MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ STATE FEEDBACK CONTROL WITH AN IMPROVED COVARIANCE BOUND

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Abstract. In this paper we consider the problem of designing a stabilizing controller which minimizes the $\mathcal{H}_2$ norm of a transfer matrix while maintaining the $\mathcal{H}_\infty$ norm of another transfer matrix below a specified level. This problem is unsolved, but we approximate the problem by a tractable convex method, and we improve on the $\mathcal{H}_2$ norm bound in the literature. Our main result shows that our formulation is less conservative and the problem can still be solved by convex programming.

Keywords. H_∞; Convex programming; State feedback; Optimal control; Robust control

1 INTRODUCTION

After reaching a mature state $\mathcal{H}_\infty$ control theory (Doyle et al, 1989), much emphasis has been placed on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem which combines the $\mathcal{H}_\infty$ control problem and the classical $\mathcal{H}_2$ (or LQG) control problem (Bernstein and Haddad, 1989; Doyle et al, 1989; Zhou et al, 1990; Khargonekar and Rotea, 1991; Ridgely et al, 1992a and b), where a stabilizing controller is sought which minimizes the $\mathcal{H}_2$ norm of a transfer matrix subject to an $\mathcal{H}_\infty$ norm bound on another transfer matrix. Such a problem is important since it represents one way of formulating a robust control problem where robustness is guaranteed by the $\mathcal{H}_\infty$ norm bound and the performance as measured by the $\mathcal{H}_2$ norm.

The first approach to this problem by Bernstein et al. (1989) utilized the fact that the solution to the Riccati equation describing the $\mathcal{H}_\infty$ norm bound constraint is an upper bound for the state covariance, and formulated a problem of minimizing the $\mathcal{H}_2$ upper bound obtained as the trace of the output covariance bound. Their approach resulted in coupled Riccati equations which are nontrivial to solve except for the "equalized" case (a single transfer matrix is considered for both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance) where the optimal controller is shown (Mustafa, 1989) to be the maximum entropy $\mathcal{H}_\infty$ controller (Glover and Mustafa, 1989). To overcome the numerical difficulty in Bernstein and Haddad (1989), Khargonekar and Rotea (1991) took an "inequality approach" and formulated an equivalent problem as a convex optimization problem. However, the $\mathcal{H}_2$ upper bound based on the $\mathcal{H}_\infty$ Riccati solution is conservative. There are no results available which solve the original (nonconservative) problem with a numerically tractable algorithm (e.g. Ridgely et al. 1992a and b).

This paper introduces a less conservative measure of $\mathcal{H}_2$ performance by adding an extra freedom (a positive scalar α) to the formulation of Khargonekar and Rotea (1991). Since our feasible set is larger than that of Khargonekar and Rotea (1991) due to the extra freedom α, our optimal $\mathcal{H}_2$ norm bound is guaranteed to be less than or equal to that of Khargonekar and Rotea (1991). Moreover, our formulation also allows convex programming. The precise problem formulation will be given in the next section.

2 PRELIMINARIES

We shall consider finite dimensional, linear, time invariant systems of the form

$$\begin{array}{rclcccc} \dot{x} & = & Ax & + & B_1 w & + & B_2 u \\ z_\infty & = & C_\infty x & & & + & D_\infty u \\ z_2 & = & C_2 x & & & + & D_2 u \end{array} . \quad (2.1)$$

For such systems we shall consider static state feedbacks, i.e., control laws of the form $u = Fx$. Let the closed loop transfer matrices ($w \mapsto z_i$) be denoted by

$$G_i(s) = (C_i + D_i F)(sI - A - B_2 F)^{-1} B_1 \quad (2.2)$$

where $i = 2$ or ∞. The set of $\mathcal{H}_\infty$ controllers is defined as

$$\mathcal{F}_\infty := \{F \in \mathcal{R}^{m\times n} : A + B_2F \text{ is stable}, \; \|G_\infty\|_\infty < 1\} \tag{2.3}$$

where $\|\cdot\|_\infty$ denotes the $\mathcal{H}_\infty$ norm. The ultimate goal of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem considered in this paper is to solve:

$$\gamma_0 := \inf\{\|G_2\|_2 : F \in \mathcal{F}_\infty\} \tag{2.4}$$

where $\|\cdot\|_2$ denotes the $\mathcal{H}_2$ norm. Recall that the $\mathcal{H}_2$ norm can be computed as the trace of the output covariance matrix $Y_{cov} := \lim_{t\to\infty} \mathcal{E}[z_2(t)z_2'(t)]$ for the white noise exogenous input w with unit intensity, where $\mathcal{E}[\cdot]$ denotes the expectation operator. It is known Ridgely et al. (1992a and b) that directly minimizing trace(Y_{cov}) subject to $F \in \mathcal{F}_\infty$ results in a computationally nontrivial problem to solve. Recent work by Khargonekar and Rotea (1991) reduced the problem to a convex programming problem by minimizing an upper bound for the $\mathcal{H}_2$ norm. This paper improves the $\mathcal{H}_2$ norm bound of Khargonekar et al. (1991) by introducing the "α-constrained covariance bound". To this end, consider the following three sets, each of which characterizes an upper bound for Y_{cov} achievable with an $F \in \mathcal{F}_\infty$.

$$\begin{aligned}\Phi_T \;&:=\; \{(F,P,X,Y):\; P>0,\; X>0,\\ &\qquad ric(P)<0\; lyap(X)<0,\; oc(X)<Y\;\}\end{aligned}$$

$$\begin{aligned}\Phi_R \;&:=\; \{(F,P,Y):\; P>0,\; ric(P)<0,\\ &\qquad oc(P)<Y\;\}\end{aligned}$$

$$\begin{aligned}\Phi_\alpha \;&:=\; \{(F,P,Y,\alpha):\; P>0,\; ric(P)<0,\\ &\qquad \alpha>0,\; lyap(\alpha P)<0,\; oc(\alpha P)<Y\;\}\end{aligned}$$

where F, P, X and Y are matrices of appropriate dimensions and α is a real number and

$$\begin{aligned} ric(P) \;&:=\; (A+B_2F)P + P(A+B_2F)' + B_1B_1' \\ &\qquad +P(C_\infty + D_\infty F)'(C_\infty + D_\infty F)P\end{aligned}$$

$$lyap(X) := (A+B_2F)X + X(A+B_2F)' + B_1B_1'$$

$$oc(X) := (C_2 + D_2F)X(C_2+D_2F)'.$$

In each set, the conditions $P > 0$ and $ric(P) < 0$ describe the necessary and sufficient condition for $F \in \mathcal{F}_\infty$, and the matrix Y corresponds to the output covariance upper bound, i.e., $Y_{cov} \le Y$. If we denote the infimum of trace(Y) over the sets Φ_T, Φ_R and Φ_α by ϕ_T, ϕ_R, and ϕ_α, respectively, then we have the following;

$$\phi_0 = \phi_T \le \phi_\alpha \le \phi_R. \tag{2.5}$$

The first equality holds since the matrix X in Φ_T is a "tight" upper bound for the state covariance. Unfortunately, infimizing trace(Y) over Φ_T is still a difficult task numerically. The inequality $\phi_0 \le \phi_R$ was first utilized by Bernstein et al. (1989), and Khargonekar and Rotea (1991) solved the problem equivalent to $\inf\{trace(Y) : (F,P,Y) \in \Phi_R\}$ by convex programming. However, this approach may be conservative since the matrix P in Φ_R is an upper bound for the state covariance which is not necessarily tight. This paper considers the set Φ_α where the state covariance upper bound X in Φ_T is restricted to those satisfying $X = \alpha P$ for some $\alpha > 0$ and $P > 0$, $ric(P) < 0$. Although this formulation is still conservative ($\phi_0 \le \phi_\alpha$), the upper bound by Khargonekar et al. is indeed improved ($\phi_\alpha \le \phi_R$) since, if $(F^*,P^*,Y^*)\in \bar{\Phi}_R$ gives the solution ϕ_R to $\inf\{trace(Y) : (F,P,Y) \in \Phi_R\}$, then the choice $(F^*,P^*,Y^*,1)\in \bar{\Phi}_\alpha$ surely yields the same value of the $\mathcal{H}_2$ norm bound as ϕ_R, where $\bar{\Phi}_R$ and $\bar{\Phi}_\alpha$ denote the closures of Φ_R and Φ_α, respectively. Moreover, any element $(F^*,P^*,Y^*,\alpha) \in \bar{\Phi}_\alpha$ yields smaller $\mathcal{H}_2$ norm bound than ϕ_R whenever $\alpha < 1$. Obviously, the set Φ_α is not convex, but we will show in the next section that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem with the improved covariance bound;

$$\phi_\alpha := \inf\{trace(Y) : (F,P,Y,\alpha) \in \Phi_\alpha\} \tag{2.6}$$

can also be solved via convex programming.

3 MAIN RESULTS

The following theorem states that the above optimization problem can be converted to that of infimizing a convex function over a convex set described by three linear matrix inequalities. A formula for the optimal state feedback gain is also given.

Theorem 1: *Let Φ_C be the convex set defined as follows; $(L,X,Y,\alpha) \in \Phi_C$ if*

$$Q + B_1B_1' < 0, \tag{3.7}$$

$$\begin{bmatrix} Q+\alpha B_1B_1' & XC_\infty' + L'D_\infty' \\ C_\infty X + D_\infty L & -\alpha I\end{bmatrix} < 0, \tag{3.8}$$

$$\begin{bmatrix} Y & C_2X + D_2L \\ XC_2' + L'D_2' & X\end{bmatrix} > 0 \tag{3.9}$$

where

$$Q := AX + XA' + B_2L + L'B_2'.$$

Then we have

$$\phi_\alpha = \inf\{trace(Y) : (L, X, Y, \alpha) \in \Phi_C\} \quad (3.10)$$

and if $\Phi_C \neq \{\phi\}$, the optimal state feedback gain which yields ϕ_α is given by

$$F^* = L^* X^{*-1}. \quad (3.11)$$

where $(L^*, X^*, Y^*, \alpha^*) \in \bar{\Phi}_C$ is the solution to the optimization problem (10) and $\bar{\Phi}_C$ is the closure of Φ_c.

Proof: Defining

$$X := \alpha P, \quad L := FX, \quad (3.12)$$

the conditions $ric(P) < 0$, $lyap(\alpha P) < 0$ and $oc(\alpha P) < Y$ in Φ_α can be expressed by

$$\begin{array}{l} AX + XA' + B_2L + L'B_2' + \alpha B_1B_1' \\ +(XC_\infty' + L'D_\infty')(\alpha I)^{-1}(C_\infty X + D_\infty L)' < 0 \end{array} \quad (3.13)$$

$$AX + XA' + B_2L + L'B_2' + B_1B_1' < 0 \quad (3.14)$$

$$(C_2X + D_2L)X^{-1}(C_2X + D_2L)' < Y. \quad (3.15)$$

Using the equivalence

$$\left.\begin{array}{c} A + BCB' < 0 \\ C > 0 \end{array}\right\} \Longleftrightarrow \left[\begin{array}{cc} A & B \\ B' & -C \end{array}\right] < 0,$$

and the conditions $\alpha > 0$ and $P > 0$ in Φ_α, (3.13) and (3.15) reduce to (3.8) and (3.9), respectively. The convexity of the set Φ_C is obvious since it is characterized by linear matrix inequalities. The state feedback gain F is obtained by solving (3.12) for F.

$\square$

If the infimum of the problem (3.10) is not attained, then the optimal state feedback gain F^* given by (3.11) does not satisfy $F^* \in \mathcal{F}_\infty$ and lies on the boundary of the set $\mathcal{F}_\infty$ where $A + B_2F^*$ has an eigenvalue on the $j\omega$-axis or $\|G_\infty\|_\infty = 1$. However, in practice, the open convex set Φ_C in the problem (3.10) can be approximated by a closed convex set $\hat{\Phi}_C(\epsilon) \subset \Phi_C$ defined by adding (or subtracting) ϵI to (3.7)-(3.9) where $\epsilon > 0$ is a sufficiently small scalar and replacing the strict inequality $<$ by $\leq$. In this case, the state feedback gain (3.11) always satisfies $F^* \in \mathcal{F}_\infty$ and attains the optimal value $\min\{trace(Y) : (L, X, Y, \alpha) \in \hat{\Phi}_C(\epsilon)\}$, which can be made arbitrarily close to ϕ_α by choosing arbitrarily small $\epsilon > 0$. To solve the convex optimization problem, standard methods such as the cutting plane technique (Geromel et al, 1991) and the ellipsoid algorithm (Rotea (to appear)) can be applied. See Beck (1991), Boyd et al. (1991) for more general review of convex programming methods relevant to control engineering.

4 CONCLUSION

We have defined a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem with an improved covariance bound. It is shown that the problem can be solved by a convex optimization programming.

Although we considered only the $\mathcal{H}_2$ norm as an additional performance measure, it is straight forward to incorporate some other specifications such as the L_2 to L_∞ gain of the closed loop system (Rotea and Khargonekar, 1991; Wilson, 1989; Zhu and Skelton, 1992) by minimizing the maximum diagonal entry of Y or the maximum singular value of Y, in which case, the objective function is still convex.

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