

Title of paper:

**REGULAR vs. SINGULAR METHODS IN CHEAP \mathcal{H}_∞ CONTROL:
A NUMERICAL STUDY.**

Authors:

Hans Henrik Niemann and Jakob Stoustrup

Mathematical Institute, Technical University of Denmark, DK-2800 Lyngby, Denmark.

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Abstract.

The numerical aspects of applying cheap control in connection with \mathcal{H}_∞ state feedback will be considered. It will be shown that the cheap control principle, applied to systems which does not satisfy the conditions for applying the regular \mathcal{H}_∞ approach, is potentially numerically unstable. Instead, the singular \mathcal{H}_∞ approach can be applied, which has a better numerical behavior in the calculation of the state feedback gain in comparison to the cheap control method. A reliable robust algorithm for solving Quadratic Matrix Inequalities, which are central in the singular \mathcal{H}_∞ approach, is presented. Further, the numerical performance of this algorithm is evaluated for some examples and compared to the cheap control solutions. Also, it is demonstrated for the examples that the singular approach leads to smaller gains.

Keywords:

Regular and Singular \mathcal{H}_∞ Theory, Cheap Control, Numerical Stability, High Gains.

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1. INTRODUCTION.

The cheap control method has been an attractive and popular way to apply the regular \mathcal{H}_∞ approach on systems which not fulfill the regularity assumption, [Petersen, 1987]. Applying the cheap control method to a singular system, i.e. the direct feedthrough terms has not maximal rank, the two Riccati equations from the regular \mathcal{H}_∞ approach, [Doyle et al., 1989], will be recovered in a simple way. The cheap control principle is based on perturbation of the real system such that the regularity conditions in the regular \mathcal{H}_∞ approach are satisfied. From a theoretical point of view, the cheap control technique is convergent in the sense that it is always possible to select a perturbation such that the \mathcal{H}_∞ design condition for the real system can be satisfied, [Stoorvogel, 1990, Scherer, 1990, Stoustrup and Niemann, 1992].

As an alternative to the cheap control method, we have the singular \mathcal{H}_∞ approach introduced by Stoorvogel and Trentelman [1990] and Stoorvogel [1992]. The singular approach directly handles systems where the direct feedthrough terms does not have maximal rank. The singular \mathcal{H}_∞ approach is somewhat more involved than the regular approach and consists of three steps. First, the unique solution to a certain Quadratic Matrix Inequality is derived followed by a transformation of the system to a minimum phase system. The transformation is based on the solutions to the Quadratic Matrix Inequality. In the last step a state feedback gain is determined by solving a \mathcal{H}_∞ Almost Disturbance Decoupling Problem (ADDP). If the direct feedthrough term has maximal rank, the Quadratic Matrix Inequality becomes equivalent with the associated Riccati equation in the regular case. Further, the state feedback gain from the regular case solves the Disturbance Decoupling Problem exactly, [Stoorvogel, 1992].

The key contribution of this paper is to show that we must be careful by using methods derived for regular systems on singular systems, albeit it is theoretical wellfounded. The reason is that numerical problems devastates the calculation of the cheap controllers as it will appears in this paper.

This paper is organised as follows. In section 2, the regular \mathcal{H}_∞ state feedback, the singular \mathcal{H}_∞ state feedback and the cheap control methods are shortly described. A simple 4th order example and a more complicated 8th order example are considered in section 3, where the cheap control and the singular \mathcal{H}_∞ approach are applied. The two methods are compared in several ways to get a clear overview of advantages and disadvantages with the methods. A discussion of the results is given in section 4.

2. \mathcal{H}_∞ STATE FEEDBACK DESIGN.

In the following \mathcal{H}_∞ state feedback design will be considered in the regular approach, and the singular approach. Further, convergence of the Riccati solutions in cheap control will be investigated.

At first, let's define the \mathcal{H}_∞ state feedback design problem.
Consider a FDLTI system:

$$\Sigma: \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}, & \mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m, \mathbf{w} \in \mathbb{R}^l \\ \mathbf{z} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, & \mathbf{z} \in \mathbb{R}^p \end{cases} \quad (1)$$

The \mathcal{H}_∞ state feedback design problem is defined as follows:

We assume that $\gamma > 0$ has been given. We wish to design, if possible, an internally stabilizing state feedback controller $\mathbf{u} = \mathbf{F}\mathbf{x}$ such that the \mathcal{H}_∞ norm of the resulting closed-loop transfer function from \mathbf{w} to \mathbf{z} is smaller than γ .

We also need an assumption on the systems invariant zeros:

Assumption 2.1.

It is assumed that the system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ has no invariant zeros in \mathbb{C}^0 .

2.1 The Regular \mathcal{H}_∞ Approach.

The regular approach taken will be based on the results in [Doyle et.al., 1989]. The regular case is based on matrix Riccati equations. In the regular case, we require that \mathbf{D} is injective. With \mathbf{D} injective, we have the following result for \mathcal{H}_∞ state feedback control [Doyle et.al., 1989]:

Lemma 2.2.

Consider a system Σ which satisfies Assumption 2.1, (\mathbf{A}, \mathbf{B}) and (\mathbf{A}, \mathbf{E}) are stabilizable and (\mathbf{C}, \mathbf{A}) is detectable. Let $\gamma > 0$ be given. Then, there exists a state feedback controller $\mathbf{u} = \mathbf{F}\mathbf{x}$ for which the resulting closed loop system is internally stable, and for which the transfer function from \mathbf{w} to \mathbf{z} has \mathcal{H}_∞ norm smaller than γ , if and only if there exist $\mathbf{P} \geq 0$ for which:

$$\mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} + \mathbf{C}^T \mathbf{C} + \gamma^{-2} \mathbf{X} \mathbf{E} \mathbf{E}^T \mathbf{X} - (\mathbf{X} \mathbf{B} + \mathbf{C}^T \mathbf{D})(\mathbf{D}^T \mathbf{D})^{-1} (\mathbf{B}^T \mathbf{X} + \mathbf{D}^T \mathbf{C}) = 0 \quad (2)$$

and

$$\Lambda(\mathbf{A} + \gamma^{-2} \mathbf{E} \mathbf{E}^T \mathbf{X} - \mathbf{B}(\mathbf{D}^T \mathbf{D})^{-1} (\mathbf{B}^T \mathbf{X} + \mathbf{D}^T \mathbf{C})) \in \mathbb{C}^- \quad (3)$$

Moreover, a state feedback gain satisfies the \mathcal{H}_∞ norm condition is given by:

$$\mathbf{F} = (\mathbf{D}^T \mathbf{D})^{-1} (\mathbf{B}^T \mathbf{X} + \mathbf{D}^T \mathbf{C}) \quad (4)$$

In this approach a singular problem, i.e. \mathbf{D} is injective, can also be treated by cheap control

techniques, where the system, Σ , is approximated by a series of systems, $\Sigma_\epsilon \rightarrow \Sigma$ as $\epsilon \rightarrow 0$, which all have full rank direct feedthrough terms. Then the solution to the original problem can be approximated by selecting ϵ sufficiently small in the perturbed problem. This approach, however, suffers from several drawbacks. From a computation point of view, we are increasing the number of optimization variable by one, since we in general have to iterate on ϵ . From a theoretical point of view, we gain much more insight by directly taking the singularities into account in our formalism, rather than considering perturbed plants. Finally, as we shall see in the sequel there are several numerical problems caused by the following fact: ϵ has to be very small for near-optimal solutions, so we are adding small numbers to large numbers, giving rise to bad condition numbers. The cheap control approach will be described in section 2.3.

2.2 The Singular \mathcal{H}_∞ Approach.

The singular \mathcal{H}_∞ approach taken will be based on the results in [Stoorvogel, 1992], which is a very general approach, and in particular it does not impose any assumption on the direct feedthrough term of the standard problem (the four block problem), as it has been assumed in the regular approach described above.

Lemma 2.3.

Consider a system Σ which satisfies Assumption 2.1. Let $\gamma > 0$ be given. Then, there exists a state feedback controller $u = Fx$ for which the resulting closed loop system is internally stable, and for which the transfer function from w to z has \mathcal{H}_∞ norm smaller than γ , if and only if there exist $P \geq 0$ for which:

$$(1) \quad F(\gamma, P) = \begin{bmatrix} C_P^T \\ D_P^T \end{bmatrix} \times [C_P \quad D_P] \geq 0$$

$$(2) \quad \text{rank } F(\gamma, P) = \text{normrank } G$$

$$(3) \quad \text{rank} \begin{bmatrix} L(\gamma, P, s) \\ F(\gamma, P) \end{bmatrix} = n + \text{normrank } G, \quad \forall s \in \mathbb{C}^+ \cup \mathbb{C}^0$$

where

$$F(\gamma, P) = \begin{bmatrix} A^T P + PA + C_2^T C_2 + \gamma^{-2} P E E^T P & PB + C^T D \\ B^T P + D^T C & D^T D \end{bmatrix} \quad (5)$$

$$L(\gamma, P, s) = [sI - A - \gamma^{-2} E E^T P \quad -B] \quad (6)$$

$$G(s) = C(sI - A)^{-1} B + D \quad (7)$$

The proof of Lemma 2.3 can be found in [Stoorvogel, 1992].

We shall refer to condition (1) as the Quadratic Matrix Inequality, and any $P \geq 0$ satisfying (1) will be called a solution to the Quadratic Matrix Inequality. Conditions (2) and (3) guarantees that a solution to the Quadratic Matrix Inequality is unique and of minimal rank.

The state feedback controllers for the \mathcal{H}_∞ problem will be given in the following in terms of the matrices for certain transformations of Σ . For this, we will need A_p given by:

$$A_p = A + \gamma^{-2} E E^T P \quad (8)$$

We shall refer to the system where A_p , C_p and D_p substitute A , C_2 and D as the full information transform of the system Σ .

Now, consider the transformed system Σ_p :

$$\Sigma_p : \begin{cases} \dot{x}_p = A_p x_p + B u_p + E w \\ z_p = C_p x_p + D_p u_p \end{cases} \quad (9)$$

where C_p and D_p are given by Lemma 2.3.

The connection between the original system Σ and the transformed system is given in the following lemma:

Lemma 2.4.

Let's use an arbitrary state feedback controller $u = Fx$. Then, the following two statements are equivalent:

1. *The state feedback controller $u = Fx$ applied to the original system Σ is internally stabilizing and the resulting closed loop transfer function from w to z is strictly proper and has \mathcal{H}_∞ norm $< \gamma$.*

2. *The state feedback controller $u = Fx$ applied to the transformed system Σ_p is internally stabilizing and the resulting closed loop transfer functions from w to z_p is strictly proper and has \mathcal{H}_∞ norm $< \gamma$.*

Proof. See [Stoorvogel, 1992].

In terms of the transformed system matrices we can compute the desired \mathcal{H}_∞ state feedback controller. The state feedback gain F must be determined such that:

Lemma 2.5.

Let A_p , B and C_p be as above. Let F be a state feedback, such that $A_p + BF$ is stable, and such that:

$$\|(C_p + D_p F)(sI - A_p - BF)^{-1}\|_\infty < \gamma / (\|E\|) \quad (10)$$

Then the state feedback controller $u = Fx$ makes the \mathcal{H}_∞ norm of the resulting closed loop transfer function smaller than γ .

The significance of Lemma 2.5 is to transform the original \mathcal{H}_∞ state feedback problem to an almost disturbance attenuation problem, which can be solved by well known methods, see e.g. [Stoorvogel, 1992]. A thorough, numerical analysis of this approach is given in [Schijfs, 1990].

2.3. Cheap Control.

The cheap control problem has been considered in various papers, see e.g. [Stoorvogel, 1990, 1992, Saberi and Sannuti, 1987, Scherer, 1990].

To obtain a regular \mathcal{H}_∞ problem when D is not injective, we can apply the cheap control technique by perturbing the C and D matrices in Σ . C_ϵ and D_ϵ are then given by:

$$C_\epsilon = \begin{bmatrix} C \\ \epsilon I \\ 0 \end{bmatrix} \quad D_\epsilon = \begin{bmatrix} D \\ 0 \\ \epsilon I \end{bmatrix}, \quad \epsilon > 0 \quad (11)$$

Applying C_ϵ and D_ϵ instead of C and D , we will recover the Riccati equation from (2). For the cheap control we have the following result:

Lemma 2.6.

Consider the system Σ with C_ϵ and D_ϵ given by (11). Then the following three statements are equivalent.

1. There exist a state feedback F for the system (2) such that $A + BF$ is stable and such that:

$$\|(C + DF)(sI - A - BF)^{-1}E\|_\infty < \gamma \quad (12)$$

2. There exist an ϵ_1 such that for all $\epsilon \in (0; \epsilon_1]$ there exist a state feedback F_ϵ such that $A + BF_\epsilon$ is stable and such that:

$$\|(C_\epsilon + D_\epsilon F_\epsilon)(sI - A - BF_\epsilon)^{-1}E\|_\infty < \gamma \quad (13)$$

3. There exist ϵ_1 such that for all $\epsilon \in (0; \epsilon_1]$ there exist $X_\epsilon > 0$ such that:

$$\epsilon^2 I + A^T X_\epsilon + X_\epsilon A + C^T C + \gamma^{-2} X_\epsilon E E^T X_\epsilon - (X_\epsilon B + C^T D)(D^T D + \epsilon^2 I)^{-1}(B^T X_\epsilon + D^T C) = 0 \quad (14)$$

and the \mathcal{H}_∞ state feedback gain in (4) take then the following form:

$$F_\epsilon = (D^T D + \epsilon^2 I)^{-1}(B^T X_\epsilon + D^T C) \quad (15)$$

Any state feedback controller F_ϵ satisfying condition 2 also satisfies condition 1. Moreover, the sequence X_ϵ is convergent, $X_\epsilon \rightarrow P$ and the limit P satisfies (1-3) in Lemma 2.3.

Proof. See [Stoustrup and Niemann, 1992].

Lemma 2.6 states that in the general case, the well known regular Riccati equation [Doyle et.al., 1989] has to be substituted with the three condition of Lemma 2.3 as $\epsilon \rightarrow 0$.

Note that the \mathcal{H}_∞ norm of the closed loop system is an increasing function of ϵ , i.e.:

$$\|G_F(s)\|_\infty \leq \|G_{F,\epsilon_1}\|_\infty \leq \|G_{F,\epsilon_2}\|_\infty \quad (16)$$

for $0 \leq \epsilon_1 \leq \epsilon_2$ where G_F is the closed-loop system with a stabilizing F . It will therefore always be possible, from a theoretical point of view, to select an $\epsilon > 0$ such that a specified \mathcal{H}_∞ norm $\gamma = \gamma^* + \delta$ is guaranteed to be satisfied. However, this will not necessarily be practice in praxis as it will be shown in next section.

3. EXAMPLES.

The cheap control and the singular \mathcal{H}_∞ method will be applied to a simple 4th order system and a more complicated 8th order system for evaluating the theoretical results shortly resumed in section 2.

All computations are done in 3.86-MATLAB by using the MATLAB Control Toolbox and the CACSD package described in [Niemann and Stoustrup, 1992a, b].

Let's consider the following 4th order system Σ :

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew \\ z = Cx + Du \end{cases}$$

where

$$A = \begin{bmatrix} 2.1 & 1. & 0 & 0 \\ 0.5 & 0 & 1. & 0 \\ 5.6 & 0 & 0 & 1. \\ 3. & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1. \\ 3. \\ -6. \\ -8. \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 5.6667 \\ 5.6667 \end{bmatrix}$$

$$C = \begin{bmatrix} 1. & 0 & 0 & 0 \end{bmatrix}, \quad D = 0$$

(A,B,C,D) has a non-minimum phase zero at $z = 2$.

γ^* is calculated to $\gamma^* = |C(2I-A)^{-1}E| = 1.0$, [Zhou, 1992].

For the cheap control approach, we change the C and D matrices in the following way:

$$C_\epsilon = \begin{bmatrix} C \\ \epsilon I \\ 0 \end{bmatrix}, \quad D_\epsilon = \begin{bmatrix} 0 \\ 0 \\ \epsilon I \end{bmatrix}$$

At first, we look at the solutions X_ϵ of the cheap control Riccati equation (14) compared to the Quadratic Matrix Inequality solution P as function of γ .

The first figure show the norm (i.e. the maximal singular value) of the Quadratic Matrix Inequality solution P as function of $\gamma^* + \gamma_6$. We can reduce γ_6 to $10^{-16} - 10^{-15}$ before numerical problems arise. This indicate a very robust algorithm for the solution of the Quadratic Matrix Inequality. Note that the smallest number which can be represented in MATLAB is $6 \cdot 10^{-17}$, so we get very close to the computer's performance in the solution of the Quadratic Matrix Inequality.

The norm of X_ϵ for different values of ϵ as function of γ are shown in fig. 3.2. (QMI is the Quadratic Matrix Inequality solution P). If we let ϵ be smaller than 10^{-5} we get numerical problems. Using $\epsilon = 10^{-5} - 10^{-4}$ gives the best Riccati approximation to the Quadratic Matrix Inequality solution for $\gamma > 1.0001$, when the norms of the solutions are compared. However, to get a more clear picture of how close the Riccati solution X_ϵ is to the Quadratic Matrix Inequality solution P, we have calculated $\|X_\epsilon - P\|$ as a function of γ for different values of

ϵ . The results is shown in fig. 3.3. The best approximation is still by applying $\epsilon = 10^{-5} - 10^{-4}$, but as a matter fact, for $\gamma_b < 10^{-4}$ we get more than 100% error. Only for reasonably big γ_b we get a good Riccati approximation of the Quadratic Matrix Inequality solution P.

In fig. 3.4 the convergence of the Riccati solution to the Quadratic matrix Inequality solution as function of ϵ is shown. For ϵ reasonably small, the norm of the Riccati solutions gets constant as long as we don't obtain numerical problems. If γ_b is reduced more than 10^{-3} , numerical problems will appear before the solutions gets "constant" norms. For $\gamma_b > 10^{-3}$ the "constant" Riccati solutions approximate the Quadratic Matrix Inequality solutions quite well in the norms, but it does not equal the Quadratic Matrix Inequality solution entrywise. This can be seen in fig. 3.5, where $\text{norm}(X_\epsilon - P)/\text{norm}(P)$ as function of ϵ has been shown for different values of γ_b . The Riccati solution X_ϵ will only tend to the Quadratic Matrix Inequality solution in the limit. However, fig. 3.5 show that this is impossible in practice, because numerical problems arise. Note that the norm of the difference between X_ϵ and P is not constant as it could have been conjectured from fig. 3.4.

In the last part of this example, we look at the existence of state feedback gains F which satisfy the specified closed loop \mathcal{H}_∞ condition. For the singular \mathcal{H}_∞ approach, we have calculated state feedback gains satisfying an \mathcal{H}_∞ constraint of $\gamma^* + 10^{-9}$ only by using a simple pole assignment method. In other examples, it has been possible to calculate state feedback controllers in the same region where the Quadratic Matrix Inequality solution exist. The existing of state feedback controllers by using cheap control as function of γ_b and ϵ is shown in fig. 3.6. Fig. 3.6 show the region where state feedback designed for the perturbed system satisfies the \mathcal{H}_∞ condition for the real system. Again, if ϵ is smaller than 10^{-5} numerical problems arise, which explain the lower part of fig. 3.6. For the upper part of fig. 3.6, there is a quite apparent boundary between the two regions.

The norm of the state feedback gains as function of γ_b is shown for different values of ϵ in fig. 3.7. As mentioned before, the state feedback gains for the singular \mathcal{H}_∞ approach (QMI) has been calculated by simple pole assignment. Note that the singular approach gives state feedback gains which are smaller than the gains calculated by the cheap control method. This is a general difference between regular and singular \mathcal{H}_∞ control. It be due to the extra freedom for calculating the state feedback gain satisfying the norm inequality (10) in the singular case. Further, note that the gains from the singular approach is smaller than the gains given by the regular approach for γ_b not too small. If the selection of (γ_b, ϵ) for the regular approach is not optimal, we will obtain a quite larger difference between the gains from the two methods.

A more complicated SISO example of order 8 is now considered. The state space matrices are generated by the build-in function rand from Matlab. The poles and zeros for the system are:

Poles of A:

-15.747 ± 34.656i
 - 6.8362
 9.2350 ± 9.9561i
 11.122 ± 5.4264i
 25.704

Zeros of (A,B,C,0):

-105.75
63.488
- 10.496 ± 13.942i
6.3935 ± 16.522i
15.563

Zeros of (A,E,C,0):

- 16.613 ± 33.637i
50.313
3.6361 ± 10.106i
15.724 ± 4.0962i

In fig. 3.8 - 3.12, we have shown the results of applying the two methods to the 8th order system. The figures show almost the same picture of the two methods as for the simple 4th order system. Our Quadratic Matrix Inequality algorithm can still be applied much closer to γ^* than the Riccati based cheap control. In this case, the numerical problems arise a decade or two earlier. We can not select ϵ smaller than 10^{-3} and γ_0 smaller than 10^{-4} to avoid numerical problems.

4. DISCUSSION.

A numerical study of regular vs. singular methods in cheap \mathcal{H}_∞ control has been presented. The theoretical results indicate that it is possible to apply the regular \mathcal{H}_∞ approach in form of cheap control of singular systems. But, as a matter of fact, this is in practice not possible due to numerical problems, as it has been demonstrated in section 3 and in [Niemann and Stoustrup, 1992a]. In cheap control, the classical numerical problem of adding small and large numbers appears. Therefore, methods which directly take care of the singularities are preferable as e.g. the singular \mathcal{H}_2 and \mathcal{H}_∞ approach. The price of using such general methods are more complicated design algorithms. However, it is possible to obtain satisfactory solutions by using the approximating methods as shown in section 3.

Design by using cheap \mathcal{H}_∞ control can, of course, still be applied far from the optimal γ^* as a shortcut to handle singular systems. A minor problem by applying cheap \mathcal{H}_∞ control is that γ -iteration is involved because of the evaluation over two parameters. When γ^* is known, it is simple to generate curves as shown in section 3 for evaluating the method on a specific problem. With γ^* unknown, we do not know how close the design is to optimum.

Our main conclusion of the research in \mathcal{H}_∞ control of singular systems reported partly in this paper, is that the singularities must be respected and the necessary carefulness must be taken in the design of controllers. Therefore, the singular approach must always be applied when the design problem is singular. The cheap control principle should be applied to near-optimal control only when the equivalent singular theory/numerical algorithms are not available.

APPENDIX A.

In this appendix we shall describe an algorithm for the solution to the Quadratic Matrix Inequality which is a basic ingredient of the singular \mathcal{H}_∞ state feedback design of section 2.

To be more precise: Given a constant $\gamma > 0$ and five matrices (A,B,E,C,D), which can be interpreted as the following dynamical system:

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew, & x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^q \\ z = Cx + Du, & z \in \mathbb{R}^p \end{cases}$$

We wish to find a positive semidefinite matrix P such that the following three are all satisfied:

$$\begin{bmatrix} A^T P + PA + C^T C + \gamma^{-2} P E E^T P & PB + C^T D \\ B^T P + D^T C & D^T D \end{bmatrix} =: \begin{bmatrix} C_P^T \\ D_P^T \end{bmatrix} \begin{bmatrix} C_P & D_P \end{bmatrix} \geq 0$$

$$\text{rank } C_P = p$$

$(A + \gamma^{-2} E E^T P, B, C_P, D_P)$ is a minimum phase system.

The algorithm below is based on [Stoorvogel, 1992]. We shall use the following notation:

$Y := \text{RAN}(X)$, where X is a matrix, gives a matrix Y whose columns are a basis for the column space of X.

$Y := \text{NULL}(X)$, where X is a matrix, gives a matrix Y whose columns are a basis for the kernel of X, $XY = 0$.

Step 1 - Calculation of the Almost Controllability Subspace.

- 1) Initialize $R_a := R_b := 0$.
- 2) $[S_1^T \ S_2^T]^T := \text{NULL}([CR_b \ D])$
- 3) $R_a := R_b S_1$..
- 4) $R_b := \text{RAN}([AR_a + BS_2, B * \text{NULL}(D)])$
- 5) Stop, if R_b did not increase its number of columns in (4), otherwise go to (2).

Step 2 - State Space Transformation.

- 6) $T_2 := R_a$
- 7) Determine T_3 such that $\text{RAN}([T_2, T_3]) = \text{RAN}(R_b)$ and the columns of $[T_2, T_3]$ are linearly independent.
- 8) Determine T_1 such that the columns of $[T_1, T_2, T_3]$ form a basis in \mathbb{R}^n .
- 9) $[S_1^T, S_2^T, S_3^T]^T := [T_1, T_2, T_3]^{-1}$

- 10) Determine F as any solution to $D^TDF = -D^TC$
- 11) $A_{11} := S_1(A+BF)T_1$
- 12) $A_{13} := S_1(A+BF)T_3$
- 13) $B_1 := S_1B$
- 14) $C_{21} := (C+DF)T_1$
- 15) $C_{23} := (C+DF)T_3$
- 16) $E_1 := S_1E$

Step 3 - Reduced Order Riccati Equation.

- 17) $A^* := A_{11} - A_{13}(C_{23}^TC_{23})^{-1}C_{23}^TC_{21}$
- 18) $R^* := \gamma^2E_1E_1^T - B_1(\text{RAN}(D^T)\text{RAN}(D))^{-1}B_1^T$
- 19) $Q^* := C_{21}^TC_{21} - C_{21}^TC_{23}(C_{23}^TC_{23})^{-1}C_{23}^TC_{21}$
- 20) Solve the algebraic Riccati equation:

$$P^*A^* + A^{*T}P^* + P^*R^*P^* + Q^* = 0$$

Step 4 - Solution to Quadratic Matrix Inequality.

- 21) $P := S_1^TP^*S_1$

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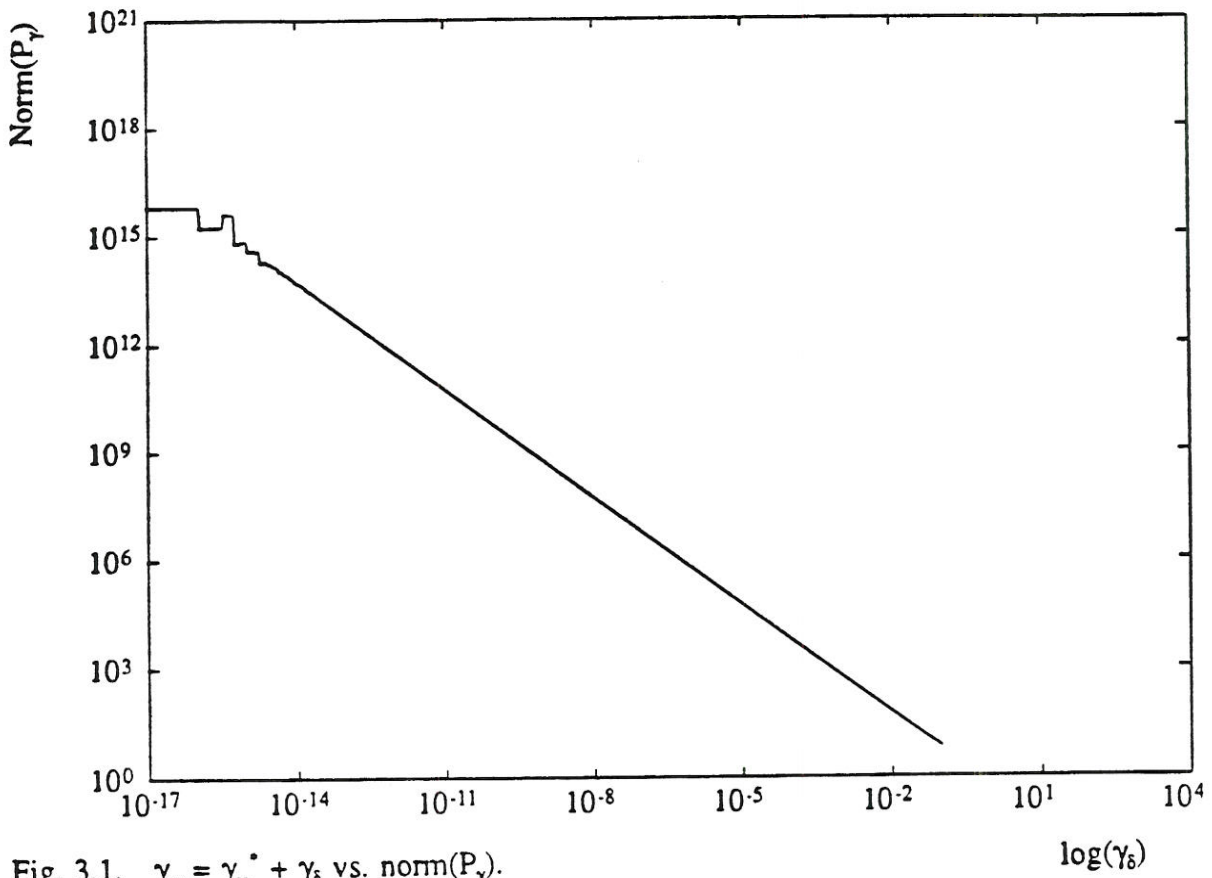


Fig. 3.1. $\gamma_\infty = \gamma_\infty^\circ + \gamma_\delta$ vs. $\text{norm}(P_\gamma)$.

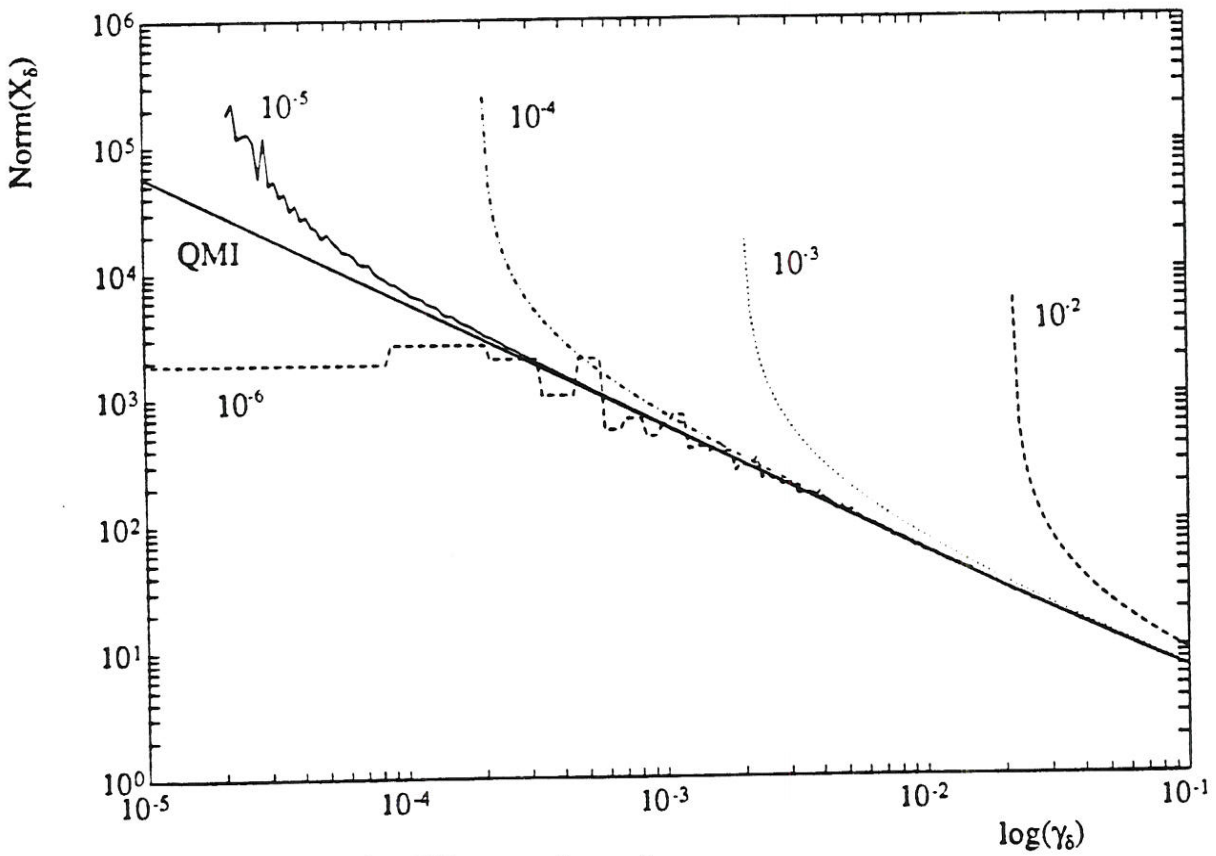


Fig. 3.2. γ_δ vs. $\text{norm}(X_\delta)$ for different values of ϵ .

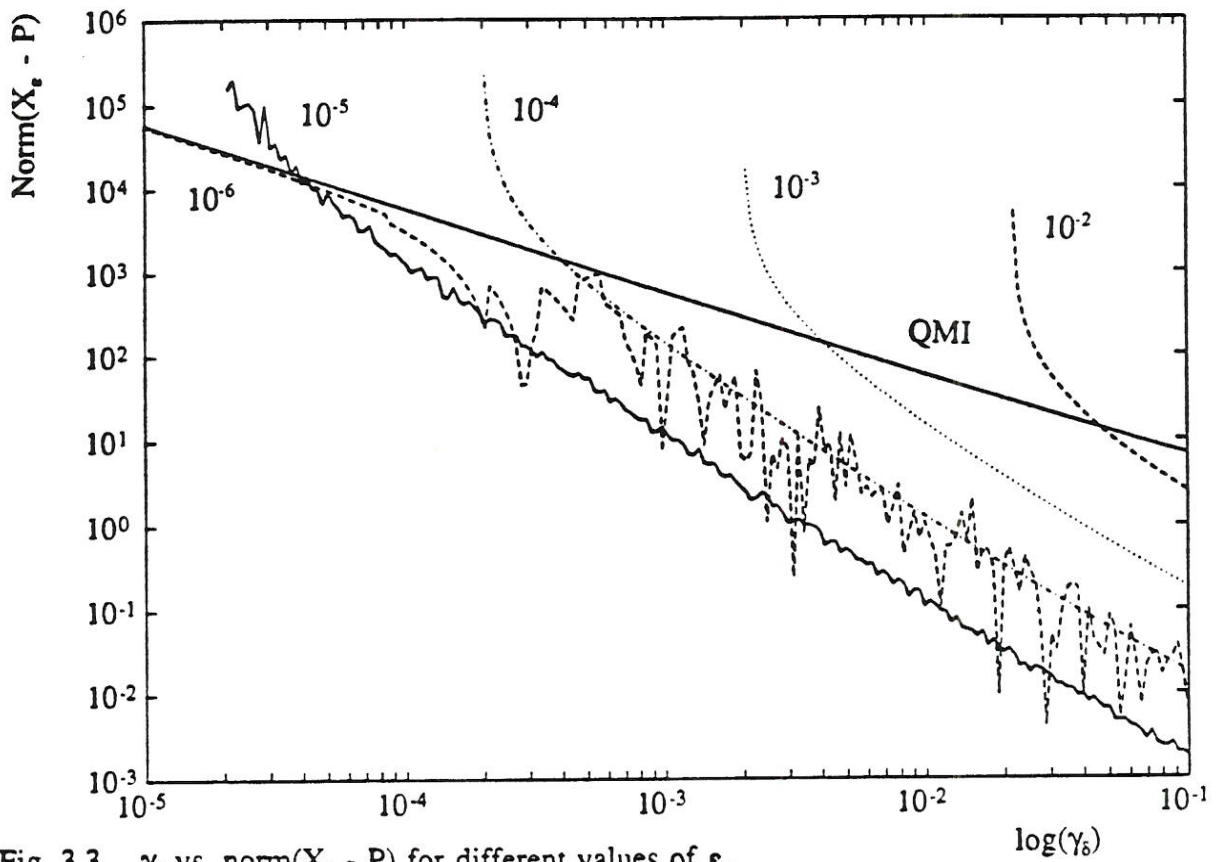


Fig. 3.3. γ_8 vs. $\text{norm}(X_8 - P)$ for different values of ϵ .

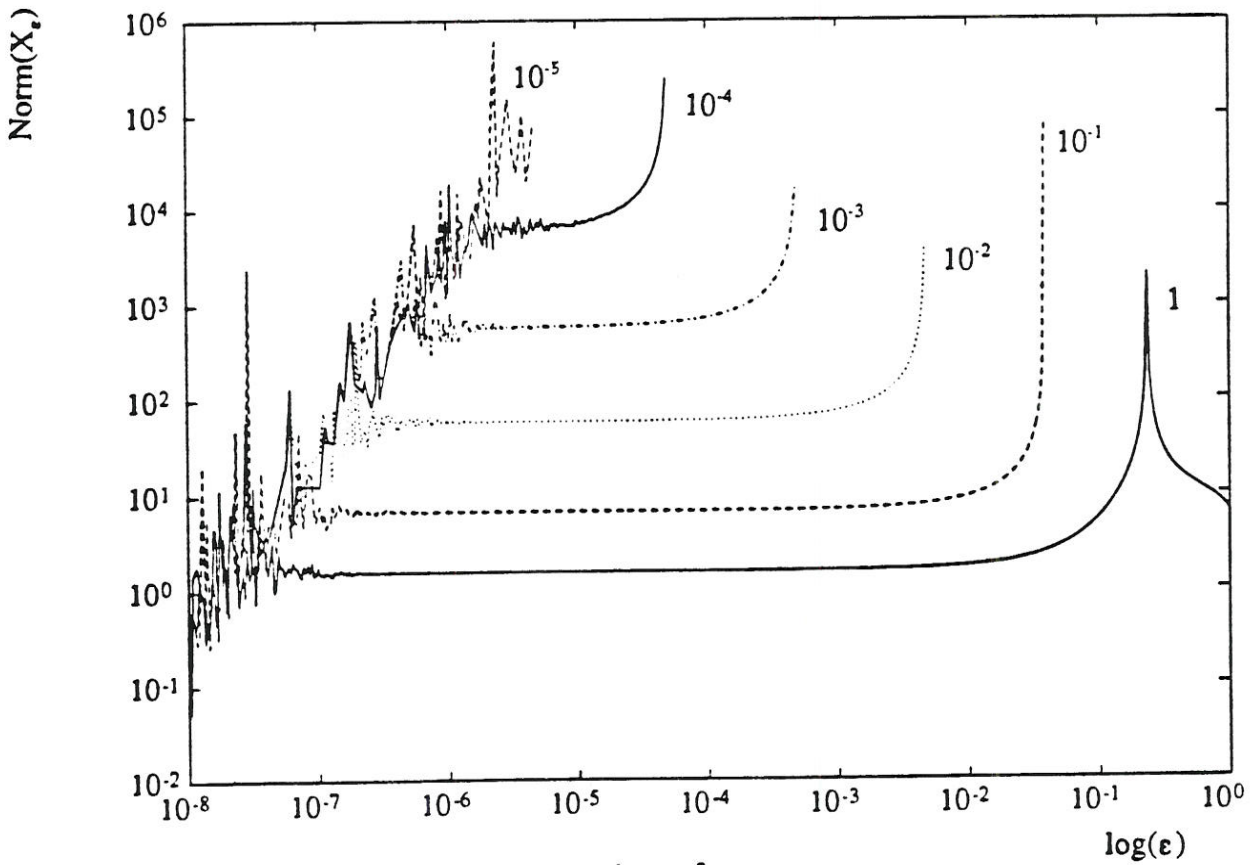


Fig. 3.4. D_ϵ vs. $\text{norm}(X_8)$ for different values of γ_∞ .

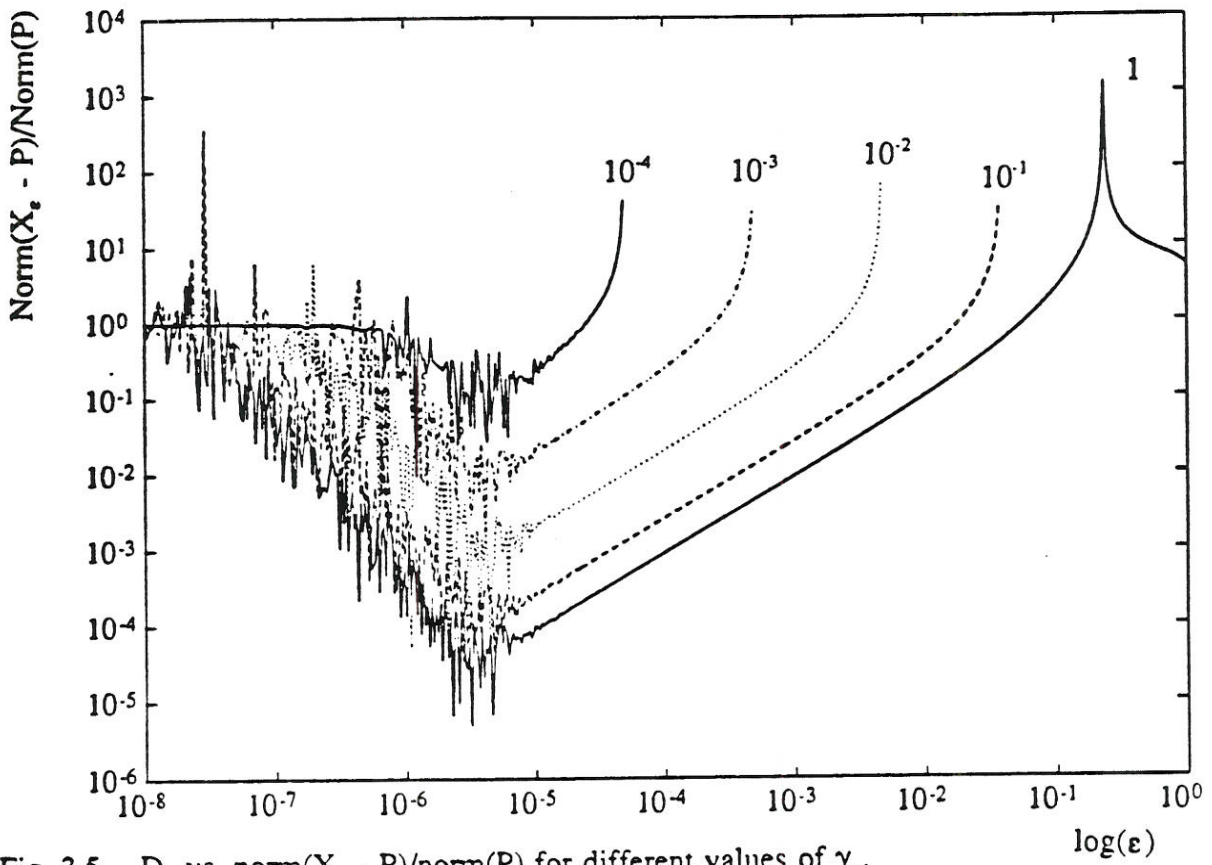


Fig. 3.5. D_ϵ vs. $\text{norm}(X_e - P)/\text{norm}(P)$ for different values of γ_∞ .

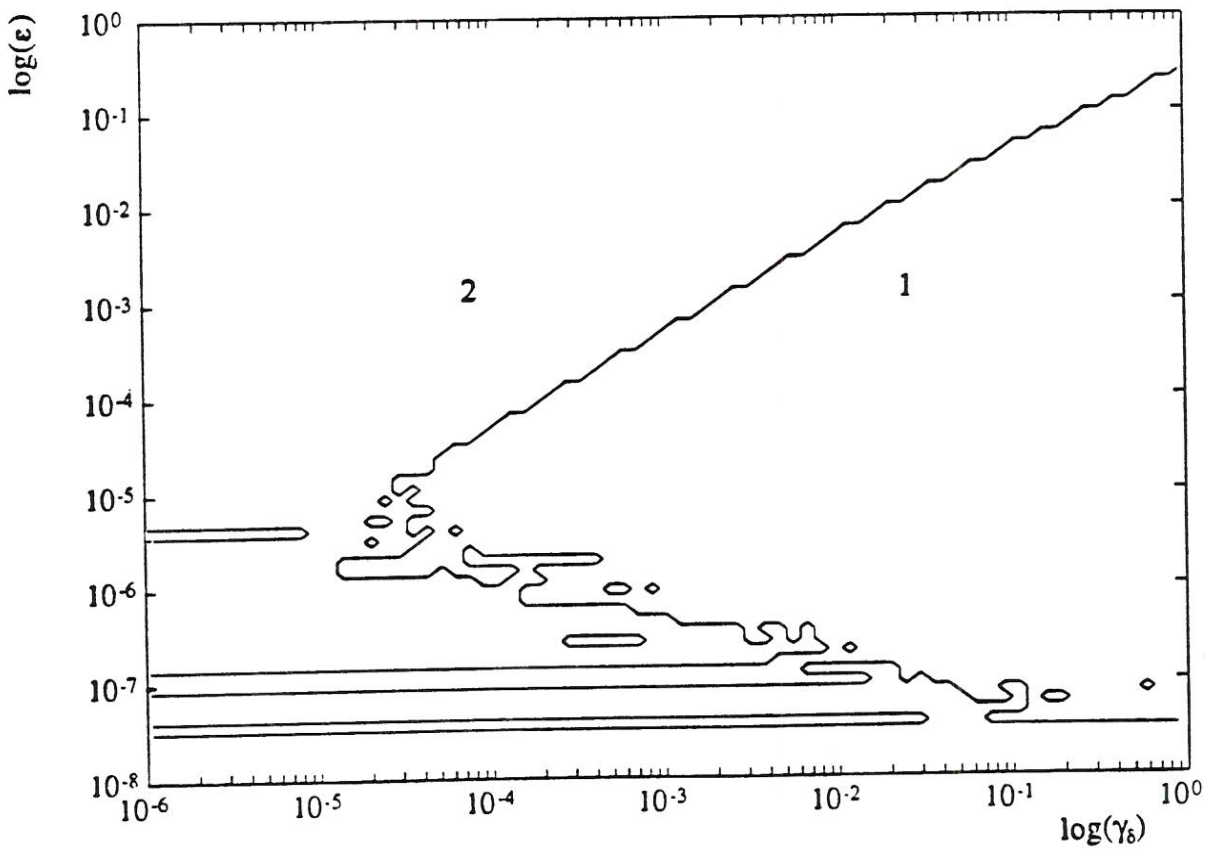


Fig. 3.6. \mathcal{H}_∞ norm condition satisfied by cheap control:
 Region 1. Satisfied.
 Region 2. Not satisfied.

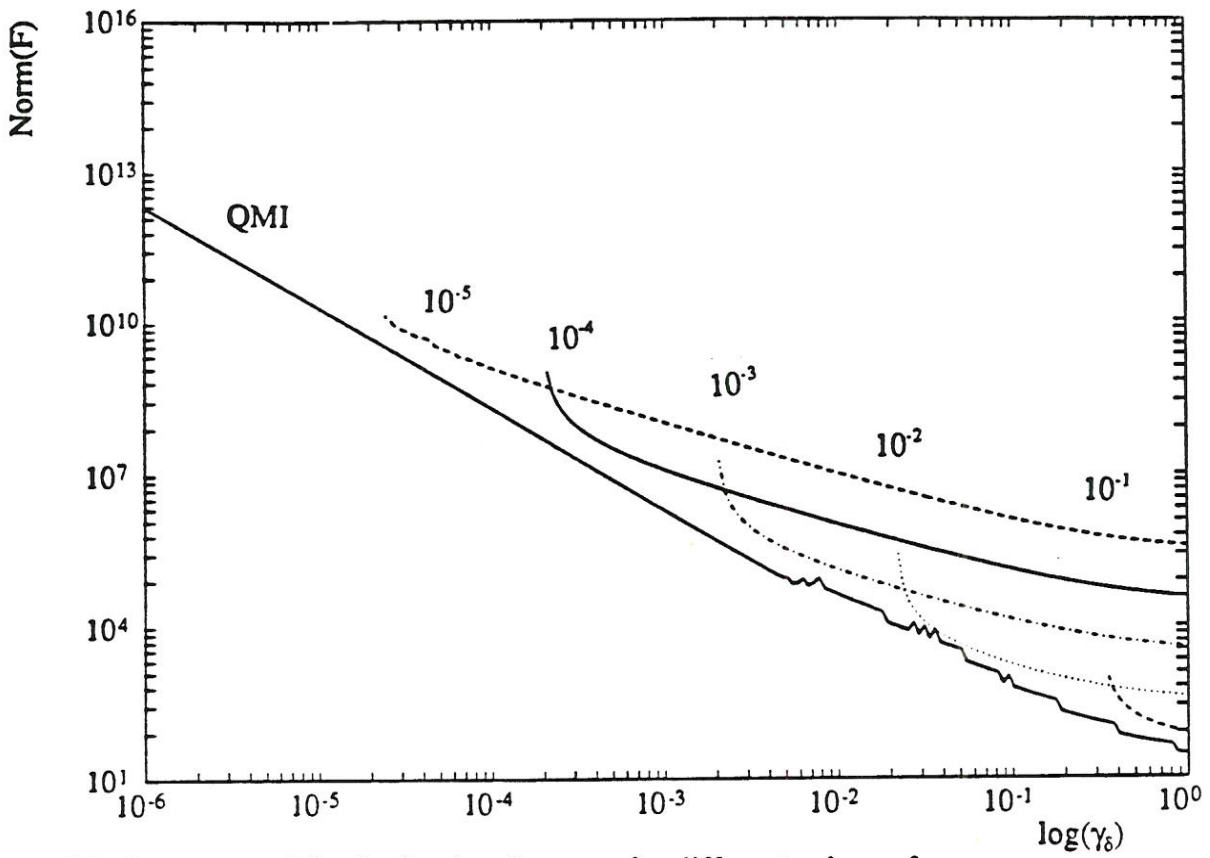


Fig. 3.7. The norm of the feedback gain vs. γ_δ for different values of ϵ .

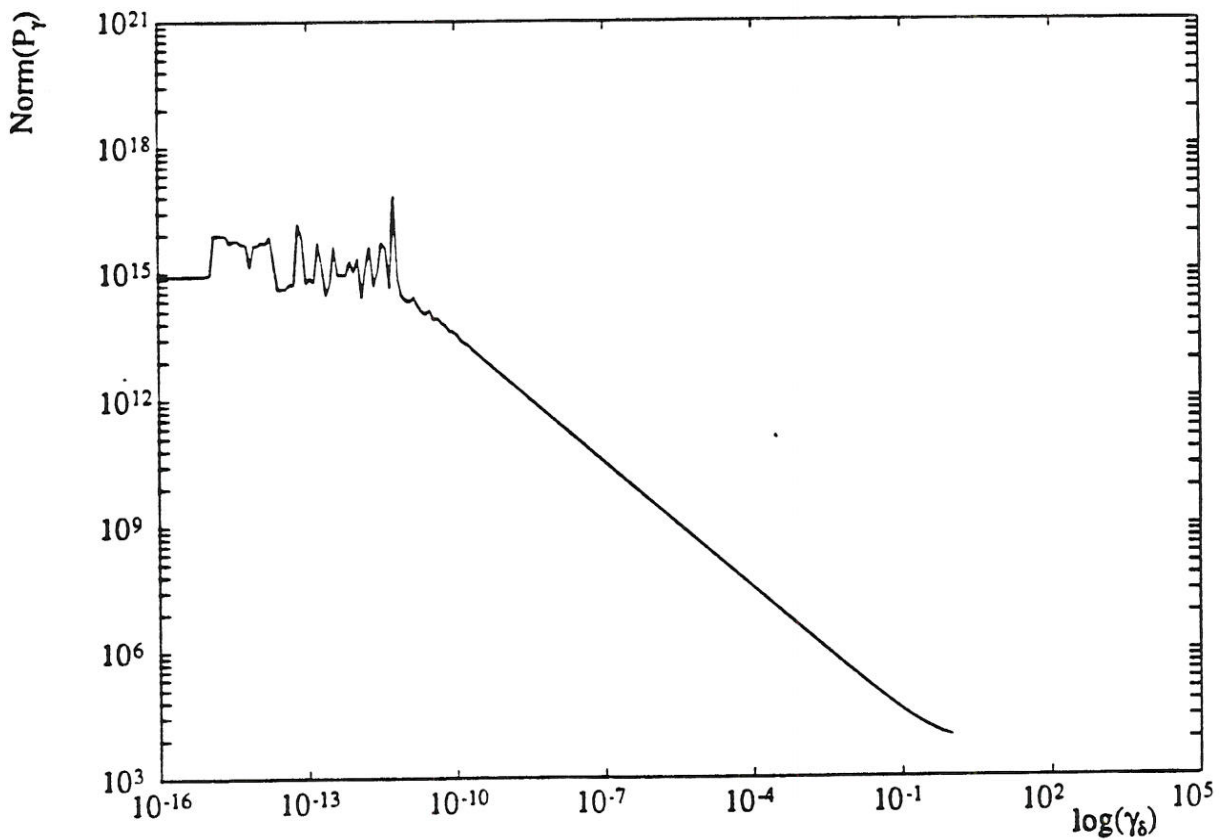


Fig. 3.8. $\gamma_\infty = \gamma_\infty^\circ + \gamma_\delta$ vs. norm(P_γ) for the QMI.

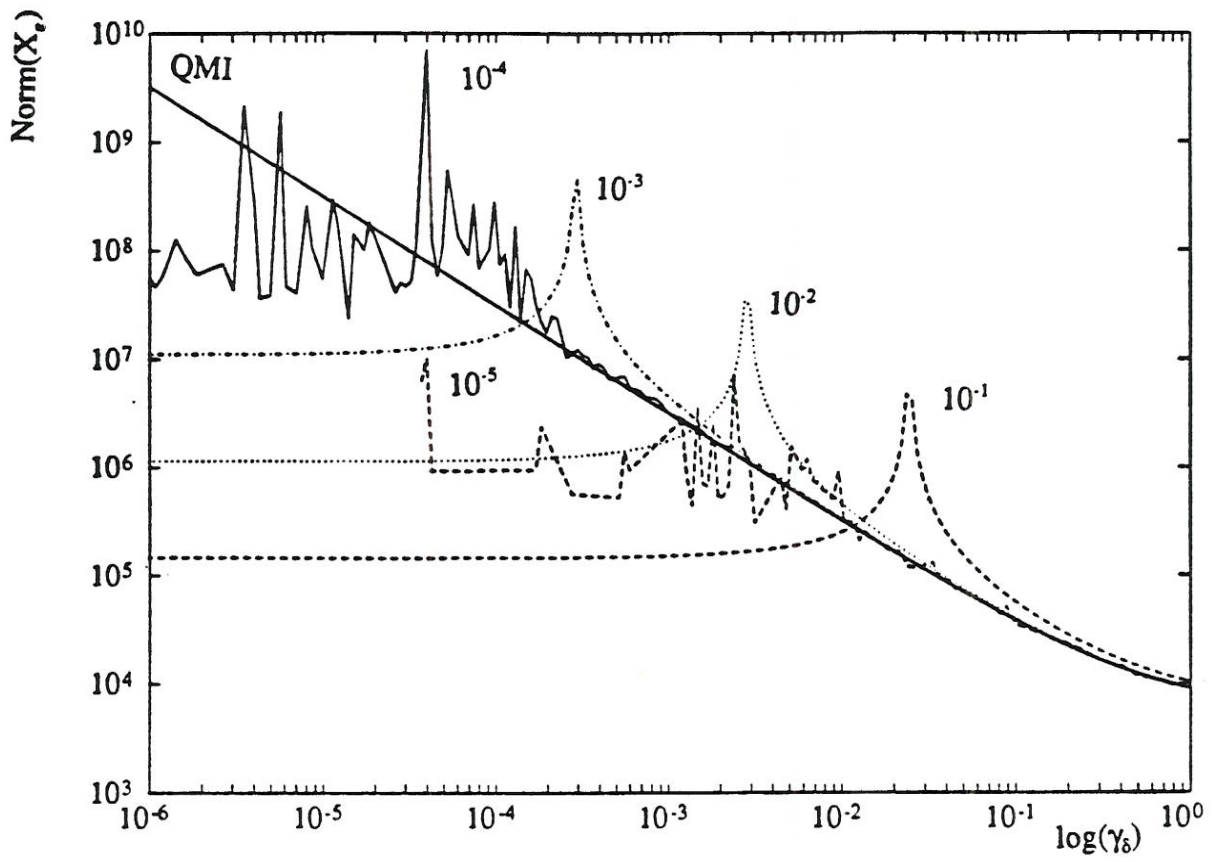


Fig. 3.9. γ_δ vs. $\text{norm}(X_\delta)$ for different values of ϵ .

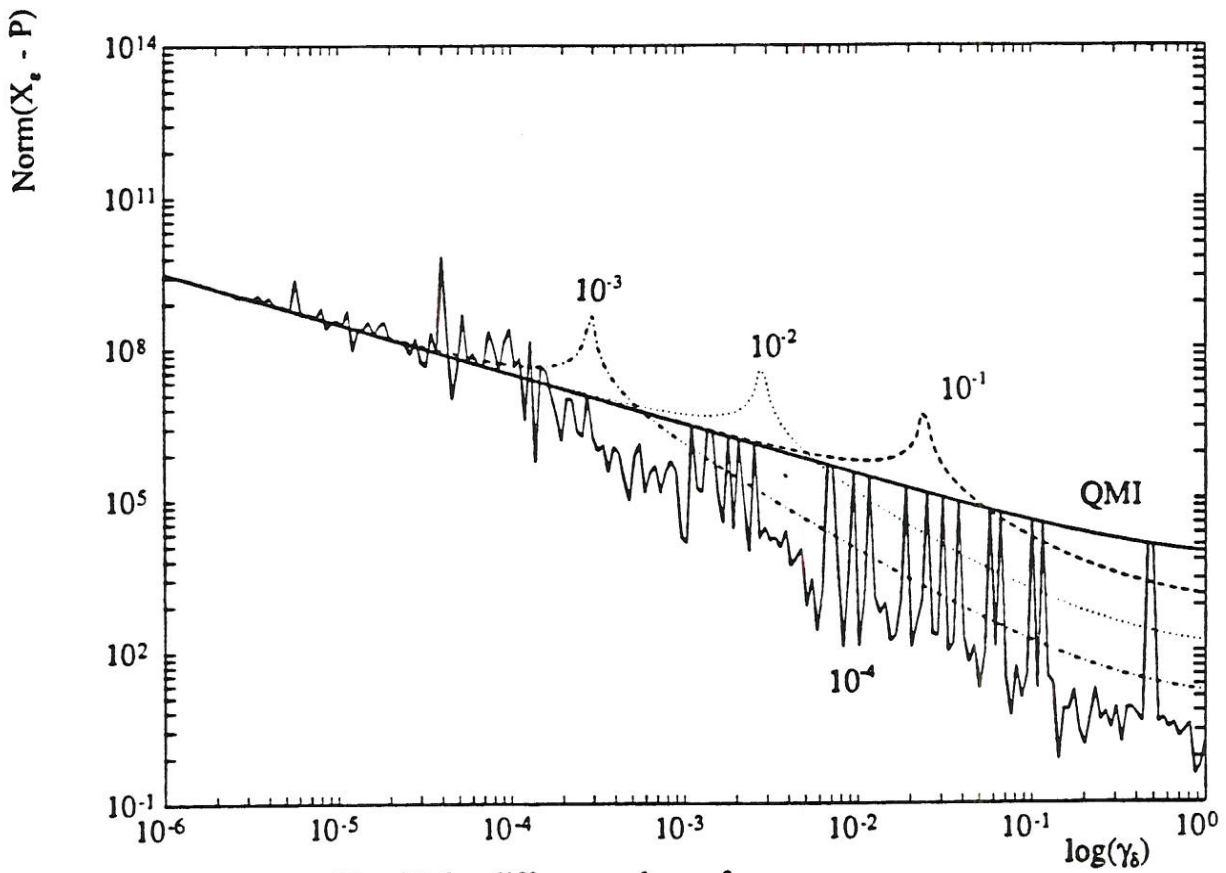


Fig. 3.10. γ_δ vs. $\text{norm}(X_\delta - P)$ for different values of ϵ .

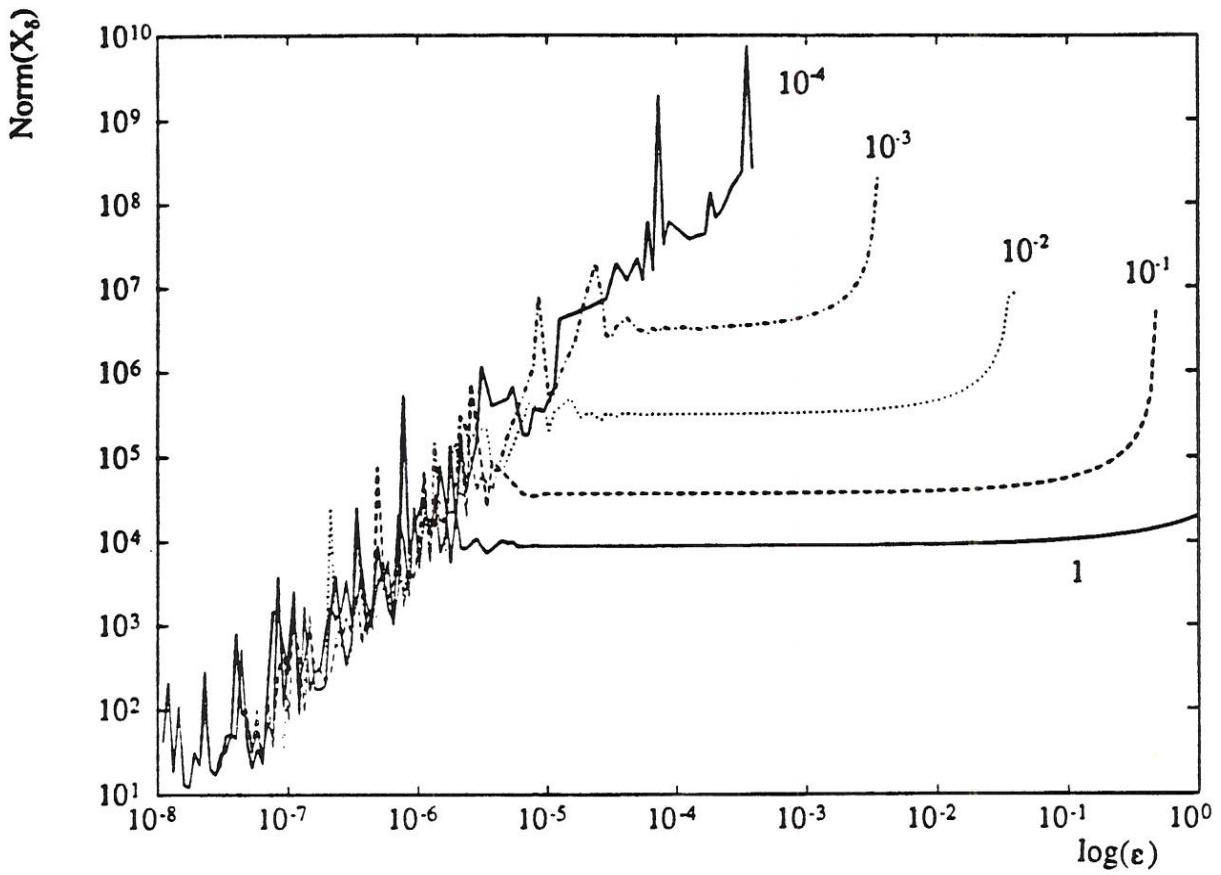


Fig. 3.11. D_ϵ vs. $\text{norm}(X_\epsilon)$ for different values of γ_ϵ .

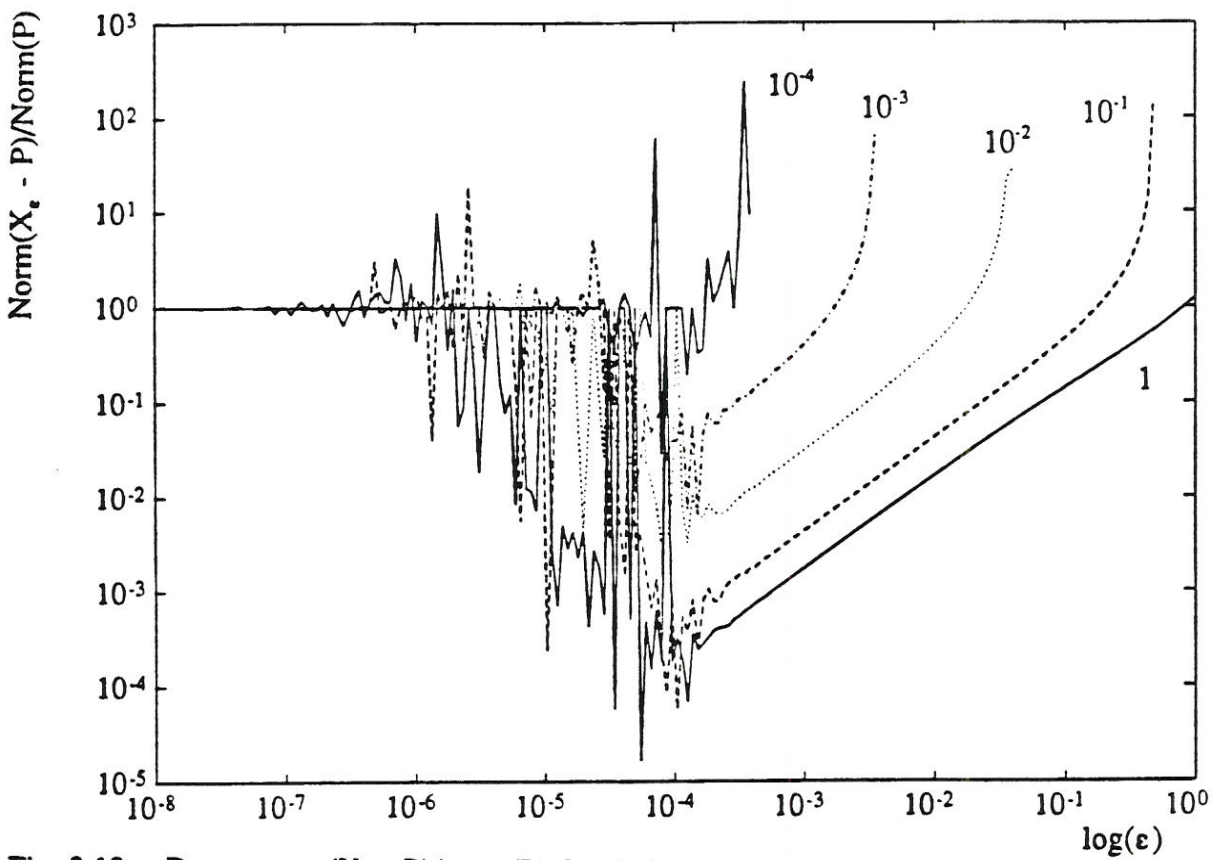


Fig. 3.12. D_ϵ vs. $\text{norm}(X_\epsilon - P)/\text{norm}(P)$ for different values of γ_ϵ .