Automatica, Vol. 29, No. 4, pp. 1149–1153, 1993

The $\mathcal{H}_\infty$ Control Problem: A State Space Approach*

Anton A. Stoorvogel

Reviewer: JAKOB STOUSTRUP
Mathematical Institute, Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark.

DURING THE LAST DECADE, much attention has been drawn to $\mathcal{H}_\infty$ control theory especially as an approach to robust compensator design. In the past years a huge number of scientific publications, and among these several monographies, were published on this and related subjects. In the late 1980s there was a breakthrough in $\mathcal{H}_\infty$ control theory, the so-called time domain or state space approach, which gave very elegant results leading to simple design techniques. There has since been a demand for a thorough textbook to describe these new methods in detail.

$\mathcal{H}_\infty$ control theory originated in the early 1980s where the control community had been aware for some time of the poor robustness properties of classical observer-based controller methods and LQG design. This led to the formulation of the *robust stability problem* which was intensely studied in the following years. There were several approaches which lead to solutions of this problem. These were based on frequency domain methods and transfer function descriptions as presented in Francis (1987).

Later on, the significance of $\mathcal{H}_\infty$ control theory to a wide variety of control problems such as for example loop shaping became apparent, since the $\mathcal{H}_\infty$ methods are well suited to treat a rather general class of design problems with frequency domain specifications.

However, the widespread popularity that $\mathcal{H}_\infty$ has attained today is mainly due to a more recent development, namely the time domain or state space methods which were developed in the late 1980s. In this line of research it became evident that solvability of the so-called $\mathcal{H}_\infty$ standard problem (which comprises the robust stability problem and several other problems as special cases) is equivalent to solvability of two algebraic Riccati equations and a coupling condition. Moreover, a complete characterization of the whole class of solutions to the $\mathcal{H}_\infty$ control problem was obtained in closed form. The present book intends to describe the current state of this approach to $\mathcal{H}_\infty$ control.

In the introductory part of the book the deficiencies of classical control with respect to robustness issues are pointed out, and the $\mathcal{H}_\infty$ control problem is introduced. It is shown, however, by means of an example that a solution to the $\mathcal{H}_\infty$ control problem does not necessarily have good stability margins. Hence, it is emphasized that the formulation of the $\mathcal{H}_\infty$ control problem itself does not guarantee robustness. Robustness is obtained only if it designed for! The robustness issue is further addressed as stabilization of uncertain systems and as graph topology convergence, and the mixed sensitivity problem is introduced as an approach to the nominal performance/robust stability problem.

The exposition given in the book requires a number of mathematical prerequisties which are collected in a separate chapter. Among these are: properties of linear continuous or discrete time systems, theory for rational matrices and theory for Hardy and Legesgue spaces. Moreover, to get a deep understanding of the methods applied in the book it is necessary to get acquainted with geometric control theory, which is also introduced.

The first main chapter deals with the regular full information $\mathcal{H}_\infty$ control problem. This is basically the problem of finding $\mathcal{H}_\infty$ state feedback laws. (However, the case where disturbances are known is also treated.) As in most of the literature it is assumed in this chapter that cost is associated with all control signals, i.e. that the direct feedthrough term from controls to output has full column rank or, in short, that the control problem is *regular*. In the regular case the state feedback problem leads to an algebraic Riccati equation. Complete and elegant proofs are given.

After stating the results with the usual regularity assumption, this assumption is dropped and the results are generalized to *singular* or *general* systems, i.e. systems without restrictions on the direct feedthrough terms. In this general case, the $\mathcal{H}_\infty$ problem is *not* directly equivalent to solvability of an algebraic Riccati equation. Instead a certain *quadratic matrix inequality* is introduced, and the main result shows that solvability of the $\mathcal{H}_\infty$ state feedback problem is equivalent to solvability of this quadratic matrix inequality and two associated rank conditions. However, in the sequel of the book it is shown that these three, by virtue of certain results from geometric theory, can be rewritten as one reduced order Riccati equation. Part of the ingredients for this are deferred to appendices and a potential reader should be aware that important material is found there. A design procedure for the state feedback problem based on geometric methods is suggested.

Also, the measurement feedback problem is treated without any assumptions on the direct feedthrough terms (in contrast to most literature on the subject). In the regular case this leads to two algebraic Riccati equations and a coupling condition. In this generalized framework this instead becomes two quadratic matrix inequalities with a coupling condition. The design method in the measurement feedback case consists of a state feedback design and its dual (an observer gain design). In contrast to the regular case, the controller is not given in closed form in terms of the solutions to the quadratic matrix inequalities, so an additional step is needed in the design procedure. Again, geometric methods are suggested, but it is also possible to use more well-known methods such as pole assignment or LQR methods for this last step.

A series of chapters treat: *differential game theory*, which gives a very nice intuition of $\mathcal{H}_\infty$ theory for more advanced readers, *the minimum entropy problem*, which is an approach to simultaneous specifications of $\mathcal{H}_\infty$ robustness and LQG performance, and *the finite horizon $\mathcal{H}_\infty$ control problem*, which deals with $\mathcal{H}_\infty$ criteria associated with time intervals rather than the whole time axis. The latter has some relevance for batch control or adaptive control. A course based on this book can easily leave these chapters out, but it seems reasonable to include this material for completeness.

The continuous time case is thus extensively elaborated. Furthermore, two chapters are devoted to the discrete time $\mathcal{H}_\infty$ control problem where it is assumed that the state feedback subproblem is left invertible and that the estimation subproblem is right invertible. Under these assumptions the

* *The $\mathcal{H}_\infty$ Control Problem: A State Space Approach* by Anton A. Stoorvogel. Prentice Hall, Englewood Cliffs, NJ (1992). ISBN 0-13-388067-2.

$\mathcal{H}_\infty$ problem reduces to two discrete time Riccati equations. The controller design based on the Riccati solution is similar to the continuous time case. After the publication of the book, discrete time $\mathcal{H}_\infty$ control has been studied in much more depth by the author of this book and others. The presented material, though, is self contained and has most of the important results in it.

The book deals with various applications of the developed theory. It shows how to set up $\mathcal{H}_\infty$ design problems in order to handle additive perturbations, multiplicative perturbations and perturbations in the realization of a system. The choice of weight matrices is not explicitly dealt with, but this probably goes beyond the scope of the book.

Finally, the book contains a worked (theoretical) example: an inverted pendulum on a cart. Design criteria are taken to be good tracking of reference signals and robustness with respect to unmodeled dynamics, parameter uncertainty and flexible modes of the pendulum, achieved with a reasonable gain and bandwidth. The design is performed by selection of two weight matrices, which each handle some of the design objectives. The example seems thoroughly worked. It demonstrates well how to set up $\mathcal{H}_\infty$ design problem and indicates what type of results can be expected.

The presented $\mathcal{H}_\infty$ approach diverges slightly from the mainstream literature of this area as indicated above. However, it seems to the reviewer that this diversion pays off well in view of the generality and homogeneity of the exposed results. Moreover, it has recently been reported that when treating singular systems, dedicated singular methods such as the one presented in the present book are by far the most reliable with respect to near optimal numerical behavior.

The back cover of the book (probably written by the publisher) reads "Written such that a broad range of control engineers can read and understand the main results" and "It is suitable for graduate students . . . and for researchers in university and industry both in engineering and mathematics". These remarks might be slightly misleading, since they do not emphasize that the book is mathematically demanding, and that a true understanding of the contents requires some effort even for advanced readers.

In spite of this, however, it is the reviewer's belief that this book will appeal to many researchers in this and related areas, to Ph.D. students with some mathematical experience and to many engineers who want more insight than is gained from using ready software for $\mathcal{H}_\infty$ control design. Albeit most readers will require several readings to understand fully the presented results, this book is excellently written and provides a smooth, comprehensive and mathematically thorough treatment of a difficult but exciting area.

Reference

Francis, B. A. (1987). *A Course in $\mathcal{H}_\infty$ Control Theory.* Springer-Verlag, Berlin.

About the reviewer

Jakob Stoustrup was born in Denmark in 1963. He received a M.Sc. Degree in EE, and a Ph.D. in mathematics both from the Technical University at Denmark. At this university he has been assistant Professor at the Mathematical Institute since 1991. His Ph.D. work concerns robust control, which has also since been a main interest. Recent work concerns $\mathcal{H}_\infty$ control theory, loop transfer recovery (LTR), systems with parametric uncertainties (in particular interval systems) and covariance control. At present he is involved in a project on robust control for finite and infinite dimensional systems, a program on partial and ordinary differential equations and a program on robust and reliable control.

Applied Robotic Analysis*

Robert E. Parkin

Reviewer: PETER KOPACEK
Technische Universität Wien, Institut für Handhabungsgeräte und Robotertechnik, Floragasse 7a, A-1040 Wien, Austria.

SINCE THE APPEARANCE of robots on the manufacturing horizon—approximately 15 years ago—the literature on robotics has grown dramatically. A lot of books, conference proceedings and special journals have been published in these years. These publications can be arranged in two main and three sub-groups for each main group. The main groups are so-called scientific books and non-scientific books. The three sub-groups depend on the authors of the books; they are from the side of mechanics and control, as well as computer sciences.

The book is based on a new approach in robot kinematics called the point-plane method. It was written in connection with a lecture for senior or first year graduate students in engineering, physics, mathematics or computer sciences. The point-plane method is different from standard notations frequently used and based on this the kinematic and kinetic equations as well as the control including path planning are discussed in the book.

The first three chapters have an introductory character and deal with relationships of vectors, points, lines and planes with transformations for translation and rotation, and with joint specification and robot classification. In the first chapter the relationships between points, lines and planes are defined in Cartesian space. In Chapter 2 the rotation of a vector about a point in a plane is discussed. These relationships are used for the classification of the joint system of robot manipulators in Chapter 3. A robot manipulator can be described by a simple open lower-pair kinematic chain of joints and rigid links. Therefore, six kinematically simple joints are defined because all the robots commercially available today can be described with these joints. In Chapter 4 the motion of kinematically simple manipulators is defined in terms of planes of motion. The capability of the robot can be deduced from these planes. The forward kinematic solution of robots is discussed in the next chapter. For robot control it is necessary to analyse kinematically the robot as a base for the controller of the robot. The kinematic analysis starts with the determination of the position of each joint given the configuration of the robot, the base position and orientation, the angles of the revolute joints, and the lengths of the sliding links of the robot. This so-called forward kinematic solution is always possible and can be determined in a closed form. These forward kinematic solutions are discussed for the revolute robot, the cylindrical robot, the polar robot and the SCARA robot, and for so-called unclassified robots. Working envelopes of these basic robots are described. In the next chapter the reverse problem: determination of the joint space of the robot from the position space is considered. This is known as solving the

* *Applied Robotic Analysis* by Robert E. Parkin. Prentice Hall, Englewood Cliffs, NJ (1991). ISBN 0-13-773391-7.