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Modified Structures for Loop Transfer Recovery Design [1]

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Abstract

In this paper, the loop transfer recovery (LTR) of P and PI observer-based controllers with two additional paths, direct output and output error feedback, are analyzed. We highlight the major differences and report new results which complement previous results in LTR theory. In particular, it is shown that proper modification of system structure leads to automatic generation of the aforementioned paths having static gains, and ELTR can be achieved by a modified full-order P observer eventhough CB is not full rank.

1 Introduction

To design a controller for the system Σ: $\{A, B, C\}$ by the LTR design methodology, we first determine a static state feedback, the target design, which satisfies our design specifications. Based on the target (full-state feedback) design gain F for the system Σ, the target loop and sensitivity transfer functions are given by

$$\begin{aligned} L_{TFL}(s) &= F(sI - A)^{-1}B, &(1)\\ S_{TFL}(s) &= (I - L_{TFL}(s))^{-1}, &(2)\end{aligned}$$

respectively. Next the LTR step is performed in which we attempt to recover the target design over a range of frequencies by a dynamic compensator $C(s)$. This step gives a full-loop, sensitivity transfer function of the form

$$S_I(s) = (I - C(s)G(s))^{-1} \tag{3}$$

where $G(s)$ represents the plant transfer function.

As a measure of the quality of the recovery, we define the sensitivity recovery error by

$$E_S(s) = S_{TFL}(s) - S_I(s). \tag{4}$$

This error is related to the so-called recovery matrix $M_I(s)$ given in [7] by the equation $E_S(s) = S_{TFL}(s)M_I(s)$.

There are various approaches available for observer-based LTR design. Once an observer structure is specified all implementations fall into two main categories: those involving structural changes to the basic observer architecture and those not. A separate publication [9] considers the latter class, more specifically LQG/LTR design of PI observers.

Consider the full-order P observer based controller having the transfer function

$$C(s) = F\,(sI - A - KC - BF)^{-1}\,K \tag{5}$$

where F and K are the regulator and observer gains, respectively. Then ELTRI is achieved if and only if $M_I(s) = 0$ where

$$M_I(s) = F(sI - A - KC)^{-1}B. \tag{6}$$

In practice, the condition $M_I(s) = 0$ can not always be satisfied exactly. Consequently, the size of $M_I(s)$ should be made small in some sense.

Let the controller be parameterized in terms of the observer gain by $K(q)$. Then to obtain ALTRI we seek a $K(q)$ such that

$$M_I(s) = F\,(sI - A - K(q)C)^{-1}\,B \to 0 \quad \text{as} \quad q \to \infty. \tag{7}$$

The literature reports a variety of methods to solve the recovery problem [6], [1], [15], [19], [12], [17], [18], [13], [8]. A few approaches consider observer based controllers having structural changes so that either ELTR or ALTR is achieved without large filter or regulator gains. In this paper we concentrate on these approaches and provide new techniques to resolve several problems associated with LTR designs. Consider the closed-loop system comprised of a plant and full-order P observer-based controller as shown in Fig. 1(a). Both closed-loop asymptotic stability and ELTRI can be achieved under the assumptions that (1) $FB = 0$, (2) the plant has all of its infinite zeros of order one (i.e., CB has full rank), and (3) the plant is left invertible and has all of its invariant zeros in the left half s-plane (i.e., the plant is minimum phase).

Since $FB = 0$ severly restricts the design of ELTRI systems, most researchers have focused attention on ALTRI methods. Here one tries to find a gain K which satisfies (7) as we discussed earlier. If the plant is left invertible and minimum phase, it can be shown that there exits such a gain which both achieves ALTRI and guarantees asymptotic stability.

The loss of robustness in observer based systems is due to the path from the control signal u to the observer via the control distribution matrix B as depicted in Fig. 1. Based on this observation Chen *et al.* [4], [5] removed this path at the outset of controller design. This technique leads to a new compensator design philosophy which is outside the realm of observer theory and, hence, the separation principle. Consequently, one must prove that closed-loop stability and LTR are simultaneously achieved. For a plant which is neither minimum phase nor left invertible, Chen *et al.* [4], [5] also established necessary and sufficient conditions for the existence of a recoverable target loop for observer-based and general compensator structures, respectively. These results motivated us to examine several alternative approaches reported in the literature, which will be referred to in the next section, and establish the connection. In this process we report new results which complement the theory.

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2 Modified Structures for P Observers

Within the framework of observer theory, attempts have been made to define alternative structures. An interesting approach which achieves ELTR, under the assumptions (2) and (3) above, is reported in [10], for P observers and in [3] for PI observers whereby an output feedback path having a gain Q, shown by the broken line of Fig. 1(a), is added to the configuration. Unfortunately, this approach leads to a coefficient matching problem and can not be used systematically. Alternatively, one may add an output estimating error feedback loop with gain P as shown by the broken line of Fig. 1(a) [11], [14].

Reference [11] considers the inserted paths with static gains and provide optimization techniques to achieve asymptotic recovery. On the other hand, when the result of reference [14] is applied for the ELTR case and compared with the one in [3], the equivalent effects of both paths are identified.

Let us elaborate on these paths individually. To avoid repetitive notation in our future development, we assume that the stabilizible and detectable system Σ is in the output identifiable form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u, \tag{8}$$

$$y = \begin{bmatrix} I_m & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{9}$$

and has an equal number of inputs and outputs (i.e., $m = r$). It should be pointed out that one can always transform a system to the above form by a similarity transformation.

Recall that a reduced-order P observer-based controller for Σ has the form

$$\Sigma_{RPC}: \quad \begin{cases} \dot{z} = \Phi z + Gy + Hu \\ \hat{x} = Mz + Ny \\ u = F\hat{x} = \bar{M}z + \bar{N}y \end{cases} \tag{10}$$

under the following constraints and associated parameters:

Constraints	Parameters
	$C = \begin{bmatrix} I & 0 \end{bmatrix}$
$\mathrm{Re}[\lambda(\Phi)] < 0$	$T = \begin{bmatrix} -L & I \end{bmatrix}$, $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$
$TA - \Phi T = GC$	$\Phi = A_{22} + LA_{12}$
$H = TB$	$G = A_{21} + LA_{11} - LA_{12}L - A_{22}L$
$MT + NC = I$	$H = B_2 + LB_1$
or	
$\bar{M}T + \bar{N}C = F$	$M = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $N = \begin{bmatrix} I \\ -L \end{bmatrix}$

Lemma 2.1 *The sensitivity recovery error and recovery matrix for the reduced-order P observer-based controller are given by*

$$\begin{aligned} E_{Sr}(s) &= S_{TFL}(s)M_{Ir}(s), &(11)\\ M_{Ir}(s) &= F_2(sI - A_{22} - LA_{12})^{-1}H, &(12)\end{aligned}$$

respectively. Furthermore, ELTRI is achieved if and only if one of the following conditions holds: $E_{Sr}(s) = 0$ or $M_{Ir}(s) = 0$; and ALTRI is obtained if and only if for all $\varepsilon > 0$ there exists a controller $C_\varepsilon(s)$ such that $\|E_{Sr,\varepsilon}(\cdot)\|_H < \varepsilon$ or equivalently $\|M_{Ir,\varepsilon}(\cdot)\|_H < \varepsilon$ where $\|\cdot\|_H$ is the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm.

Theorem 2.1 *Let the system Σ be left invertable, minimum phase and have all of its infinite zeros of order one (i.e., let CB have full rank). Then the reduced-order P observer-based controller Σ_{RPC} achieves both asymptotic stability of the closed-loop system and ELTRI if and only if $M_{Ir}(s) = 0$ or equivantly $H = TB = B_2 + LB_1 = 0$. Furthermore, Σ_{RPC} achieves both asymptotic stability of the closed-loop system and ALTRI iff the system Σ is left invertible and minimum phase.*

The above exact recovery condition implies that the reduced-order P observer-based controller transfer function

$$C_r(s) = F\left(M(sI - \Phi)^{-1}G + N\right) \tag{13}$$

has $n - m$ poles identical to LTR observer poles, which are the $n - m$ transmission zeros of the system Σ. For this reason the structure of the reduced-order P observer is unique. Note that with the previous assumptions on the system Σ, the matrix B_1 is non-singular, and we have

$$L = -B_2B_1^{-1}. \tag{14}$$

Comparing Σ_{RPC} with a full-order P observer-based controller Σ_{PC} represented by

$$\Sigma_{PC}: \quad \begin{cases} \dot{\hat{x}} = A\hat{x} + K(C\hat{x} - y) + Bu \\ u = F\hat{x} \end{cases} \tag{15}$$

yields $\hat{x} = z$, $A + KC = \Phi$, $-K = G$, $B = H$, $F = \bar{M}$, and $T = I$ which shows that the term $\bar{N} = FN$ in Σ_{RPC} distinguishes the two structures. This fact motivates one to mimic the structure of Σ_{RPC} using Σ_{PC} to achieve recovery with respect to the modified structure, hence, justifying the term Q shown by the broken line of Fig. 1(a).

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2.1 Direct Output Feedback

For static Q the transfer function between r and y is given by

$$G_{fsq}(s) = C(sI - A - BF - BQC)^{-1}B, \tag{16}$$

and the return difference matrices at the input and output points are, respectively,

$$\begin{aligned} T_i(s) &= \left(I - F(sI - A - KC)^{-1}B\right)^{-1}\left(I - (F + QC)(sI - A)^{-1}B\right), &(17)\\ T_o(s) &= \left(I - C(sI - A)^{-1}K\right)\left(I - C(sI - A - BF - BQC)^{-1}(K - BQ)\right). &(18)\end{aligned}$$

The resulting closed-loop characteristic equation is the product of $\det(sI - A - KC)$ from the stable observer or Kalman filter and $\det(sI - A - BF - BQC)$. The latter polynomial remains stable despite the size of Q provided that CB has rank m and $\Lambda(s)$ has rank $n + m$ for all s where

$$\Lambda(s) = \begin{bmatrix} sI - A - BF & B \\ C & 0 \end{bmatrix}. \tag{19}$$

Okada *et al.* [11] proposed an optimization technique to determine a Q so that stability and performance robustness requirements are satisfied while ALTRI or ELTRI is realized. However, as in the coefficient matching technique, this technique is not transparent. One usually faces the conflicting goals of recovery level, sensor noise reduction, and convergence of the algorithm. A precompensator may be used to improve the response properties with respect to parameter perturbations and disturbances. The precompensator makes the purturbed closed-loop system between r and y behave similar to that of the optimal, full-state regulator. Given such a precompensator $G_f(s)$, it is required that

$$C(sI - A - BF - BQC)^{-1}G_f(s) = C(sI - A - BF)^{-1}B. \tag{20}$$

One can easily show that

$$G_f(s) = I - QC(sI - A - BF)^{-1}B \tag{21}$$

satisfies this requirement. For arbitrary response characteristics one can use a prefilter or extended perfect model following methods [10]. The drawback of these precompensation methods is the increase in controller dimension.

2.2 Output Error Feedback

To overcome increased controller dimension, one can alternatively add an output estimating error feedback loop with gain P as shown by the broken line of Fig. 1(a). In this case the transfer function between r and y given by

$$G_{fsp}(s) = C(sI - A - BF)^{-1}B \tag{22}$$

matches the full-state feedback implementation and is indepenent of P. The need for a precompensator is therefore avoided, and the resulting system is termed a model matching system. The possibility of obtaining recovery at both the plant input and/or output makes this method advantageous; although, it is generally difficult to realize this goal with a fixed gain P, and one is required to use a dynamic gain matrix $P(s)$. To see this, the return difference matrices at the input and output points are given by

$$\begin{aligned} T_i(s) &= \left(I - (F - P_iC)(sI - A - KC)^{-1}B\right)^{-1}\left(I - F(sI - A)^{-1}B\right), &(23)\\ T_o(s) &= \left(I - C(sI - A)^{-1}K\right)\left(I - C(sI - A - BF)^{-1}(K - BP_o)\right)^{-1}, &(24)\end{aligned}$$

respectively, and we have the following result.

Theorem 2.2 *Let the system Σ be left invertible and minimum phase. Then the modified structure comprised of observer-based controller Σ_{PC} and the additional path P achieves recovery at the input or output if*

$$\begin{aligned} P_i &= N_iD_i^{-1}, &(25)\\ P_o &= D_o^{-1}N_o, &(26)\end{aligned}$$

respectively, where $N_i = F(sI - A - KC)^{-1}B$, $D_i = C(sI - A - KC)^{-1}B$, $N_o = C(sI - A - BF)^{-1}K$, $D_o = C(sI - A - BF)^{-1}B$. Furthermore, the recovery conditions for static gains become $F = P_iC$ and $K = BP_o$.

Thus, the problem of finding P_i or P_o reduces to the realization of fractional representation (25) or (26), respectively. For a static gain, recovery can be achieved by the optimization technique of [11]; however, one is faced with the conflicting goals of closed-loop stability, degree of recovery, and convergence of the recovery design algorithm. Although this technique avoids increase of controller dimension, it has the drawback of robustness degradation.

For a common static gain $P_i = P_o = P$, recovery can be achieved at both plant input and output by using a result which ties this method to LQG design. A future publication will elaborate on this result.

For a dynamic gain, assuming that exact recovery can not be obtained, we parameterise $P(s)$ and analyze the degree of recovery based on the following result borrowed from [7]. To be consistent with the notation introduced in this paper we call the observer a P-parametrized observer.

Lemma 2.2 *Assume that $P(s) \in \mathcal{RH}_\infty$ is given by the state-space representation*

$$\Sigma_P: \quad \left\{ \begin{aligned} \dot{x}_p &= A_px_p + B_pu_p \\ y_p &= C_px_p + D_pu_p \end{aligned} \right. . \tag{27}$$

Then the corresponding P observer-based controller has the following parameters:

$$\begin{aligned} D &= \begin{bmatrix} A + KC & 0 \\ B_pC & A_p \end{bmatrix}, \quad G = \begin{bmatrix} -K \\ -B_p \end{bmatrix}, \quad H = \begin{bmatrix} B \\ 0 \end{bmatrix}, \\ \tilde{M} &= \begin{bmatrix} F + D_pC & C_p \end{bmatrix}, \quad \tilde{N} = -D_p, \qquad T = \begin{bmatrix} I \\ 0 \end{bmatrix}; \end{aligned}$$

and the corresponding recovery matrix becomes

$$M_{fp}(s) = F(sI - A - KC)^{-1}B + P(s)C(sI - A - KC)^{-1}B. \tag{28}$$

Now the minimization of $M_{fp}(s)$ has the form of a standard $\mathcal{H}_\infty$ model-matching problem:

$$\|T_1 + T_2PT_3\|_\infty < \gamma \tag{29}$$

with $T_1 = F(sI - A - KC)^{-1}B$ representing the model, $T_2 = I$, and $T_3 = C(sI - A - KC)^{-1}B$. Since T_1 is strictly proper, the $\mathcal{H}_\infty$/LTR formulated here involves the so-called singular problem in $\mathcal{H}_\infty$ control theory considered by Stoorvogel [16]. For a detailed treatment of $\mathcal{H}_\infty$/LTR refer to [17] and [18]. In order to explore the basic idea, the following theorem is instrumental.

Theorem 2.3 *There exists an internally stabilizing compensator $P(s)$ which satisfies (29) if and only if there exists a $P \geq 0$ satisfying*

$$\begin{aligned} Z_\gamma(P) &= \tilde{B}\tilde{B}^T \geq 0, &(30)\\ CP &= 0, &(31)\\ \rho(\tilde{B}) &= \rho(B), &(32)\end{aligned}$$

as well as the system

$$\left\{A_k + \gamma^{-2}PF^TF, \tilde{B}, C, 0\right\} \tag{33}$$

being a minimum phase system where $A_k = A + KC$, $Z_\gamma(P) = A_kP + PA_k^T + BB^T + \gamma^{-2}PF^TFP$. Furthermore, a feasible choice for the controller is

$$P(s) = F(sI - A_k - GC)^{-1}G \tag{34}$$

where G is any matrix satisfying

$$\left\|F(sI - A_k - \gamma^{-2}PF^TF - GC)^{-1}\tilde{B}\right\|_\infty < \gamma. \tag{35}$$

For minimum phase systems, Eqns. (30)–(33) are satisfied for $P = 0$ for all γ, and for non-minimum phase systems γ has to be chosen sufficiently large. Consequently, it can be shown that the controller

$$C(s) = F(sI - A_k - GC + BF)^{-1}(K - G) \tag{36}$$

when applied to the original plant $\{A, B, C\}$ makes the norm of the recovery matrix smaller than γ. Important conclusions have been made in [18] concerning the performance of $\mathcal{H}_\infty/LTR$ as compared to traditional LTR methods and controllers found from pure $\mathcal{H}_\infty$ design. The main advantage of $\mathcal{H}_\infty/LTR$ design is that non-minimum phase systems can be treated by exactly the same technique as minimum phase systems. However, a major concern in $\mathcal{H}_\infty/LTR$ design is the high dimensionality of the controller.

To establish a link between $P(s)$ and $Q(s)$, we remove $P(s)$ from Fig. 1(a) and define a new structure with dynamic state and output feedback transfer functions $F(s)$ and $Q(s)$, respectively. The following result is an immediate consequence of Theorem 2.3.

Corollary 2.1 *Let $Q(s) = P(s)$ be the internally stabilizing compensator defined in Theorem 2.3. Then*

$$F(s) = F - P(s)C \tag{37}$$

realizes the same degree of recovery as in Theorem 2.3.

To recover robustness by keeping F constant and allowing Q to be dynamic leads to complicated pole placement procedures. An initial attempt to parametrize $Q(s)$ and apply $\mathcal{H}_\infty/LTR$ techniques encounters similar difficulties. In any case, the use of dynamic $Q(s)$ or $P(s)$ is not consistent with the objective to reduce the dimension of the feedback system. This problem could be avoided by choosing constant gains, however, it is generally difficult to find such g as we discussed in our analysis. This difficulty may be overcome by taking a slightly different approach, namely, to modify the system structure. We will show that such a modification manifests itself into a systematic way of designing the aforementioned paths. Before persuing this approach we continue our analysis for the case of a PI observer and see whether or not we get any benefits by adding extra paths as compared to the P observer case.

3 Modified Structures for PI Observers

The full-order PI observer based controller is defined originally in [2], and its more elegant modified version, which allows one to derive systematic design methods, is defined in [9] as

$$\Sigma_{PIC}: \quad \left\{ \begin{aligned} \dot{\hat{x}} &= A\hat{x} + K_P(C\hat{x} - y) + Bu + Bv \\ \dot{v} &= K_I(C\hat{x} - y) \\ u &= F\hat{x} \end{aligned} \right. . \tag{38}$$

Fig. 1(b) shows a full-order PI observer. It is clear that when $K_I = 0$, $K_P = K$ we have a conventional P observer. The system Σ_{PIC} can be represented as an extended state system:

$$\Sigma_{PIC}: \quad \left\{ \begin{aligned} \dot{z} &= A_zz + K_z(C_zz - y) + B_zu \\ u &= F_zz \end{aligned} \right. \tag{39}$$

where where the parameters are easily specified by (38). This augmentation allows methods such as LQG, eigenstructure assignment, etc. to be applied as in ordinary observer design to determine the gain K_z.

The LTR design of PI observers is studied extensively in [9]. The main advantage of PI observers is that they achieve ELTR as time tends to infinity, termed as time recovery. Another advantage over the usual full-order P observer is the need for relatively low oberserver gains. Reference [9] arrived at the conclusion that there will be no difference between full-order P and PI observers in normal asymptotic recovery (frequency recovery). In fact it was shown that the time recovery effect (integral effect) disappears in the LQG/LTR design of PI observers as q tends to infinity. However, asymptotic recovery will in general result in high observer gains. This limitation of the full-order P observer makes the PI observer interesting from a time recovery point of view. To obtain time recovery we do not necessarily need high gains. It is also important to point out that the reduced-order PI observer-based controller Σ_{RPIC} can be defined similar to the reduced-order P observer based controller Σ_{RPC} using the structure defined for PI observers.

In the following discussion by dropping the subscript "C" in Σ_{PC}, Σ_{PIC}, Σ_{RPC}, and Σ_{RPIC} we obviously mean the corresponding observer structures. Now let A1 and A2 (A1$'$ and A2$'$) denote the usual assumptions on the system Σ for ELTR (ALTR) based on full-order and reduced-order P observers, respectively. Then we may summarize one of our important results from [9] in a compact form as follows.

Theorem 3.1 *Let the system Σ satisfy A1 or A2 (A1$'$ or A2$'$). Then Σ_{PI} or Σ_{RPI} achieves ELTR (ALTR) if and only if its corresponding Σ_P or Σ_{RP} achieves ELTR (ALTR).*

Recall that obtaining ELTR ties to the inherent presence of an ouput feedback in Σ_{RP}. To achieve ELTR with Σ_P under the same assumptions made for ELTR with Σ_{RP} we need to add the extra path P or Q. However, Theorem 3.1 confirms that the effects of inserting these paths in Σ_{PI} would be equivalent to their insertion in Σ_P. Thus, we do not further discuss the modified structures for PI observers.

4 Modified System Structures for LTR

The previous section concludes that it is sufficient to work with P observers when inserting the P or Q path. The drawbacks of directly inserting these paths were discussed in Section 2. In this section we modify the system structure and show how it leads to the automatic generation of these paths in static form in order to avoid the increase in controller dimension. In particular, we concentrate on the case of direct output feedback path Q.

4.1 Exact Recovery with Full-Order Observers ($\rho(CB) = m$)

In the derivation of reduced-order P observer Σ_{RP}, the output derivative appearing in the initial observer equation plays an important role in achieving ELTR as stated in Theorem 2.1. The output derivative can be avoided by a variable transformation in the final observer equation. Here, our initial goal is to realize Q by modifying the structure of Σ in order to achieve ELTR with Σ_P under the same assumption on Σ required to achieve ELTR with Σ_{RP}. The idea is to modify the output equation by taking the derivative of the output and mimic the structure of Σ_{RP}. this leads to a modified full-order P observer structure $\tilde{\Sigma}_P$. The new output is generated by one of the following two choices:

$$\text{Choice (1)} \qquad \bar{y} = y + Q_1\dot{y},$$
$$\text{Choice (2)} \qquad \bar{y} = \begin{bmatrix} y \\ Q_2\dot{y} \end{bmatrix}.$$

Choice (1) Using the output identifiable form for the system Σ, we define our first modified system structure as

$$\Sigma_1: \quad \begin{cases} \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ \bar{y} = \bar{C}x + \bar{D}u \end{cases} \tag{40}$$

where

$$\bar{C} = \begin{bmatrix} I + Q_1A_{11} & Q_1A_{12} \end{bmatrix}, \quad \bar{D} = Q_1B_1.$$

It is not difficult to show that (a) the poles and transmission zeros of Σ_1 and Σ are the same and (b) there exists Q_1 such that $\bar{C}B$ has full rank and Σ_1 is both stabilizable and detectable.

The parameter Q_1 plays an imortant role in our development as it can be seen from the following simple system. Let

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Then by using $\bar{y} = y + \dot{y}$ *(without Q_1) the new output vector becomes* $\bar{C} = [0 \ 1]$ and $\bar{C}B$ is not full rank despite the fact that the original CB is full rank. However, for any $Q_1 \notin \{(-3 \pm \sqrt{13})/2, 1\}$ the system remains detectable and $\bar{C}B$ is full rank (non-singular). Proper selection of Q_1 can also be useful in recovery design.

Now, the modified full-order P observer-based controller $\tilde{\Sigma}_{PC1}$ can be derived for Σ_1 using the standard procedure which leads to

$$\tilde{\Sigma}_{PC1}: \quad \begin{cases} \dot{z} = \tilde{\Phi}z + \tilde{G}y + \tilde{H}u \\ u = F\hat{x} = \tilde{M}z + \tilde{N}y \end{cases} \tag{41}$$

where $z = \hat{x} + KQ_1y$, and

$$\tilde{\Phi} = A + K\bar{C}, \quad \tilde{G} = -(A + K\bar{C})KQ_1 - K, \quad \tilde{H} = B + K\bar{D},$$
$$\tilde{M} = F, \qquad \tilde{N} = -FKQ_1.$$

The corresponding recovery matrix is given by

$$M_{ff}(s) = F(sI - \tilde{\Phi})^{-1}\tilde{H}. \tag{42}$$

Theorem 4.1 *Let the system Σ be left invertible, minimum phase and have all of its infinite zeros of order one (i.e., let CB have full rank). Then the modified full-order P observer-based controller $\tilde{\Sigma}_{PC1}$ achieves both asymptotic stability of the closed-loop system and ELTRI if and only if $M_{ff}(s) = 0$ or equivalently $\tilde{H} = B + K\bar{D} = 0$. Moreover, a constructive method of obtaining $\tilde{\Sigma}_{PC1}$ to achieve both ELTRI and asymptotic stability of the closed-loop system can be given.*

Note that the condition $FB = 0$, imposed on LTR of regular full-order P observer-based controllers, is not necessary here. Note also that subsitution of $K = -B(Q_1B_1)$ in $\tilde{N}$ defines our static output feedback gain $Q = \tilde{N} = FBB_1^{-1}$ which is independent of the choice Q_1. Furthermore, the transfer function of the full-order recovery controller is given by

$$C_f(s) = F(sI - A - K\bar{C})^{-1}\tilde{G} + \tilde{N} \tag{43}$$

Example 4.1 Consider the system

$$\dot{x} = \begin{bmatrix} -1.5 & -0.5 \\ -1.5 & -2.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \tag{44}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{45}$$

with the desired target loop transfer function $L_{TFL}(s)$ realized by the full-state feedback regulator

$$u = Fx = \begin{bmatrix} -50 & -10 \end{bmatrix} x. \tag{46}$$

Using our systematic procedure we find that the pair $\{Q_1, K\} = \{2, [-0.5 \ -0.5]^T\}$ specifies the modified output as

$$\bar{y} = \begin{bmatrix} -2 & -1 \end{bmatrix} x + 2u \tag{47}$$

and the modified full-order P observer-based controller as

$$\dot{z} = \begin{bmatrix} -0.5 & 0 \\ -0.5 & -2 \end{bmatrix} z + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y, \tag{48}$$
$$u = \begin{bmatrix} -50 & -10 \end{bmatrix} z - 60y. \tag{49}$$

It is clear that although the very restrictive condition $FB = 0$ is not satisfied, we achieve exact recovery with the above full-order observer.

Choice (2) Using the output identifiable form for the system Σ, we define our second modified system structure as

$$\Sigma_2: \quad \begin{cases} \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ \bar{y} = \bar{C}x + \bar{D}u \end{cases} \tag{50}$$

where

$$\bar{C} = \begin{bmatrix} I & 0 \\ Q_2A_{11} & Q_2A_{12} \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 \\ Q_2B_1 \end{bmatrix}.$$

The corresponding modified full-order P observer-based controller $\tilde{\Sigma}_{PC2}$ can be derived in a similar fashion and is given by

$$\tilde{\Sigma}_{PC2}: \quad \begin{cases} \dot{z} = \tilde{\Phi}z + \tilde{G}y + \tilde{H}u \\ u = F\hat{x} = \tilde{M}z + \tilde{N}y \end{cases} \tag{51}$$

where $z = \hat{x} + K_2Q_2y$ with K_2 defined by proper partitioning of $K = [K_1 \ K_2]$ according to $\bar{y}$, and

$$\tilde{\Phi} = A + K\bar{C}, \quad \tilde{G} = -(A + K\bar{C})KQ_2 - K_1, \quad \tilde{H} = B + K\bar{D},$$
$$\tilde{M} = F, \qquad \tilde{N} = -FK_2Q_2.$$

An exactly analogous theorem can be stated for this case. However, to avoid repetitive material, we provide a constructive method for obtaining $\tilde{\Sigma}_{PC2}$. In particular we show that asymptotic stability of the closed-loop system and ELTRI can be treated separately. To establish this we rewrite $\tilde{\Phi}$ as

$$\tilde{\Phi} = \begin{bmatrix} \tilde{A}_1 + K_1 & \tilde{A}_2 \end{bmatrix} \tag{52}$$

where $\tilde{A}_1 = A_1 + K_2Q_2A_{11}$ and $\tilde{A}_2 = A_2 + K_2A_2A_{12}$.

Theorem 4.2 *The pair $\{A, \bar{C}\}$ is detectable if and only if the pair $\{\tilde{A}, \tilde{C}\}$ is detectable, where $\tilde{A} = [\tilde{A}_1 \ \tilde{A}_2]$ and $\tilde{C} = [I \ 0]$.*

Consequently there exists a matrix K_1, such that the composite matrix $\tilde{\Phi}$ has a prescribed set of eigenvalues. The following two-step procedure achieves asymptotic stability of the closed-loop system and ELTRI independently:

1. Obtain K_2 from the ELTRI condition $\tilde{H} = B + K\bar{D} = 0$ as $K_2 = -B(Q_2B_1)^{-1}$.

2. Obtain K_1 such that $\tilde{\Phi}$ has a prescribed set of eigenvalues, a subset of which consists of transmission zeros of Σ.

Note that ELTRO can be handled by duality.

Example 4.2 Let us consider the same system given in Ex. 4.1. Applying Step 1 we get $K_2 = [-1, -1]^T$, $Q_2 = 1$; and applying Step 2 we get

$$\tilde{A}_1 = A_1 + K_2Q_2A_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \tag{53}$$
$$\tilde{A}_2 = A_2 + K_2Q_2A_{12} = \begin{bmatrix} 0 & -2 \end{bmatrix}^T \tag{54}$$

which specifies $\{\tilde{A}, \tilde{C}\}$ in $\tilde{\Phi} = \tilde{A} + K_1\tilde{C}$ with

$$\tilde{A} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Since the pair $\{\tilde{A}, \tilde{C}\}$ is detectable one of the eigenvalues of $\tilde{\Phi}$ can be assigned arbitrarily at $\lambda = K_1 \in C^-$, and the other is the transmission zero at -2.

4.2 Exact Recovery with Full-Order Observers ($\rho(CB) < m$)

In this section we take advantage of the results of Section 4.1 to treat the difficult case of $\rho(CB) < m$. A double look at duality (recall Luenberger) reveals that one needs also to take the derivative of the input and construct $\bar{u}$ as in Choice (1) or Choice (2). Here for the sake of brevity we consider Choice (2) with

$$\bar{y} = \begin{bmatrix} y \\ Q_2\dot{y} \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} u \\ R_2\dot{u} \end{bmatrix}$$

and define the modified system as

$$\Sigma_M: \quad \begin{cases} \dot{x} = Ax + \bar{B}\bar{u} \\ \bar{y} = \bar{C}x \end{cases} \tag{55}$$

Note that we avoid the appearance of a $\bar{D}$ term in Σ_M. Since $\rho(CB) < m$, the parameters Q_2 and R_2 can be selected in such a way that $Q_2CB = 0$ and $CBR_2 = 0$. Thus, the modified system Σ_M can be represented by the triple $\{A, \bar{B}, \bar{C}\}$, where

$$\bar{C} = \begin{bmatrix} C \\ Q_2CA \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B & ABR_2 \end{bmatrix}.$$

Again it is easy to show that (a) the poles and transmission zeros of Σ_M and Σ are the same and (b) Σ_M is stabilizable and detectable.

For the special case of $CB = 0$, we choose $Q_2 = R_2 = I$. In the following we assume that $\rho(CB) < m$ while $\rho(CAB) = m$, which means that $\rho(\bar{C}\bar{B}) = m$. The modified full-order P observer-based controller $\bar{\Sigma}_{PCM}$ is given by

$$\bar{\Sigma}_{PCM}: \quad \left\{ \begin{array}{rcl} \dot{z} & = & \bar{\Phi} z + \bar{G} y + \bar{H} u \\ u & = & F\hat{x} = \bar{M} z + \bar{N} y \end{array} \right. \tag{56}$$

where $z = \hat{x} + K_2 Q_2 y - \bar{B}_2 R_2 u$ with K_2 and $\bar{B}_2$ defined by proper partitioning of $K = [K_1\ K_2]$ and $\bar{B} = [\bar{B}_1\ \bar{B}_2]$ according to $\bar{y}$ and $\bar{u}$, and

$$\begin{array}{rclrclrcl} \bar{\Phi} & = & A + K\bar{C}, & \bar{G} & = & -\bar{\Phi} K_2 Q A_2 - K_1, & \bar{H} & = & \bar{\Phi}\bar{B}_2 + R_2 + \bar{B}_1, \\ \bar{M} & = & \left(I - F\bar{B}_2 R_2\right)^{-1} F, & \bar{N} & = & \left(I - F\bar{B}_2 R_2\right)^{-1} F K_2 Q_2. \end{array}$$

provided that $\Gamma = I - F\bar{B}_2 R_2$ is non-singular.

Lemma 4.1 *The matrix $\Gamma = I - F\bar{B}_2 R_2$ is non-singular if and only if $\lambda_i\left(F\bar{B}_2 R_2\right) \neq 0$ $\forall i = 1, 2, \ldots, m$.*

It is interesting to point out the appearance of the term FAB in Γ. If $FAB = 0$ then the non-singularity of Γ is guaranteed and $\bar{M} = F$, $\bar{N} = -FK_2Q_2$. This establishes a connection between the restrictive condition $FB = 0$ in ELTRI of full-order P observers and the non-singularity condition of Γ in ELTRI of modified full-order P observers considered here. The following theorem summarizes the above result.

Theorem 4.3 *Let the system Σ be left invertible and minimum phase. Then the modified full-order P observer-based controller $\bar{\Sigma}_{PCM}$ achieves both asymptotic stability of the closed-loop system and ELTRI using a constructive method if and only if $\bar{H} = (A + K\bar{C})\,\bar{B}_2 R_2 + \bar{B}_1 = 0$ and* $\det\Gamma \neq 0$.

Example 4.3 Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \tag{57}$$

$$y = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} x \tag{58}$$

with the target feedback loop realized by $F = [-64.965\ \ -39.736\ \ -4.7458]$. Since $CB = 0$, we have $Q_2 = R_2 = I$ and the pair $\{\bar{C}, \bar{B}\}$ is given by

$$\bar{C} = \begin{bmatrix} 4 & 1 & 4 \\ 0 & 4 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & -6 \end{bmatrix}.$$

Following our constructive procedure we obtain

$$K = \begin{bmatrix} -1 & 0 \\ 6 & 0 \\ -28 & -1 \end{bmatrix} \tag{59}$$

and the modified full-order P observer-based controller $\bar{\Sigma}_{PCM}$ is specified by

$$\begin{array}{rclrcl} \bar{\Phi} & = & \begin{bmatrix} -4 & 0 & 0 \\ 24 & 6 & 1 \\ -118 & -43 & -7 \end{bmatrix}, & \bar{G} & = & \begin{bmatrix} 1 \\ -5 \\ 21 \end{bmatrix}, \\ \bar{M} & = & \begin{bmatrix} -5.2984 & -3.2408 & -0.38706 \end{bmatrix}, & \bar{N} & = & -0.38706 \end{array}$$

which achieves ELTRI. Note that a reduced-order P observer can also be designed for the system $\{A, \bar{B}, \bar{C}\}$ to achieve ELTRI. This observer is given by

$$\dot{z} = -4z + \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{y}, \tag{60}$$

$$\hat{x} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -4 & 1 \end{bmatrix} \bar{y}; \tag{61}$$

however, the derivative of the output is required to implement the controller.

5 Conclusion

This paper considered structural changes to the basic observer architecture to facilitate LTR. Both P and PI observers with additional direct output and output error feedback loops were analyzed. The advantages and drawbacks of these paths with static and dynamic gains were discussed. In this process new results have also been reported. We provided constructive methods of designing modified full-order P observers to achieve both asymptotic stability of the closed-loop system and ELTR without imposing the restrictive condition of $FB = 0$. These observers mimic the structure of reduced-order observers and realize ELTR for CB having full and non-full rank. The desirable attributes required in LTR such as small gain and low controller dimension are also fulfilled. The results of this paper can be used for non-minimum phase systems as well. However, necessary and sufficient condition to achieve ALTR should be stated. For ALTR we intend to compare the performance of our modified structures with $\mathcal{H}_\infty/LTR$ and other existing LTR techniques.

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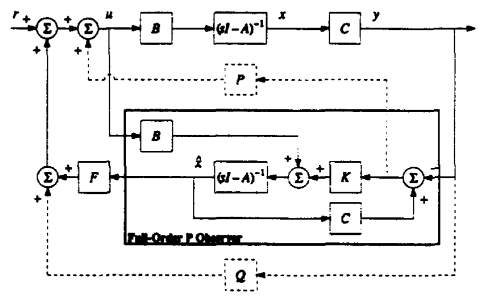
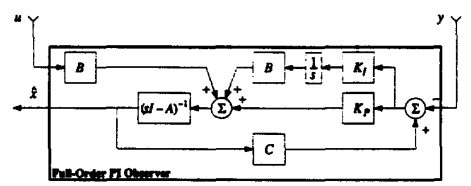


(a)



(b)

Figure 1: Modified structures based on (a) full-order P observer and (b) full-order PI observer

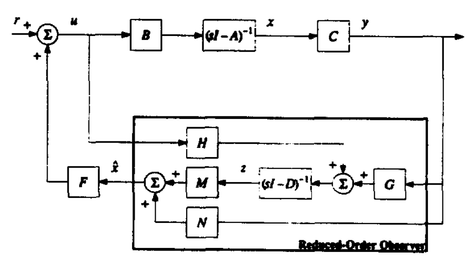


Figure 2: Reduced-order observer-based controller