Proceedings of the 31st Conference
on Decision and Control
Tucson, Arizona • December 1992

Schur Stability of Uncertain Matrices

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Abstract

This note considers the stability of uncertain matrices. It is shown that under certain structural assumptions on the uncertain matrices the Schur stability can be assured from computing the numerical radius of the vertex matrices. This result is less conservative than that of using a simultaneous Lyapunov function method. Necessary and sufficient conditions are also obtained for the stability of a class of interval matrices.

1 Introduction and Preliminaries

In this note, Schur stability of uncertain matrices is considered. The results presented are based on some simple linear algebra facts. Nevertheless, they seem to give very reasonable robust stability criterion. To present these results, we shall adopt the following standard notations. $\mathrm{Co}(S)$ denotes the convex hull of a set S. The Euclidean norm of a vector $x \in \mathcal{C}^n$ is denoted by $\|x\|$. Let $A \in \mathcal{C}^{n\times n}$. Then A^* denotes its complex conjugate, $\mathrm{spec}(A)$ denotes its spectral set, and $\rho(A)$ denotes its spectral radius. The induced 2 norm (or spectral norm) of A is denoted by $\|A\|$. A matrix $A \in \mathcal{C}^{n\times n}$ is said to be *normal* if $A^*A = AA^*$.

Definition 1 Let $A \in \mathcal{C}^{n\times n}$. The set

$$F(A) := \{x^*Ax : \ x \in \mathcal{C}^n, \ \|x\| = 1\}$$

is called the *numerical range* of A and the function

$$r(A) := \max_{x\in\mathcal{C}^n, \ \|x\|=1} |x^*Ax|$$

is called its *numerical radius.*

The numerical range of a matrix A has the following properties

Lemma 2 *[7, 8] Let $A \in \mathcal{C}^{n\times n}$. Then*

(i) $F(A)$ is a compact and convex subset in $\mathcal{C}$.

(ii) $F(\frac{A+A^*}{2}) = \mathrm{Re}\ F(A)$.

(iii) $\mathrm{spec}(A) \subset F(A)$.

(iv) $F(A) = \mathrm{Co}(\mathrm{spec}(A))$ if A is normal.

(v) $\frac{1}{2}\|A\| \le r(A) \le \|A\|$.

(vi) $\rho(A) \le r(A) \le \|A\|$. The equalities hold if A is normal.

2 Stability in State Space

Consider the stability of the following uncertain matrix

$$A_\delta := \frac{A_0 + \sum_{i=1}^m \delta_i A_i + \sum_{i<j}^m \delta_i\delta_j A_{ij} + \cdots + \delta_1\delta_2\ldots\delta_m A_{12\ldots m}}{a_0 + \sum_{i=1}^m \delta_i a_i + \sum_{i<j}^m \delta_i\delta_j a_{ij} + \cdots + \delta_1\delta_2\ldots\delta_m a_{12\ldots m}}$$

i.e, the uncertainty is entered in such a way so that for each i, A_δ can be written as

$$A_\delta = \frac{G_i(\delta_1,\ldots,\delta_{i-1},\delta_{i+1},\ldots,\delta_m) + \delta_i H_i(\delta_1,\ldots,\delta_{i-1},\delta_{i+1},\ldots,\delta_m)}{g_i(\delta_1,\ldots,\delta_{i-1},\delta_{i+1},\ldots,\delta_m) + \delta_i h_i(\delta_1,\ldots,\delta_{i-1},\delta_{i+1},\ldots,\delta_m)}$$

*Supported in part by the Danish Technical Research Council under grant nos. 16-4885-1 and 26-1830.

for some G_i, H_i, g_i, h_i where $\delta_i \in [\underline{\delta}_i, \bar{\delta}_i]$. In order to make this well defined, we assume $a_0 + \sum_{i=1}^m \delta_i a_i + \sum_{i<j}^m \delta_i\delta_j a_{ij} + \cdots + \delta_1\delta_2\ldots\delta_m a_{12\ldots m} \neq 0$ for all $\delta_i \in [\underline{\delta}_i, \bar{\delta}_i]$

Now define the vertex set of A_δ as the matrices

$$A_{\mathrm{vex}} := \{A_\delta : \ \delta_i \in \{\underline{\delta}_i, \bar{\delta}_i\}\}$$

Theorem 3 *The uncertain matrix A_δ is Schur stable if there exists a nonsingular matrix T such that $r(TAT^{-1}) < 1$ for all $A \in A_{\mathrm{vex}}$. Moreover, if A_δ is normal, then*

$$\max_{A\in A_\delta} \rho(A) = \max_{A\in A_{\mathrm{vex}}} \rho(A) = \max_{A\in A_{\mathrm{vex}}} r(A) = \max_{A\in A_{\mathrm{vex}}} \|A\|$$

Proof. To show that $r(TAT^{-1}) < 1$ for all $A \in A_{\mathrm{vex}}$ implies the stability of A_δ, we first note that, for each i

$$\begin{aligned}\max_{\delta_i\in[\underline{\delta}_i,\bar{\delta}_i]} r(TAT^{-1}) &= \max_{\delta_i\in[\underline{\delta}_i,\bar{\delta}_i]} \ \max_{x\in\mathcal{C}^n, \ \|x\|=1} \left|\frac{x^*TG_iT^{-1}x + \delta_i x^*TH_iT^{-1}x}{g_i + \delta_i h_i}\right| \\ &= \max_{\delta_i=\underline{\delta}_i, \ \delta_i=\bar{\delta}_i} r(TAT^{-1})\end{aligned}$$

Now, repeatedly applying the above results to each i, we have

$$\max_{A\in A_\delta} \rho(A) = \max_{A\in A_\delta} \rho(TAT^{-1}) \le \max_{A\in A_\delta} r(TAT^{-1}) = \max_{A\in A_{\mathrm{vex}}} r(TAT^{-1})$$

which is the conclusion of the first part of the theorem. Finally if A_δ is normal, then it follows from Lemma 2 that

$$\begin{aligned}\max_{A\in A_\delta} \rho(A) &\ge \max_{A\in A_{\mathrm{vex}}} \rho(A) = \max_{A\in A_{\mathrm{vex}}} r(A) = \max_{A\in A_{\mathrm{vex}}} \|A\| = \max_{A\in A_\delta} \|A\| \\ &\ge \max_{A\in A_\delta} \rho(A)\end{aligned}$$

□

For the normal case it is easy to see from Lemma 2 that the uncertainty matrix A_δ is (Hurwitz or Schur) stable if and only if A_{vex} is (Hurwitz or Schur) stable. A similar result is shown in [14].

An immediate consequence of the above theorem is the following corollary.

Corollary 4 *The uncertain matrix A_δ is Schur stable if there exists a nonsingular matrix T such that $\|TAT^{-1}\| < 1$ for all $A \in A_{\mathrm{vex}}$ or equivalently there exists a $P > 0$ such that*

$$A^*PA - P < 0$$

for all $A \in A_{\mathrm{vex}}$.

Proof. The conclusion follows from the fact that for any nonsingular matrix T

$$\rho(A) = \rho(TAT^{-1}) \le r(TAT^{-1}) \le \left\|TAT^{-1}\right\|$$

Now let $P = T^*T$. Then $\|TAT^{-1}\| < 1$ if and only if $A^*PA - P < 0$. □

It is interesting to note that the last condition is a discrete version of the so called *simultaneous Lyapunov stability* condition introduced in Boyd and Yang [3]. In addition, a numerical algorithm is given in [3] for finding a positive definite solution P which can not fail if such P exists.

It should be pointed out that the scaling matrix T in the above theorem is essential in improving the stability test. The results can be arbitrarily conservative without the scaling matrix. For example

$$A = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$$

is Schur stable for any n since $\rho(A) = \inf_T r(TAT^{-1}) = \inf_T \|TAT^{-1}\| = 0$. However, $r(A) \geq \frac{1}{2}\|A\| = \frac{n}{2}$ can be arbitrarily large.

Finding a T to minimize $r(TAT^{-1})$ for all $A \in A_{\text{vex}}$ is in general harder than to find a T to minimize $\|TAT^{-1}\|$ for all $A \in A_{\text{vex}}$. Hence a suboptimal choice is to use the optimal T from the minimization of $\|TAT^{-1}\|$ in the stability test criterion $r(TAT^{-1})$ for all $A \in A_{\text{vex}}$. This will still lead to a less conservative result than that of using $\|TAT^{-1}\|$.

It is also important to note that

$$\min_T \max_{A\in A_{\text{vex}}} \left\|TAT^{-1}\right\| \geq \min_T \max_{A\in A_{\text{vex}}} r(TAT^{-1}) \geq \frac{1}{2} \min_T \max_{A\in A_{\text{vex}}} \left\|TAT^{-1}\right\|$$

Hence if $\min_T \max_{A\in A_{\text{vex}}} \left\|TAT^{-1}\right\| \geq 2$, then $\min_T \max_{A\in A_{\text{vex}}} r(TAT^{-1}) \geq 1$. In this case, we do not need to compute the numerical radius of TAT^{-1} since the test fails to assure the stability.

Next we consider the stability of a special class of uncertain matrices A_δ which are called interval matrices. Let $P = [p_{ij}]$ and $Q = [q_{ij}]$ be $n \times n$ real matrices with $(i,j)^{\text{th}}$ elements p_{ij} and q_{ij} respectively. Let $M(P,Q)$ denote the following set of matrices

$$M(P,Q) = \{A = [a_{ij}] : \ p_{ij} \leq a_{ij} \leq q_{ij}\}.$$

Then the set $M(P,Q)$ is called an *interval matrix* and is said to be *Schur stable* if $\rho(A) < 1 \ \forall A \in M(P,Q)$ where $\rho(A)$ denotes the spectral radius of A.

It is easy to see that interval polynomial problems are special cases of interval matrix problems. Hence it is conceivable that the interval matrix stability problem is much harder than the interval polynomial stability problems. In particular, the stability of the vertex matrices does not imply the stability of the interval matrices. However, under certain assumptions, necessary and sufficient conditions can be obtained for the stability of interval matrices. We will demonstrate this possibility by considering a special class of interval matrices. First we recall a matrix fact. A matrix A is called a nonnegative matrix if each element of A is nonnegative. Denote $|A|$ the matrix obtained from taking the absolute value of each element of A, i.e., $|A| = [|a_{ij}|]$. Then we have the following fact, see [7, p. 491].

Fact 1 *Let $A \in \mathcal{R}^{n\times n}$. Then $\rho(A) \leq \rho(|A|)$.*

Using this fact, we have

Theorem 5 *Let $W = [w_{ij}]$ be a matrix with $w_{ij} = \max\{|p_{ij}|, |q_{ij}|\}$. Then the interval matrix $M(P,Q)$ is Schur stable if $\rho(W) < 1$. The converse is also true if either one of the following conditions holds*

(i) $0 \leq |p_{ij}| \leq q_{ij}$ holds for all i,j.

(ii) $0 \leq |q_{ij}| \leq -p_{ij}$ holds for all i,j.

The later part of the theorem can be regarded as an extreme point result since the necessary and sufficient stability condition is given in terms of the vertices of the interval matrix. Part of the results was also obtained by Shafai *et al* [13] under the assumption of P and Q nonnegative which is much more restrictive. Some related results were obtained in [4] for the Hurwitz stability, see also the references therein for other related results.

It is a trivial fact that the necessary part (of course the sufficient part as well) of the theorem still holds under similarity transformation. Hence the above stability criterion applies to any interval matrices which satisfy either condition (i) or (ii) after the similarity transformation. For example, consider the following interval matrix

$$M = \begin{bmatrix} [0.1, & 0.3] & [-0.2, & 0.1] \\ [-0.5, & -0.4] & [0.5, & 0.8] \end{bmatrix}.$$

It is clear that this interval matrix does not satisfy either condition (i) or condition (ii). However, after a similarity transformation, we have

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} M \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} [0.1, & 0.3] & [-0.1, & 0.2] \\ [0.4, & 0.5] & [0.5, & 0.8] \end{bmatrix}$$

which satisfies the condition (i). Hence the Schur stability of the interval matrix can be answered by checking the Perron eigenvalue of

$$A = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}.$$

(Actually, the Perron eigenvalue is smaller than 1, so the interval matrix is always Schur stable.) Of course, there are limitations to how much the similarity transformation can offer.

In cases where the condition given above is not necessary, scaling methods as in the last section has to be applied.

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