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The General $\mathcal{H}_\infty$ Problem with Static Output Feedback

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Abstract

In this paper we shall consider the $\mathcal{H}_\infty$ control problem using static output feedback. Recently, this problem was solved in the regular case. We shall extend this result such that no assumption concerning the direct feedthrough term in the $\mathcal{H}_\infty$ problem is made. The main result states that the $\mathcal{H}_\infty$ static output feedback control problem is solvable if and only if the general $\mathcal{H}_\infty$ static state feedback problem is solvable and, further, that a certain 'geometric Riccati type' side constraint is satisfied.

1 Introduction

Frequently, static output feedback controllers are desired in miscellaneous control engineering problems and are sometimes implemented, mostly in ad hoc design schemes. Whereas static state feedback controllers and dynamic output feedback controllers are quite well understood from a theoretical point of view and have systematic design algorithms which are easy to implement, nothing similar is true for static output feedback controllers and there is a great lack of feasible design algorithms for such controllers.

It is not clear what characterizes the closed loop systems which can be obtained by zeroth order controllers. Recently [1], the static output feedback problem has been considered in an eigenstructure assignment approach, where some important properties of static output feedback controllers has been recorded.

Under 'generic' conditions, a static output feedback controller can assign at most r closed loop poles where r is the smaller of (1) no. of states (2) no. of inputs + no. of outputs - 1, and the paper provides an algorithm for assigning these poles, whenever it is possible. There is no guarantee, however, that the unstable poles are among the assignable ones, and henceforth pole assignment strategies do not provide sufficiently general, systematic design techniques.

It has not been clear at all how to generalize the popular $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design techniques to static output feedback systems.

Recently, however, a solution to the regular static output feedback $\mathcal{H}_\infty$ problem has been given [8], in terms of simultaneous solvability of a Riccati inequality and a partial Riccati inequality (see below). The approach in [8] was based on methods from the covariance control literature [5, 6, 4].

The main restriction in this approach, besides computational aspects, is the technical assumption that the direct feedthrough term from controllers to output should have full (column) rank. In the present paper we shall try to overcome this difficulty.

2 Preliminaries

In the sequel we shall consider systems of the following form:

$$\begin{array}{rclcccc} \dot{x} & = & Ax & + & B_1 w & + & B_2 u \\ z & = & C_1 x & & & + & D_{12} u \\ y & = & C_2 x & & & & \end{array} \qquad (1)$$

for which we shall study control laws of the form $u = Ky$ for constant K which solve the $\mathcal{H}_\infty$ standard problem [3].

For technical reasons we shall assume that this system has no invariant zeros on the imaginary axis, i.e. the matrix $\begin{pmatrix} A - i\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix}$ has constant rank for all ω, but we shall make no assumptions on the rank of D_{12}.

We need to introduce the kernel of C_2 in terms of the matrix V_2 given by the singular value decomposition of C_2.

$$C_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 & V_2 \end{pmatrix}' \qquad (2)$$

The main result of [8] provides the solution to the static output feedback $\mathcal{H}_\infty$ problem for the special case where D_{12} has full column rank:

Theorem 1 *Consider the system (1). The following two statements are equivalent.*

(i) There exists a matrix $K \in \mathcal{R}^{m\times p}$, such that when applying the static output feedback law $u = Ky$, the resulting closed loop system is internally stable, and the $\mathcal{H}_\infty$ norm from w to z is smaller than γ.

(ii) There exists a positive definite solution P to the following two inequalities.

$$\begin{array}{l} A'P + PA + C_1'C_1 + \gamma^{-2}PB_1B_1'P \\ -(PB_2 + C_1'D_{12})R^{-1}(PB_2 + C_1'D_{12})' < 0 \end{array} \quad (3)$$

$$V_2'(A'P+PA+C_1'C_1+\gamma^{-2}PB_1B_1'P)V_2 < 0 \quad (4)$$

where $R := D_{12}'D_{12}$ and V_2 is given by (2).

In this paper we shall extend Theorem 1 to the general case where no assumptions are made on the rank of D_{12}. To that end, we shall need the solution to the so called singular $\mathcal{H}_\infty$ state feedback problem which can be found in [9].

Consider the following state feedback system:

$$\begin{array}{rclcccc} \dot{x} & = & Ax & + & B_1w & + & B_2u \\ z & = & C_1x & & & + & D_{12}u \\ y & = & x & & & & \end{array} \quad (5)$$

The existence of state feedback laws for (5) is characterized by the following result.

Theorem 2 *The following two are equivalent.*

1. *There exists a state feedback gain F such that $A + B_2F$ is stable and such that*

$$\left\|(C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_1\right\|_\infty < \gamma$$

2. *There exists $P \geq 0$ such that the following three hold*

(a) $$F(P) = \begin{pmatrix} F_{11}(P) & F_{12}(P) \\ F_{21}(P) & F_{22}(P) \end{pmatrix} =: \begin{pmatrix} C_P' \\ D_P' \end{pmatrix} \begin{pmatrix} C_P & D_P \end{pmatrix} \geq 0$$

where

$$\begin{array}{rcl} F_{11}(P) & = & A'P + PA + C_1'C_1 \\ & & +\gamma^{-2}PB_1B_1'P \\ F_{12}(P) & = & PB_2 + C_1'D_{12} \\ F_{21}(P) & = & B_{12}'P + D_{12}'C_1 \\ F_{22}(P) & = & D_{12}'D_{12} \end{array}$$

(b) $$m_{nr} := \operatorname{rank}\begin{pmatrix} C_P & D_P \end{pmatrix} = \max_{s\in\mathcal{C}} \operatorname{rank}\,(C_1(sI - A)^{-1}B_2 + D_{12})$$

(c) $$\operatorname{rank}\begin{pmatrix} sI - A - \gamma^{-2}B_1B_1'P & -B_2 \\ C_P & D_P \end{pmatrix} = n + m_{nr}\ , \ \forall s \in \bar{\mathcal{C}}^+$$

Whenever $P \geq 0$ exists satisfying the three conditions (2a-2c) of Theorem 2 such P can be found by solving a reduced order Riccati equation. Moreover, it can be shown that P is unique (see [9]).

3 Main Results

In this section, we shall derive necessary and sufficient conditions for solvability of the general $\mathcal{H}_\infty$ problem by static output feedback control.

First, however, we need the following preliminary result.

Lemma 1 *Consider the system*

$$\begin{array}{rclcccc} \dot{x} & = & Ax & + & B_1w & + & B_2u \\ z & = & C_{1,\varepsilon}x & & & + & D_{12,\varepsilon}u \\ y & = & x & & & & \end{array} \quad (6)$$

where $C_{1,\varepsilon} = \begin{pmatrix} C_1 \\ \varepsilon I \\ 0 \end{pmatrix}$ *and* $D_{12,\varepsilon} = \begin{pmatrix} D_{12} \\ 0 \\ \varepsilon I \end{pmatrix}$

Then the following three statements are equivalent.

1. *There exists a static state feedback F for the system (5) such that $A + B_2F$ is stable and such that $\left\|(C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_1\right\|_\infty < \gamma$*

2. *There exists ε_1 such that for all $\varepsilon \in (0;\varepsilon_1]$ there exists a static state feedback F_ε such that $A + B_2F_\varepsilon$ is stable and such that*

$$\left\|(C_{1,\varepsilon} + D_{12,\varepsilon}F_\varepsilon)(sI - A - B_2F_\varepsilon)^{-1}B_1\right\|_\infty < \gamma$$

3. *There exists ε_1 such that for all $\varepsilon \in (0;\varepsilon_1]$ there exists $P_\varepsilon > 0$ such that*

$$\begin{array}{l} \varepsilon^2 I + A'P_\varepsilon + P_\varepsilon A + C_1'C_1 + \gamma^{-2}P_\varepsilon B_1B_1'P_\varepsilon \\ -(P_\varepsilon B_2 + C_1'D_{12})R^{-1}(P_\varepsilon B_2 + C_1'D_{12})' < 0 \end{array}$$

where $R := D_{12}'D_{12} + \varepsilon^2 I$

Moreover, the sequence P_ε is convergent, $P_\varepsilon \to P$, and the limit P satisfies conditions (2a-2c) of Theorem 2.

Proof. The equivalence of Lemma 1(2) and Lemma 1(3) is the well known regular state feedback result. The equivalence of Lemma 1(1) and Lemma 1(2) is a general $\mathcal{H}_\infty$ cheap control result. The proof of the regular case carries through to the general case without any basic changes. Finally, that $P_\varepsilon \to P$, where P satisfies Theorem 2(2a-2c) is proved in [7]. □

Lemma 1 states that in the general case, the well known regular case Riccati equation [2] has to be substituted with the three conditions in Theorem 2(2a-2c) as $\varepsilon \to 0$. Our main result states that solvability of the general $\mathcal{H}_\infty$ control problem is equivalent to solvability of a generalization of the three conditions in Theorem 2(2a-2c) along with a 'geometric' condition of the same type as Theorem 1(4).

Theorem 3 *The following two are equivalent.*

1. *There exists a static output feedback gain K such that $A + B_2KC_2$ is stable and such that*
$$\|(C_1 + D_{12}KC_2)\Phi B_1\|_\infty < \gamma \ ,$$
$$\Phi(s) := (sI - A - B_2KC_2)^{-1}$$
2. *There exists $P \geq 0$ and $W > 0$ such that the following four hold*

(a) $F_W(P) = \begin{pmatrix} F_{11W}(P) & F_{12W}(P) \\ F_{21W}(P) & F_{22W}(P) \end{pmatrix}$
$$=: \begin{pmatrix} C_P' \\ D_P' \end{pmatrix} \begin{pmatrix} C_P & D_P \end{pmatrix} \geq 0$$
where
$$\begin{aligned} F_{11W}(P) &= A'P + PA + C_{1W}'C_{1W} \\ &\quad + \gamma^{-2}PB_1B_1'P \\ F_{12W}(P) &= PB_2 + C_{1W}'D_{12W} \\ F_{21W}(P) &= B_2'P + D_{12W}'C_{1W} \\ F_{22W}(P) &= D_{12W}'D_{12W} \end{aligned}$$

(b) $m_{nr} := \operatorname{rank}\begin{pmatrix} C_P & D_P \end{pmatrix}$
$$= \max_{s\in C} \operatorname{rank}\,(C_{1W}(sI - A)^{-1}B_2 + D_{12W})$$

(c) $\operatorname{rank}\begin{pmatrix} sI - A - \gamma^{-2}B_1B_1'P & -B_2 \\ C_P & D_P \end{pmatrix}$
$$= n + m_{nr} \ , \ \forall s \in \bar{C}^+$$

(d) $V_2'(A'P + PA + C_1'C_1 + \gamma^{-2}PB_1B_1'P)V_2 < 0$

where $\begin{pmatrix} C_{1W}' \\ D_{12W}' \end{pmatrix} \begin{pmatrix} C_{1W} & D_{12W} \end{pmatrix}$
$$:= \begin{pmatrix} C_1'C_1 + W & C_1'D_{12} \\ D_{12}'C_1 & D_{12}'D_{12} \end{pmatrix}$$
and V_2 is given by (2).

Proof. First, we note that the existence of a K satisfying Theorem 3(1) is equivalent to the existence of K_ε such that $A + B_2K_\varepsilon C_2$ is stable and such that
$$\|(C_{1,\varepsilon} + D_{12,\varepsilon}K_\varepsilon C_2)\Phi_\varepsilon B_1\|_\infty < \gamma \qquad (7)$$
$$\Phi_\varepsilon := (sI - A - B_2K_\varepsilon C_2)^{-1}$$
where $C_{1,\varepsilon}$ and $D_{12,\varepsilon}$ are defined as in Lemma 1. This is seen by a cheap control argument. Define $C_{1,0} := \begin{pmatrix} C_1 \\ 0 \\ 0 \end{pmatrix}$ and $D_{12,0} := \begin{pmatrix} D_{12} \\ 0 \\ 0 \end{pmatrix}$. Then,
$$\left\|(C_{1,0} + D_{12,0}KC_2)(sI - A - B_2KC_2)^{-1}B_1\right\|_\infty$$
$$= \left\|(C_1 + D_{12}KC_2)(sI - A - B_2KC_2)^{-1}B_1\right\|_\infty$$
Also, it is easy to see that
$$\left\|(C_{1,0} + D_{12,0}KC_2)(sI - A - B_2KC_2)^{-1}B_1\right\|_\infty$$
$$\leq \|(C_{1,\varepsilon} + D_{12,\varepsilon}KC_2)(sI - A - B_2KC_2)^{-1}B_1\|_\infty$$
for all ε proving that (7) implies Theorem (3)(1). Conversely, assume that
$$\left\|(C_{1,0} + D_{12,0}KC_2)(sI - A - B_2KC_2)^{-1}B_1\right\|_\infty$$
$$= \gamma - \delta \ , \ \delta > 0$$
Stability of $A + B_2KC_2$ implies boundedness of $\left\|(sI - A - B_2KC_2)^{-1}B_1\right\|_\infty =: M$. Obviously, ε_1 can be chosen such that
$$\|(C_{1,\varepsilon} + D_{12,\varepsilon}KC_2) - (C_{1,0} + D_{12,0}KC_2)\| < \frac{\delta}{2M}$$
for all $\varepsilon \in (0;\varepsilon_1]$. Hence, for all $\varepsilon \in (0;\varepsilon_1]$,
$$\left\|(C_1 + D_{12}KC_2)(sI - A - B_2KC_2)^{-1}B_1\right\|_\infty$$
$$= \left\|(C_{1,0} + D_{12,0}KC_2)(sI - A - B_2KC_2)^{-1}B_1\right\|_\infty$$
$$= \|(C_{1,\varepsilon} + D_{12,\varepsilon}KC_2)(sI - A - B_2KC_2)^{-1}B_1$$
$$-((C_{1,0} + D_{12,0}KC_2) - (C_{1,\varepsilon} + D_{12,\varepsilon}KC_2))$$
$$\times(sI - A - B_2KC_2)^{-1}B_1\|_\infty \leq \gamma - \delta + \frac{\delta}{2M}M < \gamma$$
Thus, applying Theorem 1 we obtain that Theorem 3(1) is equivalent to the existence of ε_1, $P_\varepsilon \geq 0$, $W_\varepsilon > 0$ and K such that
$$\varepsilon^2 I + A'P_\varepsilon + P_\varepsilon A + C_1'C_1 + \gamma^{-2}P_\varepsilon B_1B_1'P_\varepsilon$$
$$-(P_\varepsilon B_2 + C_1'D_{12})R_\varepsilon^{-1}(P_\varepsilon B_2 + C_1'D_{12})' = -W_\varepsilon \quad (8)$$
$$V_2'(A'P_\varepsilon + P_\varepsilon A + C_1'C_1 + \gamma^{-2}P_\varepsilon B_1B_1'P_\varepsilon)V_2 < 0 \quad (9)$$
where $R_\varepsilon := D_{12}'D_{12} + \varepsilon^2 I$ and V_2 is given by (2), for all $\varepsilon \in (0;\varepsilon_1]$.

In (8) W_ε is bounded from below by 0, but from standard Riccati theory W_ε is also bounded from above. In fact (8) is solvable for all $\varepsilon \in (0;\varepsilon_1]$ if and only if it is solvable also with the right hand side substituted with W, $W_\varepsilon \geq W \geq 0$, $\forall \varepsilon \in (0;\varepsilon_1]$. Now, rewriting (8) as

$$\varepsilon^2 I + A'P_\varepsilon + P_\varepsilon A + C_{1W}'C_{1W} + \gamma^{-2}P_\varepsilon B_1 B_1' P_\varepsilon$$

$$-(P_\varepsilon B_2 + C_{1W}'D_{12W})R_\varepsilon^{-1}(P_\varepsilon B_2 + C_{1W}'D_{12})' = 0$$

where C_{1W} and D_{12W} are defined by

$$\begin{pmatrix} C_{1W}' \\ D_{12W}' \end{pmatrix} \begin{pmatrix} C_{1W} & D_{12W} \end{pmatrix} := \begin{pmatrix} C_1'C_1 + W & C_1'D_{12} \\ D_{12}'C_1 & D_{12}'D_{12} \end{pmatrix}$$

We are now in position to apply Lemma 1 with $F = KC_2$ to obtain that $\lim_{\varepsilon\to 0} P_\varepsilon = P$, where $P \geq 0$ satisfies Theorem 3(2a-2c). Finally, Theorem 3(2d) follows from (9) as ε tends to zero from continuity of the eigenvalues of a matrix as functions of the entries, and sufficiency follows from uniqueness of P. (Actually, the proof given here does not give *strict* inequality, but this can again easily be obtained by perturbation techniques as above.) □

To obtain $W > 0$ and $P \geq 0$ satisfying Theorem 3(2a-2d) one has to iterate on W and γ. For γ sufficiently large and for W sufficiently small there exist a unique $P \geq 0$ satisfying 2a-2c which can be found by solving a reduced order Riccati equation (see [9]). Once such P has been obtained, condition 2d has to be checked. If 2d fails, one has to decrease W and/or decrease γ. This scheme will converge if and only if there exists any stabilizing static controllers at all, which might not be the case even if the system is stabilizable and detectable in the usual sense.

Given such P a static output feedback gain can easily be determined, either through a cheap control method using the approach in [8] or by finding an appropriate state feedback gain and solving a number of linear equations.

Finally, consider instead the system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \\ y &= C_2 x + D_{21} w \end{aligned} \tag{10}$$

By dualizing Theorem 3 we get the following result

Theorem 4 *The following two are equivalent.*

1. *There exists a static output feedback gain K such that $A + B_2KC_2$ is stable and such that*

$$\|C_1\Phi(B_1 + B_2KD_{21})\|_\infty < \gamma$$

$$\Phi = (sI - A - B_2KC_2)^{-1}$$

2. *There exists $P \geq 0$ and $W > 0$ such that the following four hold*

(a) $F_W(P) = \begin{pmatrix} F_{11W}(P) & F_{12W}(P) \\ F_{21W}(P) & F_{22W}(P) \end{pmatrix}$

$$=: \begin{pmatrix} B_P \\ D_P \end{pmatrix} \begin{pmatrix} B_P' & D_P' \end{pmatrix} \geq 0$$

where

$$\begin{aligned} F_{11W}(P) &= PA' + AP + B_{1W}B_{1W}' \\ &\quad +\gamma^{-2}PC_1'C_1P \\ F_{12W}(P) &= PC_2' + B_{1W}D_{21W}' \\ F_{21W}(P) &= C_2P + D_{21W}B_{1W}' \\ F_{22W}(P) &= D_{21W}D_{21W}' \end{aligned}$$

(b) $m_{nr} := \text{rank}\begin{pmatrix} B_P & D_P \end{pmatrix}$

$$= \max_{s\in\mathcal{C}} \text{rank}\,(C_2(sI - A)^{-1}B_{1W} + D_{21W})$$

(c) $\text{rank}\begin{pmatrix} sI - A - \gamma^{-2}PC_1'C_1 & B_P \\ -C_2 & D_P \end{pmatrix}$

$$= n + m_{nr}\ ,\ \forall s \in \bar{\mathcal{C}}^+$$

(d) $\hat{U}_2'(AP + PA' + B_1B_1' + \gamma^{-2}PC_1'C_1P)\hat{U}_2 < 0$

where

$$\begin{pmatrix} B_{1W} \\ D_{12W} \end{pmatrix} \begin{pmatrix} B_{1W} \\ D_{12W} \end{pmatrix} \begin{pmatrix} B_{1W}' & D_{21W}' \end{pmatrix} := \begin{pmatrix} B_1B_1' + W & B_1D_{21}' \\ D_{21}B_1' & D_{21}D_{21}' \end{pmatrix}$$

and U_2 is given by

$$C_2 = \begin{pmatrix} \hat{U}_1 & \hat{U}_2 \end{pmatrix} \begin{pmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{V}_1 & \hat{V}_2 \end{pmatrix}'$$

4 Closing Remarks

Above we have provided necessary and sufficient conditions for the existence of zeroth order $\mathcal{H}_\infty$ controllers. The result was given as simultaneous solvability of a quadratic matrix inequality, two associated rank conditions and a 'geometric' side constraint.

The conditions are constructive in the sense, that the corresponding static output gains are calculated directly in terms of the matrix P found in the necessary and sufficient conditions. The actual algorithms to determine P depend upon solutions to reduced order Riccati equations as described in [9] in combination with an optimization approach.

The main drawback of the suggested approach is probably the involved computational aspects. It is the authors belief, however, that the algorithmic complexity associated with checking the given conditions relate to the complexity of the problem itself rather than to the specific approach taken.

The advantages of the present approach in comparison to a perturbation method based on [8] are improved numerical aspects along with the more direct approach from a theoretical point of view.

Finally, the methods used above and in [8] are conjectured to apply equally well for $\mathcal{H}_2$ or LQG problems and the like, which is a subject for further research.

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