_Q y_Q \\ u_Q = C_Q\xi + D_Q y_Q \end{cases} \qquad (2.19)$$

Here $\xi \in R^q$, is the order of Q. Then the corresponding Q-observer is a Luenberger observer with the following parameters:

$$D = \begin{bmatrix} A+KC & 0 \\ -B_QC & A_Q \end{bmatrix}$$
$$G = \begin{bmatrix} B \\ 0 \end{bmatrix}$$
$$P = \begin{bmatrix} F - D_QC & C_Q \end{bmatrix} \qquad (2.20)$$
$$E = \begin{bmatrix} -K \\ B_Q \end{bmatrix}$$
$$V = D_Q$$
$$T = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Proof. Comparing terms with (2.3) we obtained the above expressions for D,G,E,P and V. The Luenberger equation TA - DT = EC, G = TB and F = PT + VC are verified by inspection. Since A + KC and A_Q are stable by assumption, also D is stable. Thus the Luenberger conditions (2.4) are satisfied.

In [15] it has been shown that the poles of the state feedback part, the full-order observer part and of Q(s) can be assigned separately.

By using the equation for the recovery matrix in (2.7) together with the general H_∞/LTR design problem for the Q-observer, Problem 2.1 turns out to be:

Problem 2.4. *The general H_∞/LTR design problem using Q-observers. With the weight matrix W(s) as in (2.9) and $\gamma > 0$ be given. Find, if possible, a Q-observer such that the weighted recovery matrix $M_I(s)$ satisfies:*

$$\|W(j\omega)M_I(j\omega)\|_\infty < \gamma \qquad (2.21)$$

or equivalently:

$$\|W(j\omega)(F(j\omega I - A - KC)^{-1}B + Q(j\omega)C(j\omega I - A - KC)^{-1}B\|_\infty \qquad (2.22)$$

The recovery matrix for the Q-observer as used in (2.22) is derived in [15].

This formulation of the general H_∞/LTR design problem based on the Q-observer architecture will be used in the sequel of this paper. In the next section a state-space solution to (2.22) is derived.

3 THE H_∞/LTR STATE-SPACE SOLUTION.

In the following we shall consider the general recovery problem formulated with an H_∞ optimality criterion. Following the approach of [15] observer-based controllers will be studied, which posses the Q-observer structure introduced in Section 2.4 based on the Youla (Q-) parameterization (Problem 2.4).

The weighted recovery matrix corresponding to the Q-observer structure given by:

$$W(s)M_I(s) = W(s)(F(sI-A-KC)^{-1}B + Q(s)C(sI-A-KC)^{-1}B) \qquad (3.1)$$

has the following standard state-space H_∞ representation (see e.g. [6] for a description of the standard H_∞ problem):

$$\Sigma_{WM,Q} : \begin{cases} \dot{x} = \begin{bmatrix} A_K & 0 \\ B_WF & A_W \end{bmatrix} x + \begin{bmatrix} 0 \\ B_W \end{bmatrix} u + \begin{bmatrix} B \\ 0 \end{bmatrix} w \\ y = \begin{bmatrix} C & 0 \end{bmatrix} x + 0w \\ z = \begin{bmatrix} D_WF & C_W \end{bmatrix} x + D_Wu \end{cases} \qquad (3.2)$$

where $A_K = A + KC$.

The H_∞/LTR design problem for a given γ is now, if possible, to design a FDLTI dynamic controller u = Q(s)y which internally stabilizes the plant, and make the H_∞ norm of the resulting closed-loop transfer function from w to z i.e. the norm of $W(s)M_I(s)$, smaller than γ.

The H_∞ problem with the state-space representation (3.2) is a so-called singular problem, because the direct feedthrough term of the $w \rightarrow y$ transfer function does not have full column rank, (it is zero) as it is required in order to apply the standard regular H_∞ theory as in e.g. [3]. Instead the approach of [18,19,20] will be taken, which is a generalization of the result in [3]. As a main difference, the singular H_∞ approach of [18,19,20] includes two certain Quadratic Matrix Inequalities (QMI) with some associated rank constraints, rather than two matrix Riccati equations known from [3]. In our case, however, it is possible to recover one of the DGKF type Riccati equations, if the direct feedthrough term of $u \rightarrow z$ transfer function is injective. The singular H_∞ theory used in this paper is shortly summarized in Appendix A.

3.1 Solutions Of The Two Quadratic Matrix Inequalities.

The solutions of the two Quadratic Matrix Inequalities (QMI) will be derived in this section for the system $\Sigma_{WM,Q}$ given by (3.2) together with the associated transformations.

From Assumption A.1 it is required that (A,B,C,0) and (A_W,B_W,C_W,D_W) has no zeros on the imaginary axis. This is assumed

throughout this section.
First, we wish to find a solution to the QMI, so we can perform the quadratic matrix- (QM) transformation. For the QMI we have the following.

Theorem 3.1. *The QMI solution.*
The Quadratic Matrix Inequality associated with the system $\Sigma_{WM,Q}$ has the solution:

$$\bar{P} = \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \tag{3.3}$$

where P is the unique matrix satisfying:

$$F\gamma(P) = \begin{bmatrix} A_W^TP + PA_W + C_W^TC_W & PB_W + C_W^TD_W \\ B_W^TP + D_W^TC_W & D_W^TD_W \end{bmatrix} \geq 0$$

$$rank[F_\gamma(P)] = normrank[W(s)],\ \forall s \in \mathbf{C}^+ \cup \mathbf{C}^0 \tag{3.4}$$

$$rank\begin{bmatrix} sI-A_W & -B \\ & F_\gamma(P) \end{bmatrix} = n_w + normrank[W(s)],\ \ \forall s \in \mathbf{C}^+ \cup \mathbf{C}^0$$

with $W(s) = C_W(sI-A_W)^{-1}B_W+D_w$.

$\bar{F}_\gamma(\bar{P})$ factorizes as:

$$\bar{F}_\gamma(\bar{P}) = \begin{bmatrix} F^TD_P^T \\ C_{2,P}^T \\ D_P^T \end{bmatrix} \times [\ D_PF \quad C_{2,P} \quad D_P\ ] \tag{3.5}$$

The proof is omitted.

Note that the solution of QMI does not depend on γ. Solvability of the general H_∞/LTR problem will effectively depend only on solvability of the transformed Dual Quadratic Matrix Inequality (DQMI). Further, in the special case when (A_W,B_W,C_W,D_W) is minimum phase, P = 0 is the unique solution for QMI.

After the solution of the QMI, the QM-transformation is performed. In general, however, the QM-transformation will be non-trivial, and amount to, see (A.6) - (A.7):

$$\bar{A}_P = \bar{A} \qquad \bar{C}_{1,P} = \bar{C}_1$$
$$\bar{C}_{2,P} = [\ D_PF \quad C_{2,P}\ ] \quad \bar{D}_{2,P} = D_P \tag{3.6}$$

On the QM-transformed system, the solution of the DQMI is now derived. The DQMI is a totally singular problem, so the dual version of Corollary A.4 is applied.

Lemma 3.2.
For the DQMI associated with the system $\Sigma_{WM,Q}$ with matrices modified as in (3.6), the solution:

$$\bar{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \quad \textit{satisfies } CY_{11} = 0 \textit{ and } CY_{12} = 0 \tag{3.7}$$

in addition to the condition (4) and (6) of Theorem A.2 is the unique solution of the DQMI.

$\bar{G}_\gamma(\bar{Y})$ factorizes as:

$$\bar{G}_\gamma(\bar{Y}) = \begin{bmatrix} \bar{E}_{P,Q} \\ 0 \end{bmatrix} \times [\ \bar{E}_{P,Q}^T \quad 0\ ] \tag{3.8}$$

Proof. Lemma 3.2 is a direct consequence of Theorem A.2.

As for the solution to the QMI, it will also be possible to simplify the solution of the DQMI in special cases, as it appears from the following Corollary.

Corollary 3.3.
If (and only if) the system (A,B,C,0) is minimum phase and invertible, Y = 0 is a solution to the transformed DQMI.

More generally, though, the DQM-transformation will result in the following matrices:

$$\bar{A}_{P,Q} = \bar{A} + \gamma^{-2}\bar{Y}\bar{C}_{2,P}^T\bar{C}_{2,P} = \begin{bmatrix} A_{P,Q}^{11} & A_{P,Q}^{12} \\ A_{P,Q}^{21} & A_{P,Q}^{22} \end{bmatrix} \tag{3.9}$$

$$\bar{B}_{P,Q} = \bar{B} + \gamma^{-2}\bar{Y}\bar{C}_{2,P}^T\bar{D}_{2,P} = \begin{bmatrix} B_{P,Q}^{11} \\ B_{P,Q}^{21} \end{bmatrix} \tag{3.10}$$

with

$$A_{P,Q}^{11} = A_K + \gamma^{-2}Y_{11}F^TD_P^TD_PF^T + \gamma^{-2}Y_{12}(PB_WF + C_W^TD_WF)$$

$$A_{P,Q}^{12} = \gamma^{-2}Y_{11}(F^TB_WP + F^TD_PC_W) + \gamma^{-2}Y_{12}C_{2,P}^TC_{2,P}$$

$$A_{P,Q}^{21} = B_WF + \gamma^{-2}Y_{12}^TF^TD_P^TD_PF + \gamma^{-2}Y_{22}(PB_WF + C_W^TD_WF)$$

$$A_{P,Q}^{22} = A_W + \gamma^{-2}Y_{12}^T(F^TB_WP + F^TD_WC_W) + \gamma^{-2}Y_{22}C_{2,P}^TC_{2,P}$$

$$B_{P,Q}^{11} = \gamma^{-2}Y_{11}F^TD_P^TD_P + \gamma^{-2}Y_{12}C_{2,P}^TD_P$$

$$B_{P,Q}^{21} = B_W + \gamma^{-2}Y_{12}^TF^TD_P^TD_P + \gamma^{-2}Y_{22}C_{2,P}^TD_P$$

Performing these transformations, we eventually obtain the controller, solving the H_∞ problem as described in the following section.

3.2 The General H_∞/LTR Controller.

Based on the transformed system in (3.6), (3.9) and (3.10), the controller Q(s) can now be calculated.

Lemma 3.4.
Assume that γ has been chosen sufficient large. Let $L = [L_1\ L_2]$ be a state feedback and $M = [M_1^T\ M_2^T]^T$ be an output injection satisfying:

$$\left\| [D_PF+D_PL_1 \quad C_{2,P}+D_PL_2] \times \begin{bmatrix} sI - A_{P,Q}^{11} - B_{P,Q}^{12}L_1 & -A_{P,Q}^{12} - B_{P,Q}^{11}L_2 \\ -A_{P,Q}^{21} - B_{P,Q}^{12}L_1 & sI - A_{P,Q}^{22} - B_{P,Q}^{21}L_2 \end{bmatrix}^{-1} \right\|_\infty < \bar{\gamma}$$

$$\bar{\gamma} = \gamma/(3\|\bar{E}_{P,Q}\|) \tag{3.11}$$

and

$$\left\| \begin{bmatrix} sI - A_{P,Q}^{11} - M_1C & -A_{P,Q}^{12} \\ -A_{P,Q}^{21} - M_2C & sI - A_{P,Q}^{22} \end{bmatrix}^{-1} \bar{E}_{P,Q} \right\|_\infty < \tilde{\gamma} \tag{3.12}$$

$$\tilde{\gamma} = \min\{\gamma/(3\|D_PL\|),\ \|\bar{E}_{P,Q}\|/\|\bar{B}_{P,Q}L\|\}$$

respectively.

Then a controller, $u = Q(s)y$, making the closed loop internally stable, and making the H_∞ norm of the transfer function from w to z smaller than γ is given by:

$$Q(s) = -[L_1\ L_2]\times\begin{bmatrix} sI - A^{11}_{P,Q} - B^{11}_{P,Q}L_1 - M_1C & -A^{12}_{P,Q} - B^{11}_{P,Q}L_2 \\ -A^{21}_{P,Q} - B^{21}_{P,Q}L_1 - M_2C & sI - A^{22}_{P,Q} - B^{21}_{P,Q}L_2 \end{bmatrix}^{-1}\times\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad (3.13)$$

Proof. Lemma 3.4 follows directly from Theorem A.5, see [18,19].

Moreover, whenever a solution exist, the state feedback L can always be chosen with a special structure.

Lemma 3.5.
The state feedback L used in the controller Q(s) Lemma 3.4 might always be chosen as:

$$L = [\ -F \quad L_2\ ] \quad (3.14)$$

where L_2 satisfy:

$$\|(C_{2,P}+D_PL_2) \times (sI-A^{22}_{P,Q}-B^{21}_{P,Q}L_2)^{-1}\|_\infty < \gamma/(3\|E_{P,Q}\|) \quad (3.15)$$

The proof is omitted.
The controller given by Lemma 3.4 is of order $n + n_w$, which means that the complete controller will be of order $2n + n_w$, if no reduction is carried out. It turns out, though, that a structural reduction can be performed without affecting the obtained H_∞-norm. The basic idea is to use the remaining freedom in the observer gain K to reduce the order to $n + n_w$. First we change the observer gain to $K^* = K + M_1$ obtaining a modified observer Σ_{obs}^*, reobtaining the transfer function of the $2n + n_w$ order controller. The 'cascading' controller is in general a little more complex than the original Q-observer. Only when D_W is injective, the cascading controller will have the structure of an Q-observer, i.e. $A_{P,Q}^{12} + B_{P,Q}^{21}L_2 = 0$. But it will always be a Luenberger observer whose parameters are described in Theorem 3.6.

Theorem 3.6.
The cascade of Σ_{obs}^* and Σ_Q^* described above is a Luenberger observer based controller with the following characteristic matrices:

$$\begin{aligned} D &= \begin{bmatrix} A + KC + M_1C & A^{12}_{P,Q} + B^{11}_{P,Q}L_2 \\ M_2C & A^{22}_{P,Q} + B^{21}_{P,Q}L_2 \end{bmatrix} \\ G &= \begin{bmatrix} B \\ 0 \end{bmatrix} \\ P &= [\ F \quad -L_2\ ] \\ E &= \begin{bmatrix} K + M_1 \\ M_2 \end{bmatrix} \\ V &= 0 \\ T &= \begin{bmatrix} I \\ 0 \end{bmatrix} \end{aligned} \quad (3.16)$$

This Luenberger observer based controller, when applied to the weighted recovery matrix as described by (3.1) makes the H_∞ norm of the closed loop transfer function from w to z smaller than γ. The controller order is of $(n + n_w)$'th order.

The proof is omitted.

When applying a weight matrix with an injective D_W, the norm inequality in (3.15) can be satisfied exactly by selecting:

$$D_PL_2 = -C_{2,P} \text{ or } L_2 = -D_P^+C_{2,P} \quad (3.17)$$

which solves an exact disturbance decoupling problem. In this case, the output injection M must satisfy:

$$\|(sI - \bar{A}_{P,Q} - M\bar{C}_1)^{-1}\bar{E}_{P,Q}\|_\infty < \gamma/\|\bar{C}_{2,P}\| \quad (3.18)$$

The reduced H_∞/LTR controller has more simple Luenberger matrices than in the general case.

Lemma 3.7.
With D_W injective, the Luenberger matrices in Theorem 3.6 take the following form:

$$\begin{aligned} D &= \begin{bmatrix} A + KC + M_1C & 0 \\ M_2C & A_W-B_WD_P^{-1}C_{2,P} \end{bmatrix} \\ G &= \begin{bmatrix} B \\ 0 \end{bmatrix} \\ P &= [\ F \quad D_P^{-1}C_{2,P}\ ] \\ E &= \begin{bmatrix} K + M_1 \\ M_2 \end{bmatrix} \\ V &= 0 \\ T &= \begin{bmatrix} I \\ 0 \end{bmatrix} \end{aligned} \quad (3.19)$$

where $M^T = [M_1^T\ M_2^T]^T$ satisfy (3.18).

Proof. Lemma 3.7 follows of Theorem 3.6 as a special case.

When applying the H_∞/LTR design method on minimum-phase systems, it is always possible to use nth order (full-order) observer based controllers for obtaining the specified H_∞-constraint γ. The full-order observer is obtained by choosing $M_2 = 0$, as stated in:

Lemma 3.8. *The Minimum Phase Case.*
The Luenberger observer, described by the following matrices:

$$\begin{aligned} D &= A + KC + M_1C \\ G &= B \\ P &= F \\ E &= K + M_1 \\ V &= 0 \\ T &= I \end{aligned} \quad (3.20)$$

will satisfy the H_∞-norm γ of the weighted recovery matrix if (A,B,C) is a minimum-phase system.

The proof is omitted.

4. CONCLUSION.

An alternative approach to the Loop Transfer Recovery design problem based on H_∞ optimization has been presented in this paper. The H_∞/LTR approach is based on the Q-parameterized observer which is a parameterized implementation of the Luenberger observer. The general H_∞/LTR design approach turns out to be an H_∞ minimization of the weighted recovery matrix, which includes other H_∞/LTR design methods based on the observer approach [22,23] as special cases.

Based on this LTR approach, sufficient conditions for obtaining LTR-controllers which satisfies specified robust stability and performance design conditions for the final feedback loop has also been derived.

State-space solutions to the general H_∞/LTR design problem has been derived by using the singular H_∞ theory approach [18,19,20]. These state-space solutions for the H_∞/LTR controller can be derived without perturbations techniques by using the singular H_∞ approach. Moreover, the order of the obtained H_∞/LTR controllers can be reduced to order $n + n_w$, (n_w is the order of the weight matrix) in the general case and to order n for minimum phase systems.

The proposed H_∞/LTR method includes several advantages compared to traditional used LTR methods. The H_∞/LTR method handles both minimum phase as well as non-minimum phase systems in the same framework. Further, the iterative nature in traditional used LTR methods is replaced in the H_∞/LTR method by a test (one shot method) of the specified recovery conditions can be satisfied by an H_∞ controller.

An investigation of selection of weight matrices in the H_∞/LTR design method for satisfying specified design conditions as e.g. in section 2.3, is subject for future research.

APPENDIX A.

The necessary preliminaries for the H_∞ methods used in this paper will be introduced in this appendix. The approach taken in based on the results in [18,19,20], the socalled singular approach. This is a general approach which includes the well known approach by Doyle et.al. [3] as a special case.

In the state space approach to H_∞ the standard problem is as follows.

Consider a finite dimensional, linear, time invariant system:

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = C_1x + D_1w \\ z = C_2x + D_2u \end{cases} \quad \text{(A.1)}$$

We assume that $\gamma > 0$ has been given. We wish to design, if possible, as internally stabilizing FDLTI compensator $u = Q(s)y$ such that the H_∞ norm of the resulting closed-loop transfer function from w to z is smaller than γ.

Assumption A.1.
It is assumed that the systems (A,B,C_2,D_2) and (A,E,C_1,D_1) have no invariant zeros in $\mathbf{C}^0$.

The main result is:

Theorem A.2.
Consider the system Σ above satisfying Assumption A.1. Let $\gamma > 0$ be given. Then, there exists an internally stabilizing FDLTI compensator $u = Q(s)y$ for which the H_∞ norm of the resulting, closed-loop transfer function from w to z is smaller than γ, if and only if there exist $P \geq 0$ and $Q \geq 0$ for which:

(1) $F_\gamma(P) \geq 0$

(2) $G_\gamma(Q) \geq 0$

(3) $\text{rank } F_\gamma(P) = \text{normrank } G$

(4) $\text{rank } G_\gamma(Q) = \text{normrank } H$

(5) $\text{rank}\begin{bmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{bmatrix} = n + \text{normrank } G, \quad \forall s \in \mathbf{C}^+ \cup \mathbf{C}^0$

(6) $\text{rank}[M_\gamma(Q,s) \;\; G_\gamma(Q)] = n + \text{normrank } H, \; \forall s \in \mathbf{C}^+ \cup \mathbf{C}^0$

(7) $\rho(PQ) < \gamma^2$

where the notation used is as follows:

$$F_\gamma(P) = \begin{bmatrix} A^TP + PA + C_2^TC_2 + \gamma^{-2}PEE^TP & PB + C_2^TD_2 \\ B^TP + D_2^TC_2 & D_2^TD_2 \end{bmatrix} \quad \text{(A.2)}$$

$$G_\gamma(Q) = \begin{bmatrix} AQ + QA^T + EE^T + \gamma^{-2}QC_2^TC_2Q & QC_1^T + ED_1^T \\ C_1Q + D_1E^T & D_1D_1^T \end{bmatrix} \quad \text{(A.3)}$$

$$L_\gamma(P,s) = [\; sI - A - \gamma^{-2}EE^TP \quad -B \;]$$

$$M_\gamma(Q,s) = \begin{bmatrix} sI - A - \gamma^{-2}QC_2^TC_2 \\ -C_1 \end{bmatrix} \quad \text{(A.4)}$$

$$G(s) = C_2(sI - A)^{-1}B + D_2, \quad H(s) = C_1(sI - A)^{-1}E + D_1 \quad \text{(A.5)}$$

The proof of Theorem A.2 can be found in [181]. We shall refer to condition (1) as the Quadratic Matrix Inequality (QMI), and any $P \geq 0$ satisfying (1) will be called a solution to QMI. Analogously we shall call (2) the Dual Quadratic Matrix Inequality (DQMI), and refer to solutions of DQMI any $Q \geq 0$ satisfying (2). Conditions (3) and (5) guarantees that a solution to QMI is unique and of minimal rank (and dually for DQMI with (4) and (6)). (7) is a typical H_∞ coupling condition, which also appears in [3].

Further, we shall need a couple of corollaries.
Corollary A.3. The regular case.
Assume that D_2 is injective. Then (1), (3) and (5) is satisfied if and only if

$$A^TP + PA + C_2^TC_2 + \gamma^{-2}PEE^TP - (PB + C_2^TD_2)(D_2^TD_2)^{-1}(B^TP + D_2^TC_2) = 0$$

and

$$\Lambda(\, A + \gamma^{-2}EE^TP - B(D_2^TD_2)^{-1}(B^TP + D_2^TC_2)) \subset \mathbf{C}^-$$

Corollary A.4. The totally singular case.
Assume $D_2 = 0$. Then (1) is equivalent to:

$$A^TP + PA + C_2^TC_2 \; \gamma^{-2}PEE^TP \geq 0$$

where P satisfies $PB = 0$.

The two corollaries for admissible controllers will be given in the following in terms of the matrices for certain transformations of Σ. First we define $C_{2,P}$ and D_P by the following factorization:

$$F_\gamma(P) = [\; C_{2,P} \;\; D_P \;]^T \times [C_{2,P} \;\; D_P \;] \quad \text{(A.6)}$$

Moreover, we will need the following matrices:

$$A_P = A + \gamma^{-2}EE^TP; \quad C_{1,P} = C_1 + \gamma^{-2}D_1E^TP \quad \text{(A.7)}$$

$$Y = (I - \gamma^{-2}QP)^{-1}Q \quad \text{(A.8)}$$

$$A_{P,Q} = A_P + \gamma^{-2}YC_{2,P}^TC_{2,P}, \quad B_{P,Q} = B + \gamma^{-2}C_{2,P}D_P \quad \text{(A.9)}$$

We shall refer to the system where A_P, $C_{1,P}$, $C_{2,P}$ and D_P substitute A, C_1, C_2 and D_2 as the QM-transformed of the system Σ. The DQMI for the QM-transformed system becomes:

$$\tilde{G}_\gamma(Y) = \begin{bmatrix} A_P Y + Y A_P^T + E E^T + \gamma^{-2} Y C_{2,P}^T C_{2,P} Y & Y C_{1,P}^T + E D_1^T \\ C_{1,P} Y + D_1 E^T & D_1 D_1^T \end{bmatrix}$$

$$\tilde{G}_\gamma(Y) = [\, E_{P,Q}^T \;\; D_{P,Q}^T \,]^T \times [\, E_{P,Q}^T \;\; D_{P,Q}^T \,] \geq 0 \qquad \text{(A.10)}$$

Substituting $A_{P,Q}$, $B_{P,Q}$, $E_{P,Q}$ and $D_{P,Q}$ for the corresponding variables in previous system will be referred to as the DQM-transformation.

In terms of these transformed system matrices we can compute the desired H_∞ controller:

Theorem A.5.
Let $A_{P,Q}$, $B_{P,Q}$ and $C_{1,P}$ be as above. Let L be a state feedback, such that $A_{P,Q} + B_{P,Q}L$ is stable, and such that:

$$\|(C_{2,P}+D_P L)(sI-A_{P,Q}-B_{P,Q}L)^{-1}\|_\infty < \gamma/(3\|E_{P,Q}\|) \qquad \text{(A.11)}$$

Let M be an output injection, such that $A_{P,Q} + MC_{1,P}$ is stable and further:

$$\|(sI - A_{P,Q} - MC_{1,P})^{-1}(E_{P,Q} + MD_{P,Q})\|_\infty < \tilde{\gamma} \qquad \text{(A.12)}$$

where

$$\tilde{\gamma} = \min\{\gamma/(3\|D_P L\|), \|E_{P,Q}\|/\|B_{P,Q}L\|\}$$

Then the controller:

$$u = -L(sI - A_{P,Q} - B_{P,Q}L - MC_{1,P})^{-1}My \qquad \text{(A.13)}$$

makes the H_∞ norm of the resulting closed loop transfer function from w to z smaller than γ.

The significance of Theorem A.5 is to transform the original H_∞ problem to two disturbance attenuation problems, which can be solved by well known methods, see e.g. [18,19,20].

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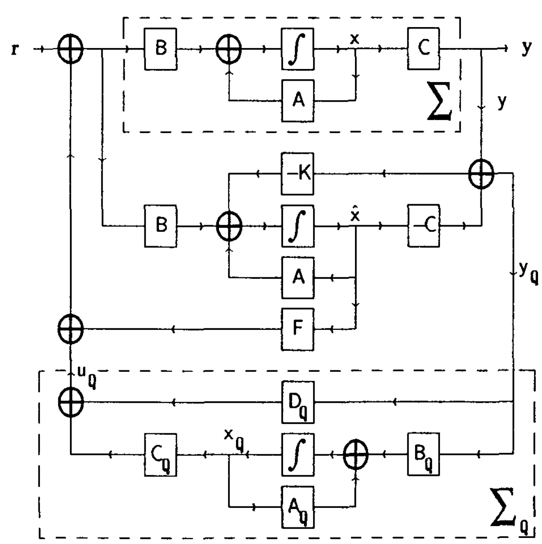

Fig. 2.1. The Q-observer.