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PROPORTIONAL INTEGRAL OBSERVER USED IN RECOVERY DESIGN

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ABSTRACT.
The contribution of this paper is to formulate a design problem for Proportional Integral (PI) observers which facilitate their use in recovery design. It is shown that the PI-observers make it possible to obtain time recovery, i.e. exact recovery for $t \to \infty$. An Loop Transfer Recovery (LTR) design method based on LQG design is derived which make it possible to obtain both time recovery and frequency-domain (normal) recovery at the same time. An example demonstrates this facility.

1. INTRODUCTION.

Since the first paper by Doyle and Stein [3] dealing with Loop Transfer Recovery (LTR) appeared, a lot of papers has been written in this area for both continuous-time and discrete-time systems, see e.g. [2,5,6,7,8].

In [2] the PI-observer has been introduced in connection with LTR design. The results derived in that paper are based on an extension of the LTR results for the full-order observer in [3] and later generalized in [5]. The pay-off of the PI-observer in connection with LTR design is the time recovery effect. Under mild conditions the PI-observer will result in exact recovery as time tends to infinity, named time recovery. One advantage by using the PI-observer in LTR design is that it is possible to obtain time recovery by using relatively low observer gains compared with LTR design of a full-order observer.

The key contribution of this paper is to derive a new formulation of the PI-observer, which will make it possible to derive systematic design methods for the observer, such as LQG and pole placement based methods. Further, based on this formulation and the LQG/LTR design method for full-order observers, a recovery design method for the PI-observer is derived.

2. THE PI-OBSERVER.

Consider a FDLTI system Σ described by a minimal state-space realization (A,B,C):

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r \\ y = Cx & y \in \mathbb{R}^m \end{cases} \tag{1}$$

with $m \geq r$, $n > m$, (A,B) is stabilizable, (C,A) is detectable and C,B of full rank. Now let the plant be controlled by an observer-based controller containing a state-feedback:

$$u = F\hat{x} + r \tag{2}$$

where F is the state feedback gain and $\hat{x}$ the state estimate. The states are estimated by using a PI-observer (the dual version of the PI-state feedback [1]):

$$\Sigma_O: \begin{cases} \dot{\hat{x}} = A\hat{x} + K(C\hat{x} - y) + Bu + Bv \\ \dot{v} = H(C\hat{x} - y) \\ u = F\hat{x} + r \end{cases} \tag{3}$$

where $H \in \mathbb{R}^{r \times m}$ is the I-gain and $K \in \mathbb{R}^{n \times m}$ is the P-gain.
The stability condition requires that the eigenvalues of R:

$$R = \begin{bmatrix} A_K & B \\ HC & 0 \end{bmatrix}, \quad A_K = A + KC \tag{4}$$

have negative real parts.
In this configuration, the design freedom has (n+m) x m parameters for placing the n+m observer poles. Moreover, it is possible to derive systematic design methods for the PI-observer by considering the closed loop system as an extended state system. The PI-observer can be represented by:

$$\Sigma_e: \begin{cases} \dot{z} = A_z z + K_z(C_z z + y) + B_z u \\ u = F_z z \end{cases} \tag{5}$$

$$A_z = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \; B_z = \begin{bmatrix} B \\ 0 \end{bmatrix}, \; C_z = [\, C \;\; 0 \,], \quad F_z = [\, F \;\; 0 \,], \; K_z = \begin{bmatrix} K \\ H \end{bmatrix} \tag{6}$$

Methods as LQG, eigenstructure-assignment etc. can now be applied for the observer design, as for ordinary observer design by designing K_z.

3. RECOVERY DESIGN USING PI-OBSERVERS.

Recovery design using the PI-observer has been treated in [2] without any design methods. The PI-observer for the LTR problem has also been considered with LTR of the Luenberger observer [5].

First, let us introduce the LTR design methodology based on recovery errors as in [5].
To design a controller for the system Σ by the LTR methodology, we first determine a static state feedback, the target design, which satisfies our design specifications. The design specifications, such as robustness and performance conditions, are assumed to be reflected to the plant input node [6]. The resulting target sensitivity transfer function becomes:

$$S_{TFL}(s) = (I - F(sI - A)^{-1}B)^{-1} \tag{7}$$

where F is the target state feedback design.
Second, the LTR step is performed, where the target design is recovered over the range of frequencies by a dynamic compensator C(s), given a full loop sensitivity transfer function of the form:

$$S_I(s) = (I - C(s)G(s))^{-1} \tag{8}$$

As a measure of the quality of the recovery, we define the sensitivity recovery error by:

$$E_S(s) = S_{TFL}(s) - S_I(s) \tag{9}$$

Applying a PI-observer, the recovery error can then be rewritten as [5]:

$$\begin{aligned} E_S(s) &= S_{TFL}(s)M(s) \\ M(s) &= sF(s^2I - s(A+KC) - BHC)^{-1}B \end{aligned} \tag{10}$$

M is called the recovery matrix [5]. Note that the recovery matrix given by (10) might equal zero in the steady state ($s \to 0$). We denote this as *time recovery.* The necessary and sufficient condition for obtaining time recovery is that the largest invariant subspace of the matrix $A_K^{-1}BHC$ corresponding to the eigenvalue 0 contained in the controllable subspace of the pair ($A_K^{-1}BHC$, $A_K^{-1}B$) is itself contained in the unobservable subspace of the pair (F, $A_K^{-1}BHC$). A matrix test for this can be found in [4]. Generically, the condition is that HC must have full row rank.

The steady-state property of the PI-observer indicates some advantages in the LTR design in comparison to the normal full-order observer, see below.

<u>Standard LQG design of PI-observers.</u>
The LQG observer design is determined by the following Riccati

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equation:

$$A_zP + PA_z^T + \Gamma - PC_z^T\Sigma^{-1}C_zP = 0 \tag{11}$$

$$K_z = \begin{bmatrix} K \\ H \end{bmatrix} = PC_z^T\Sigma^{-1}, \quad \Gamma = L^TL \geq 0, \quad \Sigma > 0 \tag{12}$$

The design parameters are Γ and Σ.
Rewriting (12) as:

$$K_z = \begin{bmatrix} K \\ H \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} C^T \\ 0 \end{bmatrix} \Sigma^{-1} = \begin{bmatrix} P_{11}C^T\Sigma^{-1} \\ P_{12}^TC^T\Sigma^{-1} \end{bmatrix} \tag{13}$$

shows that H has full rank if CP_{12} has full rank, which will result in obtaining time recovery.
The condition for H to have full rank can be derived from the Riccati equation in (11). The Riccati equation is equivalent to 4 (effectively 3) equations given by:

$$\begin{aligned} AP_{11}+P_{11}A^T+BP_{12}^T+P_{12}B^T+P_{11}C^T\Sigma^{-1}CP_{11}+L_1^TL_1 &= 0 \\ AP_{12}+BP_{22}+P_{11}C^T\Sigma^{-1}CP_{12}+L_1^TL_2 &= 0 \\ P_{12}^TC^T\Sigma^{-1}CP_{12}+L_2^TL_2 &= 0 \end{aligned} \tag{14}$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \begin{bmatrix} L_1^T \\ L_2^T \end{bmatrix} \begin{bmatrix} L_1 & L_2 \end{bmatrix} \geq 0 \tag{15}$$

From (14) we have that CP_{12} has full rank if and only if Γ_{22} is positive definite. Using a standard LQG design of a PI-observer will result in time recovery if the weight matrix Γ_{22} is positive definite.

<u>LQG/LTR design of PI-observers.</u>
Let's instead apply the LQG/LTR method to the PI-observer. Let the weight matrices in (11) and (12) be given by [3]:

$$\begin{aligned} \Gamma &= \Gamma_0 + q^2B_zVB_z^T, \quad \Gamma_0 \geq 0, \; V > 0, \; 0 \leq q < \infty \\ \Sigma &= \Sigma_0, \qquad\qquad \Sigma_0 > 0 \end{aligned} \tag{16}$$

As q approaches the limit, the observer gain behaves as [3]:

$$\frac{K_z}{q} \to B_zV^{1/2}\Theta\, \Sigma^{-1/2} = \begin{bmatrix} B \\ 0 \end{bmatrix} V^{-1/2}\Theta\, \Gamma^{-1/2}, \text{ as } q \to \infty \tag{17}$$

where Θ is an orthogonal matrix. Equation (17) shows that the I-gain H is zero in the limit. Hence, the PI-observer reduces to a normal full-order observer without time recovery effects.

<u>Modified LQG/LTR design of PI-observers.</u>
Using the LQG/LTR weight matrices (16) in the last equation in (14) gives:

$$P_{12}(q)^TC^T\Sigma^{-1}CP_{12}(q) + L_2^TL_2 = 0 \tag{18}$$

As a consequence of (18) it can be shown that as q approach infinity $L_2^TL_2/q^2 \to 0$, $\Sigma^{1/2}CP_{12}/q \to 0$ and the integral effect in the observer will disappear.
However, by introducing a scalar parameter $\alpha(q)$ $(0 \leq \alpha(q) < \infty)$ in (18) as:
make it possible explicit to control the limitation of the integral effect in

$$P_{12}^T(q)C^T\Sigma^{-1}CP_{12}(q) + (1 + \alpha(q))L_2^TL_2 = 0 \tag{19}$$

the observer as q approach infinity in the recovery design. The modified LQG/LTR weight matrices is then given by:

$$\begin{aligned} \Gamma &= \Gamma_0(\alpha) + q^2B_zVB_z, \quad \alpha \geq 0, \; q \geq 0 \\ \Sigma &= \Sigma_0, \qquad\qquad \Sigma_0 > 0 \\ \Gamma_0(\alpha) &= \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & (1+\alpha)\Gamma_{22} \end{bmatrix}, \quad \Gamma_0(0) \geq 0, \; \Gamma_{22} > 0 \end{aligned} \tag{20}$$

The q parameter is related to the frequency-domain recovery properties whereas α is related to the time recovery properties.

4. EXAMPLE.

Consider the minimum phase system (A,B,C) described by:

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [\,2 \;\; 1\,]$$

A target design (state-feedback) for Σ is given by [3]:

$$u = Fx \quad \text{with} \quad F = [\,-50 \;\; -10\,]$$

Let's use the modified LQG/LTR design method for the PI-observer. Let the weight matrices Γ_0 and Σ be given by:

$$\Gamma_0 = I_{3x3}\,, \quad \Sigma_0 = I_{3x3}\,, \quad \Gamma_0(3,3)(\alpha) = (1+\alpha)\Gamma_0(3,3)$$

where α is the tuning parameter of the integral effect in the LTR design. By using q = 1000, fig. 4.1 shows directly the effect on the recovery matrix of an increasing α-parameter (a LQG/LTR design of a full-order observer with q = 1000 is also shown in fig. 4.1). Further, it is seen from fig. 4.1 that the gain of the recovery matrix at high frequencies is independent of the selected α-parameter. However, if we increase α, the norm of the observer gain will also increase in the same way as when q is increased.

5. CONCLUSION.

An LTR design method based on LQG has been derived for PI-observers. The LQG/LTR design method for full-order observers has been modified by including an extra design parameter which make it possible to design the integral effect explicitly. The advantage of using an PI-observer in the recovery design instead of full-order observers is that it is possible both to reduce the maximal gain of M(s) (by increasing q) and to reduce the gain of M(s) at low frequencies (by increasing α). The future research in LTR design of PI-observers must also include the non-minimum phase case, where asymptotic recovery is in general impossible. The limitating effects of RHP zeros on the recovery design of the PI-observers, (the equivalent of the results in [8] for the full-order observer) must be derived.

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Fig. 4.1: LQG/LTR design of PI-observer