Proceedings of the 31st Conference on Decision and Control
Tucson, Arizona · December 1992

WP9 - 16:40

LQG DESIGN OF DISCRETE-TIME PI-OBSERVERS

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ABSTRACT.
Two versions of the discrete-time Proportional Integral (PI)-observer are introduced. The observers are given in a form which makes it possible to use standard LQG design techniques. Further, the PI-observers are applied in recovery design, where it is possible to obtain time recovery for $t \to \infty$. Loop Transfer Recovery (LTR) design methods based on LQG are derived for the PI-observers.

1. PRELIMINARY.

The continuous-time PI-observer has been introduced in [2] and later used in [6,8] in connection with LTR design. Time recovery has also been introduced in [6,8]. In the discrete-time it is possible to derive two versions of the PI-observer: A prediction PI-observer and a filtering PI-observer equivalent to the full-order observer case [4,5,7,11].

1.1 Discrete-time PI-observers.

Consider the discrete-time FDLTI system Σ represented by the state space realization (A,B,C):

$$\Sigma: \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

with the transfer function:

$$G_D(z) = C(zI - A)^{-1}B = C\Phi_D(z)B \tag{2}$$

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ and A,B and C are matrices of appropriate dimensions. The system Σ is assumed to be stabilizable, detectable and without poles or zeros at the origin. Further, it is assumed that CB has full rank.

The two versions of the discrete-time PI-observers are considered in the following. The discrete-time prediction PI-observer is equivalent to the continuous-time version. Therefore we can directly formulate the prediction PI-observer in the following state-space description:

$$\Sigma_{PPI}: \begin{cases} \hat{z}(t+1) = Az(t) + K(C\hat{z}(t) - y(t)) + Bu(t) + Bv(t) \\ v(t+1) = v(t) + H(C\hat{z}(t) - y(t)) \\ u(t) = F\hat{z}(t) \end{cases} \tag{3}$$

where z is the state estimate.
The PI-observer internally stabilizes Σ if and only if [6,8]:

$$\rho\left(\begin{bmatrix} A+KC & B \\ HC & I \end{bmatrix}\right) < 1 \quad \text{and } \rho(A+BF) < 1 \tag{4}$$

The discrete-time PI-observer can also be formulated in a filtering version as in the full-order observer case [4,5,11]. In a prediction observer, the feedback signal u(t) is based on measurement signals up to the time t-1, whereas the feedback signal u(t) is based on measurement signals up to the time t in a filtering observer.

The state-space description of the filtering PI-observer can be derived from the full-order filtering observer by including an integral term:

$$\Sigma_{FPI}: \begin{cases} \hat{z}(t+1) = A\hat{z}(t) + K(C\hat{z}(t) - y(t)) + Bu(t) + Bv(t) \\ v(t+1) = v(t) + H(C\hat{z}(t) - y(t)) \\ u(t) = F_fA\hat{z}(t) + F_fK(C\hat{z}(t) - y(t)) \end{cases} \tag{5}$$

where $F = F_fA$.

For deriving systematic design methods for the PI-observers, the observers are formulated in a more compact form, the dual of the PI-state feedback [1]:

$$\Sigma_{D,O}: \begin{cases} \xi(t+1) = A_x\xi(t) + K_x(C_x\xi(t) - y(t)) + B_xu(t) \\ u(t) = F_x\xi(t) + F_dy(t) \end{cases} \tag{6}$$

$$\text{with } A_x = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \quad B_x = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad C_x = \begin{bmatrix} C & 0 \end{bmatrix} \quad K_x = \begin{bmatrix} K \\ H \end{bmatrix}$$
$$F_x: \begin{cases} [\, F \quad 0 \,],\ F_d = 0 & \text{for the pred. observer} \\ [\, F_f(A+KC) \quad 0 \,],\ F_d = -F_fK & \text{for the filt. observer} \end{cases} \tag{7}$$

1.2 Recovery in discrete-time.

First, the recovery matrix are introduced for the PI-observers. Second, time recovery are defined and conditions for obtaining time recovery with the PI-observers is given.

Let the sensitivity recovery error be defined as the difference between the target design and the full-loop design, [8]:

$$E_S(z) = S_{TFL}(z) - S_I(z) = S_{TFL}(z)M_I(z) \tag{8}$$

with $M_I(s)$ (the recovery matrix) given by:

$$\begin{aligned} M_I(z) &= \begin{bmatrix} F & 0 \end{bmatrix} \begin{bmatrix} zI-A-KC & -B \\ -HC & zI-I \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= (z-1)F(zI(z-1)-(A+KC)(z-1)-BHC))^{-1}B \end{aligned} \tag{9}$$

for the prediction PI-observer and

$$\begin{aligned} M_I(z) &= \begin{bmatrix} F_f(A+KC) & 0 \end{bmatrix} \begin{bmatrix} zI-A-KC & -B \\ -HC & zI-I \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= (z-1)F_f(A+KC)(zI(z-1)-(A+KC)(z-1)-BHC)^{-1}B \end{aligned} \tag{10}$$

for the filtering PI-observer.

Further, we define time recovery for discrete-time systems in the following way:

Definition 1.1. *Let $M_I(z)$ be the recovery matrix. Time recovery is obtained if and only if:*

$$M_I(1) = 0 \tag{11}$$

The significance of time recovery is that the recovery error tends to zero as t tends to infinity, $E_S(\infty) = 0$, (it is straight forward to show this. Necessary and sufficient conditions for obtaining time recovery is given in:

Theorem 1.2. *Time recovery is obtained with a prediction PI-observer if and only if the largest invariant subspace of the matrix $(I-A-KC)^{-1}BHC$ contained in the controllable subspace of the pair $((I-A-KC)^{-1}, (I-A-KC)^{-1}B)$ corresponding to the eigenvalue $z = 1$ is itself contained in the unobservable subspace of the pair $(F, (I-A-KC)^{-1}BHC)$.*
Proof. Omitted.

Further, we have the following corollary in connection to Theorem 1.2, which give a simple matrix condition to check for time recovery.

Corollary 1.3.
Let the Jordan normal form of the matrix $(I-A-KC)^{-1}BHC$ be given by:

$$T^{-1}((I-A-KC)^{-1}BHC)T = \begin{pmatrix} J_0 & 0 \\ 0 & \bar{J} \end{pmatrix},\ T = \begin{pmatrix} T_1 & T_2 \end{pmatrix},\ T^{-1} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \tag{12}$$

[1]This work is supported in part by the Danish Technical Reseach Council under grant no. 16-4885-1 and grant no. 5.26-1830.

Where J_0 contains all Jordan blocks associated with the eigenvalue $z = 1$. Then time recovery is obtained if and only if

$$FT_1(S_1(I-A-KC)^{-1}B, J_0S_1(I-A-KC)^{-1}B, \ldots, J_0^{n-1}S_1(I-A-KC)^{-1}B) = 0 \qquad (13)$$

For the filtering PI-observer, the only difference is that the target design gain F in Theorem 1.2 must be substituted by $F_f(A+KC)$.

2. DISCRETE-TIME LQG AND LQG/LTR DESIGN.

Applying the compact form of the PI-observers in (6), the LQG design is determined by the following Riccati equation:

$$P = A_xPA_x^T - A_xPC_x^T(\Sigma + C_xPC_x^T)^{-1}C_xPA_x^T + \Gamma$$
$$\Sigma \geq 0,\ \ \Gamma = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\begin{bmatrix} L_1 & L_2 \end{bmatrix} \geq 0 \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \qquad (14)$$

$$K_x = -A_xPC_x^T(\Sigma + C_xPC_x^T)^{-1}$$
where
$$= \begin{bmatrix} -AP_{11}C^TD^{-1} - BP_{12}^TC^TD^{-1} \\ -P_{12}^TC^TD^{-1} \end{bmatrix},\quad D = \Sigma + CP_{11}C^T \qquad (15)$$

The integral gain $H = [0\ I]K_x$ has full rank if and only if CP_{12} has full rank. Rewriting the Riccati equation in (14) into 4 (effectively 3) equations gives:

$$0 = -P_{11} + AP_{11}A^T + BP_{12}^TA^T + AP_{12}B^T + BP_{22}B^T - AP_{11}C^Td^{-1}CP_{11}A^T$$
$$-AP_{11}C^Td^{-1}CP_{12}B^T - BP_{12}^TC^Td^{-1}CP_{11}A^T - BP_{12}^TC^Td^{-1}CP_{12}B^T + \Gamma_{11} \qquad (16)$$
$$0 = -P_{12} + AP_{12} + BP_{22} - AP_{11}C^Td^{-1}CP_{12} - BP_{12}^TC^Td^{-1}CP_{12} + \Gamma_{12}$$
$$0 = -P_{12}^TC^Td^{-1}CP_{12} + \Gamma_{22}$$

From the last equation in (16) it can be seen directly that CP_{12} has full rank if and only if $L_2L_2^T = \Gamma_{22}$ is positive definite. Moreover, Γ_{22} is the only part of Γ which has influence on H through P_{12}. As in the continuous-time case, we will obtain time recovery with LQG design generically.

Discrete-time LQG/LTR design is derived in the same way as for continuous-time systems [3,10], execpt that the design problem can be solved with a weight matrix Σ on the measurement signals equal to zero. Moreover, the solution to the Riccati equation with $\Sigma = 0$ has been given in explicit forms in [9].

LQG/LTR design for full-order observer is computed by using $\Gamma = BB^T$ and $\Sigma = 0$, [4,11]. If the LQG/LTR design of the PI-observers is done in the same way, we get (related to the Riccati equation in (14)):

$$\Gamma_x = B_xB_x^T,\ \ \Sigma_x = 0\ \ K_x = \begin{bmatrix} -AB(CB)^{-1} \\ 0 \end{bmatrix} \qquad (17)$$

assuming (A,B,C) is minimum phase. Again, as in the continuous-time case, the integral effect vanish in the traditional LQG/LTR design. However, by modifying Γ in (14) so the integral effect is preserved in the PI-observer, time recovery can then be obtained as the result. To do this, we first need the following result, [9]:

Lemma 2.1. *The singular stationar Riccati equation ($\Sigma = 0$) is given by:*

$$P = APA^T + \Gamma - APC^T(CPC^T)^{-1}CPA^T \qquad (18)$$

With $\Gamma = LL^T$ is the observer gain K the given by (for (A,L,C) minimum-phase):

$$K = -AL(CL)^{-1} \qquad (19)$$

It is now simple to modify the LQG/LTR method, so it is possible to handle the PI-observer suitable.
Let's assume that (A,B,C) is minimum phase and let L be given by:

$$L = \begin{bmatrix} B \\ L_2 \end{bmatrix} \qquad (20)$$

The minimum phase condition for the system (A_x, L, C_x) is:

$$\mathrm{rank}\begin{bmatrix} zI-A & -B & B \\ 0 & zI-I & L_2 \\ C & 0 & 0 \end{bmatrix} = n+2m, \quad \forall z: \rho(z) \geq 1 \qquad (21)$$

From (21) we have directly that $I-L_2$ must have all it's eigenvalues inside the unit circle for (A_x, L, C_x) to be minimum phase.

Using $\Gamma = LL^T$ with L given by (20) instead of $L = B_x$ as the weight matrix in the LQG/LTR design, result in the following observer gain by using Lemma 2.1:

$$K_x = -A_xL(C_xL)^{-1} = \begin{bmatrix} -AB(CB)^{-1} - BL_2(CB)^{-1} \\ -L_2(CB)^{-1} \end{bmatrix} \qquad (22)$$

With $L_2 = 0$ in (22) we get the full-order LQG/LTR design.

The integral weight L_2 is now selected such that time recovery is obtained and the recovery matrix is minimized.

The recovery matrix for the prediction observer in (9) together with the above observer gain in (22) is now rewritten into a Taylor series:

$$M_I(z) = z^{-1}[\,F\quad 0\,]\sum_{i=0}^{\infty}\begin{bmatrix} A-AB(CB)^{-1}C-BL_2(CB)^{-1} & B \\ -L_2(CB)^{-1}B & I \end{bmatrix}^i z^{-i}\begin{bmatrix} B \\ 0 \end{bmatrix}$$
$$= \frac{FB}{z}\left(I - \frac{L_2}{z}\sum_{i=0}^{\infty}\frac{(I-L_2)^i}{z^i}\right) \qquad (23)$$

The last equation arrived by using $(A-AB(CB)^{-1}C)B = 0$.

An obvious choice for the integral weight L_2 is to use $L_2 = I$ which gives the following recovery matrix:

$$M_I(z) = \frac{FB}{z}\left(I - \frac{I}{z}\right) \qquad (24)$$

With this choice of L_2 we have:

$$M_I(1) = 0,\ \ \max M_I(z) = 2 \times \max M_{IFO}(z),\ M_{IFO}(z) = \frac{FB}{z} \qquad (25)$$

where M_{IFO} is the recovery matrix for the full-order observer ($L_2 = 0$). In the minimum phase case, the discrete-time full-order filtering observer will always give exact recovery, [4], because $F_f(A+KC) = 0$. The time recovery effect from the PI-observer is therefore unnecessary in this case.

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3 Parameterization of All Periodic Covariance Controllers

Define the following matrices

$$\mathcal{A}(n) = \begin{bmatrix} [I - B(n)B^+(n)]L(n) \\ [I - M^+(n)M(n)]A^T(n)L^{-T}(n) \end{bmatrix} = U_{\mathcal{A}}(n) \begin{bmatrix} S_{\mathcal{A}}(n) & 0 \\ 0 & 0 \end{bmatrix} V_{\mathcal{A}}^T(n) \; ; \quad (11a)$$

$$\mathcal{B}(n) = \begin{bmatrix} [I - B(n)B^+(n)]A(n)T(n) \\ [I - M^+(n)M(n)]T^{-T}(n) \end{bmatrix} = U_{\mathcal{A}}(n) \begin{bmatrix} S_{\mathcal{A}}(n) & 0 \\ 0 & 0 \end{bmatrix} V_{\mathcal{B}}^T(n) \; ; \quad (11b)$$

$$r_{\mathcal{A}}(n) = \text{rank}[\mathcal{A}(n)] \quad , \quad r_u(n) = (n_x + n_c) - r_{\mathcal{A}}(n) \; . \quad (11c)$$

Then the following Theorem presents the parameterization of all the covariance controllers which assign the given periodic covariance X(n).

Theorem 2

Suppose that the given periodic covariance X(n) *is assignable, then the set of all controllers are parameterized by arbitrary matrices* $Z(n) \in \mathbb{R}^{(n_u+n_c)^2}$ $(n = 0, 1, \dots, T_p-1)$ *and arbitrary orthonormal matrices* $U(n) \in \mathbb{R}^{r_u \times r_u}$ $(n = 0, 1, \dots, T_p-1)$ *as*

$$G(n) = B^+(n)\{L(n)V_{\mathcal{A}}(n) \begin{bmatrix} I & 0 \\ 0 & U(n) \end{bmatrix} V_{\mathcal{B}}(n)T^{-1}(n) - A(n)\}M^+(n)$$

$$+ Z(n) - B^+(n)B(n)Z(n)M(n)M^+(n) \; . \quad (12)$$

For a proof of the above Theorem, see [10].

Corollary 4

In the case where $B_p(n)$ *and* $M_p(n)$ *are of full rank. The freedom* Z(n) $(n = 0, 1, \dots, T_p-1)$ *in (12) disappears and* G(n) *is given by*

$$G(n) = B^+(n)\{L(n)V_{\mathcal{A}}(n) \begin{bmatrix} I & 0 \\ 0 & U(n) \end{bmatrix} V_{\mathcal{B}}^T(n)T^{-1}(n) - A(n)\}M^+(n) \quad (13)$$

Corollary 5

Suppose that the given $X_p(n)$ *is assignable via state feedback control and* $B_p(n)$ *is of full rank, then the set of all controllers which assign this* $X_p(n)$ *to the system is parameterized by arbitrary orthonormal matrices* $U_s(n) \in \mathbb{R}^{n_u \times n_u}$ $(n = 0, 1, \dots, T_p-1)$

$$G(n) = B_p^+(n)\{L(n)V_{\mathcal{A}_s}(n) \begin{bmatrix} I & 0 \\ 0 & U_s(n) \end{bmatrix} V_{\mathcal{B}_s}(n)T^{-1}(n) - A_p(n)\} \; , \quad (14)$$

where

$$\mathcal{A}_s(n) = [I - B_p(n)B_p^+(n)]L(n) = U_{\mathcal{A}_s}(n) \begin{bmatrix} S_{\mathcal{A}_s}(n) & 0 \\ 0 & 0 \end{bmatrix} V_{\mathcal{A}_s}(n) \; , \quad (15a)$$

$$\mathcal{B}_s(n) = [I - B_p(n)B_p^+(n)]A_p(n)T(n) = U_{\mathcal{A}_s}(n) \begin{bmatrix} S_{\mathcal{A}_s}(n) & 0 \\ 0 & 0 \end{bmatrix} V_{\mathcal{B}_s}(n) \quad (15b)$$

4 Example

Consider the periodically time-varying system (1) with period $T_p = 3$, where the system matrices are given below.

$$A(0) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \; ; \; A(1) = \begin{bmatrix} 0 & 1 \\ -1 & 1.1 \end{bmatrix} \; ; \; A(2) = \begin{bmatrix} 0 & 1.0 \\ -1 & 1.2 \end{bmatrix} \; ; \quad (16)$$

$$B(0) = D(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; ; \; B(1) = D(1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \; ; \; B(2) = D(2) = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} \; ;$$

The covariance matrices of the input cyclo-stationary process $w_p(k,n)$ are

$$W(0) = 1 \; ; \; W(1) = 2 \; ; \; W(2) = 3 \; . \quad (17)$$

Firstly we consider the state feedback case. Let the steady state periodic covariance to be assigned be

$$X(0) = \begin{bmatrix} 12.4794 & 8.9958 \\ 8.9958 & 14.1822 \end{bmatrix} \; ; \quad (18a)$$

$$X(1) = \begin{bmatrix} 14.1822 & 2.5932 \\ 2.5932 & 3.1675 \end{bmatrix} \; ; \quad (18b)$$

$$X(2) = \begin{bmatrix} 3.1675 & 0.6613 \\ 0.6613 & 12.4794 \end{bmatrix} \; . \quad (18c)$$

Check the assignability condition (8). The norm of the left side of equation (8) is 0, 0 and 0.3553e–14 for $n = 0, 1, 2$. Hence, the given covariance function is assignable numerically. In this case the dimension of the free unitary matrices U(n) is 1. By taking

$$U(n) = 1 \; , \; n = 0, 1, 2 \; .$$

we obtain the following state feedback controller which assigns the given covariance function (18)

$$G(0) = [0.5 \; -0.5] \; ; \; G(1) = [0.2 \; -0.2] \; ; \; G(2) = [0.3 \; -0.3] \; , \quad (19)$$

and by taking

$$U(n) = -1 \; , \; n = 0, 1, 2 \; .$$

the control gain we obtain is different to (19), where

$$G(0) = [1.5000 \; -1.1343] \; ; \quad (20a)$$

$$G(1) = [0.8000 \; -0.6912] \; ; \quad (20b)$$

$$G(2) = [1.0333 \; -0.3389] \; . \quad (20c)$$

Hence the control gains which assign the same periodic covariance (18) are not unique.

5 Conclusion

For discrete periodically time-varying systems, the set of all assignable periodic covariances is characterized by two explicit conditions. The set of all the covariance controllers which assign the given assignable periodic covariance to the periodically time-varying system is parameterized with T_p arbitrary orthonormal matrices U (n) $(n = 0, 1, \dots, T_p-1)$.

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