

$\mathcal{H}_\infty$ /LTR OBSERVER BASED CONTROLLERS: DIRECT AND INDIRECT DESIGN METHODS.

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**1. INTRODUCTION.**

This paper describes two  $\mathcal{H}_\infty$  design methods for observer based controllers, where the controllers are required to satisfy specific Loop Transfer Recovery (LTR) conditions. The two  $\mathcal{H}_\infty$ /LTR design methods presented are based on the main results in the Ph.D. thesis by Stoustrup [13], where a more thorough treatment of the methods can be found.

In the original setting, LTR was intimately related to LQG design methods of full order observers Doyle and Stein [4] for the design of robust observer based control systems. Later, however, other design methods such as eigenstructure assignment techniques for both full order as well as for minimal order observers Sogaard-Andersen [15, 16] etc. have also proved to be efficient LTR design methods.

LTR design is the last step in a three step procedure for the design of robust observer based controllers. In the first step, the design specifications, i.e. robust stability and performance specifications, are formulated. The second step is a state feedback (target) design, which has to satisfy the design specifications, followed by the LTR-step where the target loop is recovered by using a dynamic measurement based controller Athans [1].

Recently, Moore and Tay [7] pioneered a new approach to the LTR problem. Their approach is based on an  $\mathcal{H}_\infty$  optimization of a suitably chosen recovery function for a controller structure, the Q-observer, consisting of a standard full order observer with an additional dynamic feedback structure attached at the estimation error node. The approach presented in [7] suffers from a number of drawbacks. First, the approach handles only the minimum phase part of a system, and for systems with RHP zeros no guaranteed norm bounds can be given for the overall system. Moreover, the resulting controller orders turn out to be at least  $2n$ , which is unnecessarily large and due to the fact that the authors use frequency domain methods rather than the state space methods, which has meanwhile proven more powerful.

In this paper we propose two alternative approaches to the  $\mathcal{H}_\infty$ /LTR problem. In Section 2 a general formulation of the LTR problem based on recovery errors Niemann, Sogaard-Andersen and Stoustrup [9] is shortly introduced along with the Q-observer based controller. The Q-observer is an implementation of the well known Youla (Q-) parameterization. Further, the  $\mathcal{H}_\infty$ /LTR problem is stated in two different formulations, the indirect and the direct  $\mathcal{H}_\infty$ /LTR problem. The indirect  $\mathcal{H}_\infty$ /LTR problem, which is strongly related to the LQG/LTR problem, is solved in Section 3 by using the so called singular  $\mathcal{H}_\infty$  state space approach Stoorvogel [11], Stoorvogel and Trentelman [12]. Solutions to the direct  $\mathcal{H}_\infty$ /LTR problem, which was studied by [7], are derived in Section 4, followed by an example in Section 5, where the two proposed  $\mathcal{H}_\infty$ /LTR methods are compared to the LQG/LTR method. A discussion is made in Section 6.

**2. LOOP TRANSFER RECOVERY: AN INTRODUCTION.**

In this section we shall shortly introduce the Loop Transfer Recovery (LTR) design method. Further, we shall introduce the Q-observer based controller, which is a certain implementation of the Youla (Q-) parameterization.

Let us consider a finite dimensional, linear, time invariant (FDLTI) plant model, represented by a state space realization  $(A, B, C, 0)$ :

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ z = Cx \end{cases} \quad (2.1)$$

with transfer function  $G(s) = C(sI-A)^{-1}B$ , where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $z \in \mathbb{R}^p$ , with  $m > p$  and  $A, B$  and  $C$  are matrices of appropriate dimensions. The system is assumed to be stabilizable, detectable and left invertible. Moreover, we shall make the technical assumption, that  $\Lambda(A) \cap \mathbb{C}^0 = \emptyset$ . Note, however, that this can always be achieved by applying a preliminary static output feedback. Furthermore, this preliminary static output feedback can be chosen arbitrarily small.

To design a controller for the system  $\Sigma$  by the LTR methodology, we first determine a (static) state feedback, the target design, which satisfies our design specifications. The design specifications, such as robustness and performance, are assumed to be reflected to the input node Athans [1], Stein and Athans [10] Sogaard-Andersen [15]. The resulting target loop transfer function becomes  $G_{TFL} = F(sI-A)^{-1}B$ , where  $F$  is the target (state feedback) design. Second, the LTR step is performed, where the target design is recovered over the range of frequencies by a dynamic compensator  $C(s)$ , giving a full loop transfer of the form  $G_I(s) = C(s)G(s)$ .

The associated sensitivity transfer functions are given by:

$$S_{TFL}(s) = (1 - G_{TFL}(s))^{-1}, S_I(s) = (1 - G_I(s))^{-1} \quad (2.2)$$

Using these transfer functions, we can define the sensitivity recovery error.

**DEFINITION 2.1.** The sensitivity recovery error  $E_S$  is defined by:

$$E_S(s) = S_{TFL}(s) - S_I(s) \quad (2.3)$$

Other types of recovery errors are considered in [9]. Note that Definition 2.1 is independent of the selected controller type. In this paper, however, we shall restrict our attention to the Q-observer based controller.

The objective in the rest of this paper is to describe how the norm of the recovery error can be made small when applying the Q-observer, using  $\mathcal{H}_\infty$  methods.

**2.1. The Q-Observer.**

One approach to characterize general controllers is the Youla parameterization of all stabilizing controllers. Briefly, the principle in the well-known Youla (or Q-) parameterization is to take any stabilizing controller which is thereafter fixed, and then make a certain interconnection structure. Now, the class of all stabilizing controllers are parameterized by applying the class of all  $\mathcal{H}_\infty$  systems at the interconnection nodes. In Boyd et al. [2] it has been shown, that the construction shown in Fig. 2.1 is an implementation of the Youla parameterization. In the sequel we shall denote this particular structure as the Q-Observer.

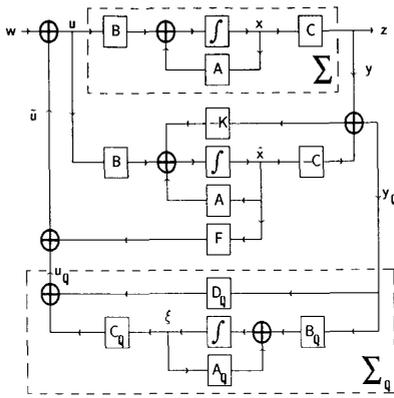


Fig. 2.1. The Q-Observer.

In the subsequent sections, we shall need the following result from [9]:

**LEMMA 2.2.** Assume that  $Q \in \mathcal{X}_\infty$ , with a state space representation, say,

$$\Sigma_Q : \begin{cases} \dot{x}_q = A_q x_q + B_q u_q \\ z_q = C_q x_q + D_q u_q \end{cases} \quad (2.4)$$

Here  $x_q \in \mathbb{R}^q$ , where  $q$  is the order of  $Q$ . Then the corresponding Q-observer is a Luenberger observer with the following parameters:

$$D = \begin{bmatrix} A+KC & 0 \\ -B_q C & A_q \end{bmatrix}, \quad G = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad P = \begin{bmatrix} F-D_q C & C_q \\ 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} -K \\ B_q \end{bmatrix}, \quad V = D_q, \quad T = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

with the Luenberger observer given by the following equations:

$$\Sigma_L : \begin{cases} \dot{z} = Dz + Gu + Ey \\ \tilde{u} = Pz + Vy \end{cases} \quad (2.5)$$

It is easily verified that  $T, D, E, G, P, V$  satisfy the Luenberger conditions Luenberger [6]:  $\Lambda(D) \subset \tilde{C}$ ,  $\Lambda A - D\tilde{T} = EC$ ,  $G = \tilde{T}B$  and  $F = P\tilde{T} + VC$ .

When the Q-observer based controller is applied, the sensitivity recovery error of Definition 2.1 can be written in a more convenient form.

**LEMMA 2.3.** Define

$$M_1(s) = F(sl-A-KC)^{-1}B - Q(s)C(sl-A-KC)^{-1}B \quad (2.6)$$

then

$$E_S(s) = S_{TFL}(s) \cdot M_1(s)$$

**PROOF.** The proof of Lemma 2.3 can be found in [9].

The matrix valued function  $M_1(\cdot)$  is called the *Recovery Matrix*, and will be central in the indirect  $\mathcal{X}_\infty$ /LTR method in Section 3 below.

### 2.2. $\mathcal{X}_\infty$ Formulation of the Loop Transfer Recovery Problem.

The  $\mathcal{X}_\infty$  version of the LTR design problem can be stated in a two different ways which will turn out to have slightly different answers.

**PROBLEM 1.** The Indirect  $\mathcal{X}_\infty$ /LTR Problem. Let  $\gamma > 0$  be given. Find, if possible, a FDLTI system  $Q(s)$  such that:

$$\|M_1(s)\|_\infty < \gamma \quad (2.7)$$

or, equivalently,

$$\|F(sl-A-KC)^{-1}B - Q(s)C(sl-A-KC)^{-1}B\|_\infty < \gamma \quad (2.8)$$

is achieved, and the closed loop system is internally stable. Here  $\|\cdot\|_\infty$  is the  $\mathcal{X}_\infty$  norm.

Problem 1 is the subject of Section 3.

**PROBLEM 2.** The Direct  $\mathcal{X}_\infty$ /LTR Problem. Let  $\gamma > 0$  be given. Find, if possible, a FDLTI system  $Q(s)$  such that when applied in a Q-observer:

$$\|E_S(s)\|_\infty < \gamma \quad (2.9)$$

or, equivalently,

$$\|(1+F(sl-A-BF)^{-1}B)(F(sl-A-KC)^{-1}B + Q(s)C(sl-A-KC)^{-1}B)\|_\infty < \gamma$$

is achieved, and the closed loop system is internally stable.

Problem 2 is the subject of Section 4.

### 3. THE INDIRECT $\mathcal{X}_\infty$ /LTR DESIGN METHOD.

Solutions to the indirect  $\mathcal{X}_\infty$ /LTR problem, Problem 1, will be derived in this section.

With  $A_K = A+KC$ , the recovery matrix (2.6) corresponding to the Q-observer structure has the following standard state space  $\mathcal{X}_\infty$  representation (see e.g. Francis and Doyle [5] for a description of the standard  $\mathcal{X}_\infty$  problem):

$$\Sigma_M : \begin{cases} \dot{x} = A_K x + 0 \cdot u + B w \\ y = C x + 0 \cdot w \\ z = F x - I \cdot u \end{cases} \quad (3.1)$$

To solve the corresponding  $\mathcal{X}_\infty$  problem we proceed along the lines of [11, 12], summarized in Appendix A. In this approach we have to solve two certain quadratic matrix inequalities, see Appendix A. First, we note that the Quadratic Matrix Inequality (see Appendix A) is regular, and hence we can write down the Riccati equation immediately (Corollary A.3). It turns out that the solution to the Riccati equation is trivial:

**LEMMA 3.1.**  $P=0$  is the unique matrix satisfying:

$$A^T P + PA - PBB^T P = 0 \text{ and } \Lambda(A - BB^T P) \subset C$$

**PROOF.**  $P=0$  is clearly a solution, since  $A_K$  is stable. Uniqueness is proved in [13].

For  $P=0$ , the coupling condition (7) of Theorem A.1 vanishes. Hence, as a consequence of Theorem A.2 solvability of the  $\mathcal{X}_\infty$ /LTR problem always is equivalent to the existence of a solution  $Q \geq 0$  to the Dual Quadratic Matrix Inequality (see Appendix A), which is characterized by the following lemma.

**LEMMA 3.2.** Problem 1 is solvable if and only if there exists  $Q \geq 0$  such that the following conditions are all satisfied:

- $A_K Q + Q A_K^T + B B^T + \gamma^{-2} Q F^T F Q =: \tilde{B} \tilde{B}^T \geq 0$
- $\text{rank } \tilde{B} = \text{rank } B$
- $C Q = 0$
- $(A_K^T + \gamma^{-2} Q F^T F, \tilde{B}, C)$  is a minimum phase system

**PROOF.** Lemma 3.2 is proved by applying Theorem A.2 to  $\Sigma_M$ .

In Stoustrup and Niemann [14] it has been shown how these four conditions can be combined to yield a single algebraic Riccati equation for  $Q$ , which can be solved by the standard methods.

Since  $(A_K^T + \gamma^{-2} Q F^T F, \tilde{B}, C)$  is a minimum phase system, it is possible to design an output injection  $G$  for this system such

that  $A_K + \gamma^2 Q F^T F + GC$  is a stability matrix and the  $\mathcal{X}_\infty$  norm of  $(sI - A_K - \gamma^2 Q F^T F - GC)^{-1} \bar{B}$  can be made smaller than any specified level. Specifically, we wish to consider a  $G$  for which:

$$\|F(sI - A_K - \gamma^2 Q F^T F - GC)^{-1} \bar{B}\|_\infty < \gamma \quad (3.2)$$

The problem (3.2) is a (dual) disturbance attenuation problem which has been met by several approaches in literature, as e.g. Willems [18], Trentelman [17].  $G$  can be found by any of these methods. With such  $G$  we have the following result:

**LEMMA 3.3.** Assume  $Q \geq 0$  satisfies (a-d) in Lemma 3.2. Then a controller  $u = Q(s)y$  solving Problem 1 is given by:

$$Q(s) = -F(sI - A_K - GC)^{-1}G \quad (3.3)$$

i.e. when applying the control law  $u = Q(s)y$  to the system  $\Sigma_M$  the closed loop system is internally stable, and the  $\mathcal{X}_\infty$  norm of the transfer function from  $w$  to  $z$  is smaller than  $\gamma$ .

**PROOF.** Lemma 3.3 follows by applying Theorem A.5 to  $\Sigma_M$ .

By means of Lemma 3.3 the indirect  $\mathcal{X}_\infty$ /LTR problem has thus been reduced to the well known almost disturbance decoupling problem.

The  $Q$ -observer based controller, constructed by applying  $\Sigma_M$  to the plant and the preliminary observer, has dynamic order  $2n$ . It turns out though that a structural reduction can always be carried out. The result of this reduction is just a standard full order observer based controller similar to the preliminary controller but with a modified gain. The result is as follows.

**THEOREM 3.4.** Let a  $Q$ -Observer based controller be given by a preliminary full order observer  $\Sigma_{obs}$  with stabilizing gain  $K$  and an  $\mathcal{X}_\infty$  controller  $\Sigma_Q$  as in Lemma 4.3:

$$Q(s) = -F(sI - A_K - GC)^{-1}G \quad (3.4)$$

Furthermore, let a full order observer based controller  $\Sigma_{obs}^*$  with observer gain  $K^* = K + G$  be given. Then for both controllers the Recovery Matrix equals:

$$M_1(s) = F(sI - A - K^*C)^{-1}B \quad (3.5)$$

Moreover, the  $Q$ -observer based and the full order observer based controllers can both be realized as Luenberger observer based controllers with the following choices of parameters:

**$Q$ -Observer:**

$$D = \begin{bmatrix} A+KC & 0 \\ -GC & A+KC+GC \end{bmatrix}$$

$$G = \begin{bmatrix} B^T & 0^T \end{bmatrix}^T$$

$$P = \begin{bmatrix} F & -F \end{bmatrix}$$

$$E = \begin{bmatrix} -K^T & G^T \end{bmatrix}^T$$

$$V = 0$$

$$T = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$

**Full Order Observer:**

$$D = A + KC + GC$$

$$G = B$$

$$P = F$$

$$E = -K - G$$

$$V = 0$$

$$T = I$$

**PROOF.** The result can be found in [13].

#### 4. THE DIRECT $\mathcal{X}_\infty$ /LTR DESIGN METHOD.

In the following we shall consider the direct  $\mathcal{X}_\infty$ /LTR problem, Problem 2, using the sensitivity recovery error. Using the  $Q$ -observer introduced in Section 2, the sensitivity recovery error is given by:

$$E_S(s) = S_{TFL}(s)M_1(s) = (I + F\Phi_F B)(F\Phi_K B + Q(s)C\Phi_K B) \quad (4.1)$$

The state space formulation of equation (4-1) is given by:

$$\Sigma_{S,Q} : \begin{cases} \dot{x} = \begin{bmatrix} A+KC & 0 \\ BF & A+BF \end{bmatrix} x + \begin{bmatrix} 0 \\ B \end{bmatrix} u + \begin{bmatrix} B \\ 0 \end{bmatrix} w \\ y = \begin{bmatrix} C & 0 \end{bmatrix} x + 0w \\ z = \begin{bmatrix} F & F \end{bmatrix} x + Iu \end{cases} \quad (4.2)$$

For simplification, we will denote  $A+KC$  by  $A_K$  and  $A+BF$  by  $A_F$ . In this case Assumption A.1 amounts to the condition that  $(A, B, C, 0)$  has neither zeros nor poles on the imaginary axis. This is assumed throughout this section.

First, we find a solution to the Quadratic Matrix Inequality and the associated QM-transformation (see Appendix A).

**THEOREM 4.1.** For the system  $\Sigma_{S,Q}$  described by eq. (4.2), the solution of the QMI with the associated rank conditions, is:

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix}, \quad (4.3)$$

where  $P$  is the unique stabilizing solution to the algebraic Riccati equation,

$$A^T P + PA - PBB^T P = 0 \quad (4.4)$$

$\bar{F}_\gamma(P)$  factorizes as:

$$\bar{F}_\gamma(\bar{P}) = \begin{bmatrix} F^T \\ PB + F^T \\ I \end{bmatrix} \times \begin{bmatrix} F & B^T P + F & I \end{bmatrix} \quad (4.5)$$

Performing the QM-transformation, we get the following matrices:

$$\bar{A}_P = \bar{A}, \quad \bar{C}_{1,P} = \bar{C}_1, \quad \bar{C}_{2,P} = \begin{bmatrix} F & B^T P + F \end{bmatrix}, \quad \bar{D}_P = \bar{D}_2 \quad (4.6)$$

**PROOF.** See [14, 13].

Note that the solution of the QMI does not depend on  $\gamma$ .

Further, in the special case where  $A$  is stable,  $P = 0$  is the unique solution, and the resulting QM-transformation is the identity.

Now, a solution  $\bar{Y}$  to the DQMI for the transformed system has to be found in order to determine the corresponding DQM-transformation.

**LEMMA 4.2.** Let the matrices  $\bar{A}_P, \bar{B}, \bar{E}, \bar{C}_{1,P}, \bar{C}_{2,P}, \bar{D}_1$  and  $\bar{D}_P$  be as in eqs. (4.2) and (4.6). Then the solution of the associated DQMI,

$$\bar{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \quad (4.7)$$

satisfies the conditions  $CY_{11} = 0$  and  $CY_{12} = 0$ .

In the case where  $(A, B, C, 0)$  is invertible and minimum phase,  $\bar{Y} = 0$  is the unique solution of the DQMI. In this case no second transformation is needed.

For  $\bar{Y} \neq 0$ , the DQM-transformation proceeds as follows:

$$\bar{A}_{P,Q} = \bar{A}_P + \gamma^{-2} \bar{Y} C_{2,P}^T, \quad \bar{B}_{P,Q} = \bar{B} + \gamma^{-2} \bar{Y} C_{2,P}^T \bar{D}_P,$$

$$\bar{G}_\gamma(\bar{Y}) = \begin{bmatrix} \bar{E}_{P,Q} \\ 0 \end{bmatrix} \times \begin{bmatrix} \bar{E}_{P,Q}^T & 0 \end{bmatrix}$$

After the QM- and DQM-transformations, the controller  $u = Q(s)y$  described in Theorem A.5 can now be designed in order to satisfy the two norm inequalities in eqs. (A.11) and (A.12). It is readily seen that (A.11) is satisfied for:

$$L = -\tilde{C}_{2,P}$$

since this solves an (exact) disturbance decoupling problem.

Then with  $M = [M_1^T \ M_2^T]^T$  satisfying (A.12) we have:

**LEMMA 4.3.** Let  $M$  be as above and let  $P$  be the stabilizing solution to the Riccati equation (4.4). Then an admissible controller for the above  $\chi_\infty$  problem is given by:

$$Q(s) = F(sI - A_K - M_1 C)^{-1} M_1 + (B^T P + F)(sI - A + BB^T P)^{-1} M_2 C (sI - A_K - M_1 C)^{-1} M_1 + (B^T P + F)(sI - A + BB^T P)^{-1} M_2 \quad (4.8)$$

**PROOF** The lemma follows directly by substituting the above matrices in Theorem A.5.

The controller derived in Lemma 4.3 has dynamic order  $2n$ . When inserted in the overall controller structure, as described in Section 2, we get a controller of order  $3n$ , if no reduction is carried out. It turns out, though, that a structural reduction can be performed without affecting the obtained  $\chi_\infty$  norm. The basic idea is to use the remaining freedom in the observer gain  $K$  designed in Section 2 to obtain some of the desired controller dynamics. By doing this we get the following result.

**THEOREM 4.4.** Let the transfer function of the feedback system  $\Sigma_\delta^*$  be given by:  $Q^*(s) = (B^T P + F)(sI - A + BB^T P)^{-1} M_2$ .

When applying  $\Sigma_\delta^*$  to a  $Q$ -observer configuration with observer gain  $K^* = K + M_1$ , the  $\chi_\infty$  norm of the transfer function from  $w$  to  $z$  equals the  $\chi_\infty$  norm obtained when applying  $\Sigma_\delta^*$  described by Lemma 4.3 to a similar system with observer gain  $K$ .

**PROOF.** See [14].

For minimum phase systems it can be seen that an  $n$ 'th order admissible controller is obtained by choosing  $M_2=0$ :

**THEOREM 4.5.** The cascade of  $\Sigma_{obs}^*$  and  $\Sigma_\delta^*$  (described above) is a Luenberger observer, described by the following matrices:

**Nonminimum phase systems:**

$$D = \begin{bmatrix} A + KC + M_1 C & 0 \\ M_2 C & A - BB^T P \end{bmatrix}$$

$$G = \begin{bmatrix} B^T & 0 \end{bmatrix}^T$$

$$P = \begin{bmatrix} F & B^T P + F \end{bmatrix}$$

$$E = \begin{bmatrix} (K + M_1)^T & M_2^T \end{bmatrix}^T$$

$$V = 0$$

$$T = \begin{bmatrix} I & 0 \end{bmatrix}^T$$

**Minimum phase systems:**

$$D = A + KC + M_1 C$$

$$G = B$$

$$P = F$$

$$E = K + M_1$$

$$V = 0$$

$$T = I$$

Moreover, the closed loop transfer function obtained by applying this Luenberger observer has  $\chi_\infty$  norm smaller than  $\gamma$ .

**PROOF.** See [13].

Note that the overall controller is of order  $2n$  in the non-minimum phase case, and  $n$  in the minimum phase case. The reason, why the controller reduction from  $3n$  (resp.  $2n$ ) to  $2n$  ( $n$ ) is possible, is the remaining freedom in the preliminary observer design. The dynamics from this observer will be cancelled by the  $\chi_\infty$  controller and substituted by a more feasible one.

Further, note that only the output injection  $M$  in the  $\chi_\infty$ /LTR controller depends on  $\gamma$ ,  $L$  doesn't.

## 5. A DESIGN EXAMPLE.

Let the system  $\Sigma$  be given by the state space model  $(A, B, C)$ :

$$A = \begin{bmatrix} -5.5000 & 0.6325 \\ 0.6325 & 0.0000 \end{bmatrix}, B = \begin{bmatrix} 1.00 \\ 0.00 \end{bmatrix}, C^T = \begin{bmatrix} 1.0000 \\ -1.5811 \end{bmatrix}$$

The system is open loop unstable, and has a RHP zero at  $z = 1.00$ .

As the target state feedback let us choose

$$F = [1.00 \ -10.2175]$$

A standard LQG/LTR design gives the following observer gain:

$$K_{LQG}^T = [-668.30 \ 206.95]$$

For the indirect  $\chi_\infty$ /LTR design method described in Section 3, the infimally achievable  $\chi_\infty$  norm of the recovery matrix is  $\gamma^* = 0.9$ . As a bound for the  $\chi_\infty$  problem we have selected  $\gamma = 1.0 \sim 0.0$  dB  $> \gamma^*$ . By the design technique outlined in Section 3, we find:

$$K_{Ind}^T = [19.023 \ 20.818]$$

as the observer gain in a controller which satisfies the  $\chi_\infty$  bound  $\gamma$ .

The recovery matrices achieved by these two 2'nd order controllers are shown in Fig. 5.1.

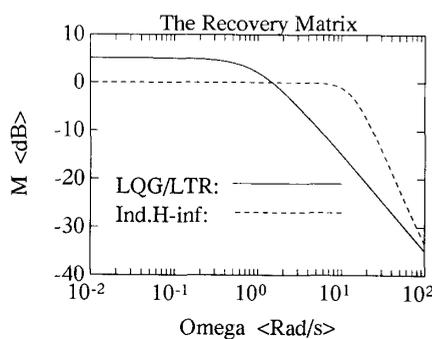


Fig. 5.1.

Now, we apply the direct  $\chi_\infty$ /LTR method to Problem 2. First a preliminary observer design is determined:

$$K_{Obs}^T = [0.1115 \ 0.5132]$$

For this problem, the infimally achievable  $\chi_\infty$  norm turns out to be  $\gamma^* = 0.55$ , so let us choose  $\gamma = 0.65 \sim -3.74$  dB  $> \gamma^*$  as a bound for the  $\chi_\infty$ /LTR problem.

The the 4'th order controller which can be computed following the line of Section 4 has the following  $\chi_\infty$ /LTR gain:

$$M^T = [8.1907 \ 15.0243 \ -13.0782 \ -7.6391]$$

The Luenberger matrices for the controller can be computed applying Theorem 4.5. The solution to the singular Riccati equation is:

$$P = \begin{bmatrix} 0.1436 & 1.2650 \\ 1.2650 & 11.1436 \end{bmatrix}$$

An explicit expression for the solution to the totally singular Riccati equation can be found in [13].

The sensitivity recovery errors for the LQG/LTR design, the indirect  $\chi_\infty$ /LTR design and the direct  $\chi_\infty$ /LTR design are

shown in Fig. 5.2. The direct  $\mathcal{X}_\infty$ /LTR design has a maximum which is 6.5 dB lower than the the maximum of the indirect  $\mathcal{X}_\infty$ /LTR design and 4.5 dB lower than the LQG/LTR design.

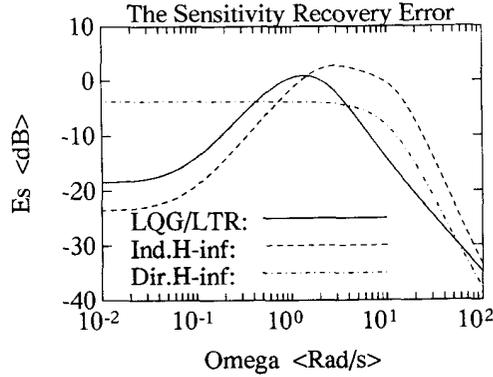


Fig. 5.2.

The algorithms for determining the  $\mathcal{X}_\infty$ /LTR controller parameters are included in the MATLAB toolbox: ' $\mathcal{X}_\infty$  and Loop Transfer Recovery Design', Niemann and Stoustrup [8], which is available on request (incl. floppy) to the authors.

## 6. DISCUSSION.

Two  $\mathcal{X}_\infty$ /LTR design methods have been introduced as state space solutions to the direct and the indirect  $\mathcal{X}_\infty$ /LTR problems.

The indirect  $\mathcal{X}_\infty$ /LTR design method generates an (at most)  $n$ 'th order controller which makes the  $\mathcal{X}_\infty$  norm of the recovery matrix smaller than a prespecified constant  $\gamma$ .

The direct  $\mathcal{X}_\infty$ /LTR design method makes the  $\mathcal{X}_\infty$  norm of the sensitivity recovery error smaller than a prespecified constant  $\gamma$ . The resulting controller is of order at most  $2n$ .

For both methods, the controllers have alternative implementations. The controllers can be constructed either as an  $\mathcal{X}_\infty$  appendage to an existing standard full order observer based controller or, alternatively, as a lower order combination of the two. The former has its significance in on-line tuning procedures, since the  $\mathcal{X}_\infty$  part can be appended in a continuous way to an existing controller.

The design examples in Fig. 5.1 and 5.2 are seen to have more 'flat' responses than the LQG curves, thus giving rise to smaller maxima. Inherent to the properties of the  $\mathcal{X}_\infty$  norm, the indirect method can always be applied to achieve a smaller maximum of the recovery matrix than the LQG method. Likewise can the direct method be applied to make the maximum of the sensitivity recovery error as small as desired, only bounded by the theoretical limit imposed by RHP zeros.

Note also that in the example, the gains of the  $\mathcal{X}_\infty$ /LTR controllers are smaller than the gain of the LQG/LTR controller. In general the gains of an  $\mathcal{X}_\infty$ /LTR controller will be moderate, when the  $\mathcal{X}_\infty$  specification is not chosen too strict, i.e. to close to  $\gamma^*$ .

The main limitation of the suggested methods is due to the fact that  $\mathcal{X}_\infty$  methods generally try to average out the errors over the whole frequency range. This situation is not desirable for LTR problems, since the acceptable errors might be low for instance at low frequencies (performance specs.). Hence, to overcome this limitation, it would be sometimes be reasonable to incorporate weighting functions in the problem formulation. This can easily be done, but at the cost of more controller states.

## APPENDIX A.

The necessary preliminaries for the  $\mathcal{X}_\infty$  methods used in this paper will be introduced in this appendix. The approach taken is based on the results in [11, 12], the so called singular approach. This is a very general approach which includes the well known approach by Doyle et al. [3] as a special case.

In the state space approach to  $\mathcal{X}_\infty$  the standard problem is as follows:

Consider a finite dimensional, linear, time invariant system:

$$\Sigma : \begin{cases} \dot{x} = A x + B u + E w \\ y = C_1 x + D_1 w \\ z = C_2 x + D_2 u \end{cases} \quad (A.1)$$

We assume that  $\gamma > 0$  has been given. We wish to design, if possible, an internally stabilizing FDLTI compensator  $u = Q(s)y$  such that the  $\mathcal{X}_\infty$  norm of the resulting closed-loop transfer function from  $w$  to  $z$  is smaller than  $\gamma$ .

**ASSUMPTION A.1.** It is assumed that the systems  $(A, B, C_2, D_2)$  and  $(A, E, C_1, D_1)$  have no invariant zeros in  $\mathbb{C}^0$ .

The main result is:

**THEOREM A.2.** Consider the system  $\Sigma$  above satisfying Assumption 3.1. Let  $\gamma > 0$  be given. Then, there exists a FDLTI compensator  $u = Q(s)y$  for which the  $\mathcal{X}_\infty$  norm of the resulting closed-loop transfer function from  $w$  to  $z$  is smaller than  $\gamma$ , if and only if there exist  $P \geq 0$  and  $Q \geq 0$  for which:

- (1)  $F_\gamma(P) \geq 0$
- (2)  $G_\gamma(Q) \geq 0$
- (3)  $\text{rank } F_\gamma(P) = \text{normrank } G$
- (4)  $\text{rank } G_\gamma(Q) = \text{normrank } H$
- (5)  $\text{rank} \begin{bmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{bmatrix} = n + \text{normrank } G, \quad \forall s \in \mathbb{C}^* \cup \mathbb{C}^0$
- (6)  $\text{rank} [M_\gamma(Q, s) \ G_\gamma(Q)] = n + \text{normrank } H, \quad \forall s \in \mathbb{C}^* \cup \mathbb{C}^0$
- (7)  $\rho(PQ) < \gamma^2$

where the notation used is as follows:

$$F_\gamma(P) = \begin{bmatrix} A^T P + P A + C_2^T C_2 + \gamma^{-2} P E E^T P & P B + C_2^T D_2 \\ B^T P + D_2^T C_2 & D_2^T D_2 \end{bmatrix} \quad (A.2)$$

$$G_\gamma(Q) = \begin{bmatrix} A Q + Q A^T + E E^T + \gamma^{-2} Q C_2^T C_2 Q & Q C_1^T + E D_1^T \\ C_1 Q + D_1 E^T & D_1 D_1^T \end{bmatrix} \quad (A.3)$$

$$L_\gamma(P, s) = [sI - A - \gamma^{-2} E E^T P - B]$$

$$M_\gamma(Q, s) = \begin{bmatrix} sI - A - \gamma^{-2} Q C_2^T C_2 \\ -C_1 \end{bmatrix} \quad (A.4)$$

$$G(s) = C_2 (sI - A)^{-1} B + D_2, \quad H(s) = C_1 (sI - A)^{-1} E + D_1 \quad (A.5)$$

The proof of Theorem A.2 can be found in [11]. We shall refer to condition (1) as the *Quadratic Matrix Inequality* (QMI), and any  $P \geq 0$  satisfying (1) will be called a solution to QMI. Analogously we shall call (2) the *Dual Quadratic Matrix Inequality* (DQMI), and refer to solutions of DQMI any  $Q \geq 0$  satisfying (2). Conditions (3) and (5) guarantees that a solution to QMI is unique and of minimal rank (and dually for DQMI with (4) and (6)). (7) is a typical  $\mathcal{X}_\infty$  coupling condition, which also appears in [3].

Further, we shall need a couple of corollaries.

**COROLLARY A.3.** The Regular Case. Assume that  $D_2$  is injective. Then (1), (3) and (5) is satisfied if and only if

$$A^T P + P A + C_2^T C_2 + \gamma^{-2} P E E^T P - (P B + C_2^T D_2) (D_2^T D_2)^{-1} (B^T P + D_2^T C_2) = 0$$

and

$$\Lambda(A + \gamma^2 EE^T P - B(D_2^T D_2)^{-1}(B^T P + D_2^T C_2)) \subset \mathbb{C}^-$$

**COROLLARY A.4.** The Totally Singular Case. Assume  $D_2 = 0$ . Then (1) is equivalent to:

$$A^T P + PA + C_2^T C_2 + \gamma^2 PEE^T P \geq 0$$

where  $P$  satisfies  $PB = 0$ .

The two corollaries have straightforward duals, which we shall also utilize in the sequel.

Expressions for admissible controllers will be given in the following in terms of the matrices for certain transformations of  $\Sigma$ . First we define  $C_{2,P}$  and  $D_P$  by the following factorization:

$$F_\gamma(P) = \begin{bmatrix} C_{2,P} & D_P \end{bmatrix}^T \times \begin{bmatrix} C_{2,P} & D_P \end{bmatrix}, \quad (A.6)$$

Moreover, we will need the following matrices:

$$A_P = A + \gamma^2 EE^T P, \quad C_{1,P} = C_1 + \gamma^2 D_1 E^T P \quad (A.7)$$

$$Y = (I - \gamma^2 QP)^{-1} Q \quad (A.8)$$

$$A_{P,Q} = A_P + \gamma^2 Y C_{2,P}^T C_{2,P}, \quad B_{P,Q} = B + \gamma^2 Y C_{2,P}^T D_P \quad (A.9)$$

We shall refer to the system where  $A_P, C_{1,P}, C_{2,P}$  and  $D_P$  substitute  $A, C_1, C_2$  and  $D_2$  as the QM-transform of the system  $\Sigma$ . The DQMI for the QM-transformed system becomes:

$$\begin{aligned} \tilde{G}_\gamma(Y) &= \begin{bmatrix} A_P Y + Y A_P^T + EE^T + \gamma^{-2} Y C_{2,P}^T C_{2,P} Y & Y C_{1,P}^T + ED_1^T \\ C_{1,P} Y + D_1 E^T & D_1 D_1^T \end{bmatrix} \\ &= \begin{bmatrix} E_{P,Q}^T & D_{P,Q}^T \end{bmatrix}^T \times \begin{bmatrix} E_{P,Q}^T & D_{P,Q}^T \end{bmatrix} \geq 0 \quad (A.10) \end{aligned}$$

Substituting  $A_{P,Q}, B_{P,Q}, E_{P,Q}$  and  $D_{P,Q}$  for the corresponding variables in the previous system will be referred to as the DQM-transformation.

In terms of these transformed system matrices we can compute the desired  $\mathcal{X}_\infty$  controller:

**THEOREM A.5.** Let  $A_{P,Q}, B_{P,Q}$  and  $C_{1,P}$  be as above. Let  $L$  be a state feedback  $L$ , such that  $A_{P,Q} + B_{P,Q}L$  is stable, and such that:

$$\|(C_{2,P} + D_P L)(sI - A_{P,Q} - B_{P,Q}L)^{-1}\|_\infty < \gamma/(3 \cdot \|E_{P,Q}\|) \quad (A.11)$$

Let  $M$  be an output injection, such that  $A_{P,Q} + B_{P,Q}L + MC_{1,P}$  is stable and further:

$$\|(sI - A_{P,Q} - B_{P,Q}L - MC_{1,P})^{-1}(E_{P,Q} + MD_{P,Q}L)\|_\infty < \tilde{\gamma} \quad (A.12)$$

where

$$\tilde{\gamma} = \min\{ \gamma/(3 \cdot \|D_P L\|), \|E_{P,Q}\|/\|B_{P,Q}L\| \}$$

Then the controller:

$$u = -L(sI - A_{P,Q} - B_{P,Q}L - MC_{1,P})^{-1}My \quad (A.13)$$

makes the  $\mathcal{X}_\infty$  norm of the resulting closed loop transfer function from  $w$  to  $z$  in  $\Sigma$  smaller than  $\gamma$ .

The significance of Theorem A.5 is to transform the original  $\mathcal{X}_\infty$  problem to two disturbance attenuation problems, which can be solved by well known methods, see e.g. [11, 17, 18].

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