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Stability Radius Optimization and Loop Transfer Recovery for Uncertain Dynamic Systems

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Abstract

The difficult problem of robust stabilization and performance of dynamic systems under structured uncertainty motivates one to look into alternative solutions. Due to the conservatism and limitation associated with two lines of research; namely, robust recovery of LQR in LQG design and quadratic stabilization, we introduce a new feasible solution to this problem. The main idea is to design controllers for robust stabilization of uncertain systems such that the resulting closed-loop systems are structurally constraint to certain classes of systems with desirable properties. The special classes of systems considered in this paper are nonnegative and metzlerian. We make use of previously reported results for these classes of systems and introduce a new design approach called stability radius optimization loop transfer recovery, SRO/LTR, which can be regarded as a two step procedure similar to the LQG/LTR method. However, the flexibility offered by the SRO allows constraints to be imposed on the structure of the resulting closed-loop target feedback loop.

1 Introduction

The stability robustness can be investigated based on perturbations in the transfer function models or parameter variations in the state space models. It has been recognized that the robust stability for one form does not necessarily mean the robust stability for the other since the relation between corresponding perturbations is complex. Consequently, in LQR, the robustness with respect to parameter variations in the state space models is quite different from the robustness with respect to the multiplicative perturbation in the transfer function models. It is known that the LQR may become unstable even with small parameter variations and the stability of LQR with Observer or Kalman filter, LQG, may be very sensitive to small parameter variations even when the LQG/LTR [1] or any other alternative LTR method [2], [3] is used. So, as a first step of our analysis, it is important to discuss the bounds of allowable parameter variations in LQR and LQG. Let us consider the continuous-time system described by

$$\dot{x}(t) = A_\Delta x(t) + B_\Delta u(t) \quad (1)$$

$$y(t) = C_\Delta x(t) \quad (2)$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$ and the matrices A_Δ, B_Δ, C_Δ are of appropriate dimensions. We assume that the nominal system is controllable and observable and consider perturbation ΔA on the matrix A only, i.e. $A_\Delta = A + \Delta A$, $B_\Delta = B$, $C_\Delta = C$. The observer or Kalman filter may be expressed by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)) \quad (3)$$

and with the control law $u(t) = F\hat{x}(t)$, the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BF + \Delta A & -BF \\ \Delta A & A - KC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (4)$$

where $e(t) = x(t) - \hat{x}(t)$. Note that F and K are obtained through the well-known algebraic Riccati equations or by other alternative methods.

Using Lyapunov theory one can provide robust stability bounds for LQR and LQG [4]-[8]. It has been shown [7],[8] that under certain restrictive assumptions the bounds of allowable parameter variations in the LQG can become as large as those bounds of the LQR in the state space models. Although these bounds are conservative and less conservative bounds are available [9], they are useful in the design of LQR and LQG for systems under structured uncertainty. It has also been established that the robustness of the LQG can be the same as that of LQR regardless of the structural assumption (matching conditions) for model uncertainty provided that the regulator and observer gains have special forms. The conservativeness issue makes one to look into an alternative robust stabilization solution, which can be tied to the properties of the linear quadratic regulator solution. In this connection, the quadratic stabilization results established in [10], [11],[12] play an important role. See also [13], [14], [15], [16], [17] for further development in this direction. Based on [12] a recent publication [18] considers two types of uncertainty structures; namely, norm-bounded and convex-bounded uncertainties, and establish the relationship between quadratic stabilizability of linear systems and the existence of a positive definite solution to a set of modified Riccati equations. Also, [17] considers the constraint stabilization and performance for general uncertain interval systems as an extension of [15]. This structurally constraint stabilization takes into account the nice properties of metzlerian systems and establish strong results in connection to quadratic stabilization.

In this paper we avoid quadratic stability idea and introduce a new design strategy called stability radius optimization loop transfer recovery (SRO/LTR). The motivation behind this new idea lies in the fact that recently, in light of several outstanding results (see [19], [20] and references therein), direct stability radius formulae are obtained for the classes of nonnegative and metzlerian systems [21]. This allows one to determine the supremum of the stability radius which can be achieved by linear state feedback control law in terms of the feedback gain. The constraints in this optimization problem are formulated through stability conditions and special structure associated with metzlerian systems. The solution of this stability radius optimization (SRO) plays the same role as target feedback loop design in LQG/LRT method. However, SRO allows additional constraints to be imposed in the optimization problem. It is clear that similar to LQG/LTR design methodology we may add an observer structure to SRO (whenever the states are not available) in such a way that robustness properties are preserved to the extend possible. This leads to SRO/LTR design methodology.

2 Stability Radii for Nonnegative and Metzlerian Systems

This section considers the special classes of nonnegative and metzlerian systems and provide procedures to compute their real and complex stability radii. First, we briefly discuss the nonnegative and metzlerian stabilizations for general uncertain discrete and continuous systems, respectively. Then, under the assumption that the system is nonnegative or metzlerian stable, we give direct formulae for their stability radii computations.

2.1 Nonnegative and Metzlerian Stabilizations

The nonnegative stabilization of uncertain interval discrete systems has been treated in [22]. In order to have a flavour of structurally constraint stabilization, let us consider the dual results of [22] for the continuous-time metzlerian system. For the sake of parallel treatment, we assume interval characterization of uncertain system (1) where only A is uncertain and is defined by the set

$$M(P,Q) = \{A(a_{ij}) : P(p_{ij}) \le A(a_{ij}) \le Q(q_{ij})\} \quad (5)$$

for all $p_{ij} \le a_{ij} < q_{ij}$, $i,j = 1,\ldots,n$.

Definition: The interval set $M(P,Q)$ becomes metzlerian set $\hat{M}(P,Q)$ if and only if $a_{ii} < 0$, $a_{ij} > 0$.

Theorem 2.1 *Let (1) be metzlerian system characterized by $\hat{M}(P,Q)$. Then (1) is stable for all $A \in \hat{M}(P,Q)$ if and only if $|-Q(\alpha)| > 0$, where $\alpha \in N$ is the order of the leading principal minor.*

It should be noted that strong stability bound can be found for this class of systems similar to the dual case of nonnegative systems. This will not be given here, rather, we state the following result.

Theorem 2.2 *The feedback control law $u = Fx$ robustly stabilize the general interval system (1) characterized by $M(P,Q)$ and makes the closed-loop system metzlerian if and only if the matrices*

$$\hat{P} = P + BF \quad (6)$$

$$\hat{Q} = Q + BF \quad (7)$$

are metzlerian and that the leading principal minors of the matrix $-\hat{Q}$ are positive.

Thus the set of robustly stabilizing state feedback controllers maintaining metzlerian structure can be characterized by

$$\hat{M}(F) = \{F \in R^{m\times n} : \hat{P}(F)_{ii} < 0,\ \hat{P}(F)_{ij} > 0\ \hat{Q}(F)_{ii} < 0,\ \hat{Q}(F)_{ij} > 0,\ |-\hat{Q}(\alpha)| > 0\} \quad (8)$$

If the feedback solution to the set (8) exists, then one can formulate the problem in terms of a mathematical programming with any desirable objective function.

2.2 Real and Complex Stability Radii

Consider the discrete and continuous time systems:

$$x(k+1) = Ax(k), \quad k \in N \tag{9}$$
$$\dot{x}(t) = Ax(t), \quad t \in R_+ \tag{10}$$

which are subjected to parameter perturbations. It is assumed that the unperturbed system is stable, i.e. $A \in F^{n\times n}$, $\lambda(A) \subset C_g$ where the good stability regions for (10) and (9) are defined by $C_g = \{s \in C; Re(s) < 0\}$ and $C_g = \{z \in C; |z| < 1\}$, respectively. Perturbations of the form $\tilde{A} = A + \Delta A$ where ΔA represented by single perturbation structure $\Delta A = D\Delta E$ with given $D \in F^{n\times l}$, $E \in F^{q\times n}$ and unknown $\Delta \in F^{l\times q}$ is considered. The structured stability radius of A with respect to perturbation structure (D, E) is defined in [19] as

$$r_F(A, B, C) = \inf\{\|\Delta\| : \Delta \in F^{l\times q},\ \lambda(A + D\Delta E) \cap C_b \neq 0\} \tag{11}$$

where F is R or C and C_b is the unstable (bad) region. If both structure matrices D and E are identity matrices, we obtain the unstructured stability radius $r_F(A, I, I) = r_F(A)$. If A, D E are real, we obtain two stability radii, r_R or r_C, according to whether real ($F = R$) or complex ($F = C$) perturbations are considered. They are known as real and complex stability radii, respectively. The complex unstructured stability radius of A, as alternatively defined in [20], using spectral norm is given by

$$r_C(A) = \inf\{\bar{\sigma}(\Delta) : \Delta \in C^{n\times n} \text{ and } A + \Delta \text{ is unstable}\} \tag{12}$$

The computation of $r_C(A)$ is not difficult and can be obtained by

$$r_C(A) = \inf_{\omega\in R} \underline{\sigma}(A - j\omega I_n) \tag{13}$$

using available algorithms reported in the literature (see [19] and the references therein). Consequently for the special classes considered in this paper one can use the same algorithms to obtain $r_C(A)$. However, using the properties of these special classes, it is possible to compute the stability radius more efficiently. It is well-known that

$$r_R(A) \geq r_C(A) \tag{14}$$

and that the computation of real stability radius is difficult and needs extra care. Consequently lower bounds are established for real unstructured stability radius $r_R(A)$ in [20] and references therein. It was claimed in [20] that the reported new lower bound may turn out to be equal to the real stability radius. Motivated by this claim, we investigated in [21] the computation of real stability radius for other special matrices in order to widen the class of matrices for which the validity of their claim is confirmed. Although for the general case, it has recently been proven [24] that the new bound is in fact equal to the real stability radius, its computation however requires frequency sweeping and it is not trivial. Nevertheless an algorithm is provided to compute the real stability radius.

In subsequent sections we consider the computations of stability radii for important classes of nonnegative and metzlerian matrices. Strong necessary and sufficient conditions exist for the stability of these matrices [22], [23]. We will make use of these results and the spectral norm to compute the real stability radii. We show that simple charactarizations of real stability radii for these special classes of matrices lead to direct formulae for their computations.

2.3 Direct Formula for Stability Radius of Nonnegative Systems

The system (9) is nonnegative if $A \geq 0$ elementwise, i.e. $a_{ij} \geq 0$. Let the perturbed nonnegative system matrix be described by

$$\tilde{A} = A + D\Delta E \tag{15}$$

The following theorem on nonnegative matrices is essential for the development of our result. We state this theorem with respect to $\tilde{A}$ defined in (15).

Theorem 2.3 *The nonnegative system (9), with perturbation (15) is stable if and only if all the leading principal minors of $I - \tilde{A}$ are positive, i.e.*

$$|I - \tilde{A}|_\alpha > 0 \tag{16}$$

where $\alpha \in N$ is the order of the principal minor.

In a recent publication [21] we gave a crude analysis of real stability radius associated with the above system. Here, we provide a more compact treatment of it and summarize the results without proofs. Consider the nonnegative system whose stability radius is defined by

$$r_R^n(A, D, E) = inf\{\|\Delta\| \quad : \quad \lambda(A + D\Delta E) \cap C_b \neq 0, \quad A + D\Delta E \geq 0\}$$

Note that the condition $A + D\Delta E \geq 0$ is needed in order to ensure nonnegativity of the entire set. Obviously, A has to be stable in order for the problem to be meaningful (otherwise, $r_R^n(A, D, E) = 0$), i.e. $A \geq 0$, $\rho(A) < 1$. This will be assumed throughout.

Theorem 2.4 *Let A be C_g stable. Then the real stability radius of the nonnegative system can be charactarized by*

$$r_R^n(A, D, E) = inf\{\|\Delta\| \quad : \quad det(I - E(I - A)^{-1}D\Delta) = 0\} \tag{17}$$

Suppose that $\|\ \|$ is the 2 - norm , then from theorem 2.4, the following results can be derived.

Corollary 2.1 *Let A be C_g stable. Then the real stability radii are given by the following direct formulae depending on the associated charactarization of Δ.*
__Case 1__ *: Real unstructured Δ*

$$r_R^n(A, D, E) = \frac{1}{\bar{\sigma}(E(I - A)^{-1}D)} \tag{18}$$

__Case 2__ : $\Delta = diag(\delta_1, \delta_2, \ldots \delta_n)$

$$r_R^n(A, D, E) = \frac{1}{\max_{Q\in\mathbf{Q}}\ \rho_R(E(I - A)^{-1}DQ)} \tag{19}$$

where

$$\mathbf{Q} = \{diag(\delta_1, \delta_2, \ldots \delta_n) : \delta_i = \pm 1, i = 1, \ldots, n\}$$

$$\rho_R(M) = max_i\ \{|\lambda_i| : \lambda_i \in R, det(\lambda_i I - M) = 0\}$$

__Case 3__ *: Define $\|\Delta\| = max_{i,j}|\delta_{i,j}|$, where*

$$\Delta = \begin{pmatrix} p_{11}\delta_{11} & \cdots & p_{1n}\delta_{1n} \\ p_{21}\delta_{21} & \cdots & p_{2n}\delta_{2n} \\ \vdots & \vdots & \vdots \\ p_{11}\delta_{11} & \cdots & p_{1n}\delta_{1n} \end{pmatrix}, p_{i,j} \geq 0$$

$$r_R^n(A, D, E) = \frac{1}{max_{Q\in\mathbf{P}}\rho_R(E(I - A)^{-1}DQ)} \tag{20}$$

where

$$\mathbf{P} = \left\{ \begin{pmatrix} p_{11}\delta_{11} & \cdots & p_{1n}\delta_{1n} \\ p_{21}\delta_{21} & \cdots & p_{2n}\delta_{2n} \\ \vdots & \vdots & \vdots \\ p_{11}\delta_{11} & \cdots & p_{1n}\delta_{1n} \end{pmatrix}, \delta_{ij} = \pm 1 \right\}$$

2.4 Direct Formula for Stability Radius of Metzlerian Systems

The system (10) is metzlerian if $a_{ii} < 0$, $a_{ij} \geq 0$. Similar to theorem 2.3 we can state a theorem for the perturbed form of continuous-time metzlerian system (10) as follows.

Theorem 2.5 *The metzlerian system (10) with perturbation (15) is stable if and only if all the leading principal minors of $-\tilde{A}$ are positive, i.e.*

$$|-\tilde{A}|_\alpha > 0 \tag{21}$$

where $\alpha \in N$ is the order of the principal minor.

The proof of theorem 2.5 follows immediately from theorem 2.3 or characterization of closely related M matrices.

Fact 1 Any stable metzlerian matrix A can be split as $A = -\rho I + R$, where $R \geq 0, \rho \geq \rho(R)$.
So, for metzlerian system $\tilde{A} = A + D\Delta E$, we can write

$$\tilde{A} = A + D\Delta E = -\rho I + R + D\Delta E$$

Furthermore, in order for $\tilde{A}$ to remain metzlerian, it is necessary that $R + D\Delta E \geq 0$. It follows from these observation that

$$\begin{aligned} r_R^m(A, D, E) &= \inf\{\|\Delta\| : \quad \det(-\rho I + R + D\Delta E) = 0\} \\ &= \inf\{\|\Delta\| : \quad \det(I - \Delta E(\rho I - R)^{-1}D) = 0\} \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \max_i Re\lambda_i(\tilde{A}) &= -\rho + \max_i Re\lambda_i(R + D\Delta E) \\ &= -\rho + \rho(R + D\Delta E) \quad (\text{since } R + D\Delta E \geq 0) \end{aligned}$$

Remark Up to this point, the problem with metzlarian system has been transformed into one with nonnegative systems. So, the above formulae derived for nonnegative systems can be applied to the metzlerian systems directly with appropriate modifications. To avoid this conversion, one can simply replace $I - A$ by $-A$ in the derived formulae to obtain equivalent results for the metzlerian case.

3 Stability Radius Optimization

Let us go back to the controlled perturbed system (1) with $\Delta A = D\Delta E$ i.e.

$$\dot{x} = (A + D\Delta E)x + Bu \tag{22}$$

Our goal is to determine the supremum of the stability radii which can be achieved by linear state feedback $u = Fx$:

$$r_F^s(A, D, E, B) = \sup\{r_F(A + BF, D, E) : F \in C^{m\times n}, \lambda(A + BF) \subset C_g\} \tag{23}$$

If $F = C$, then the problem is called complex stability radius maximization. On the other hand, if $F = R$, then the problem is called real stability radius maximization. For the former case, by definition

$$r_C^s(A + BF, D, E, B) = r_C^s(A, D, E, B)\ , \quad \lambda(A) \subset C_g \tag{24}$$

and the optimization problem (23) can be shown to be equivalent to the H_∞ norm minimization problem

$$\min \|G_F\|_\infty \ \textit{subject to} \ \lambda(A+BF) \subset C_g \tag{25}$$

where $G_F(s) = E(sI - A - BF)^{-1}D$ and $\|G_F\|_\infty = \max_{\omega \in R} \|G_F(j\omega)\|$. The problem of finding a stabilizing feedback controller such that the resulting closed-loop transfer function has an H_∞ norm strictly less than a given bound γ, is known as the H_∞-optimal control problem. Many papers appeared in the past on this subject and certainly the above equivalency establishes a link between two areas of complex stability radius optimization and H_∞ optimal control problem. In summary, since it is possible to characterize the complex stability radius via an associated Riccati equation, one can also characterize the supremal achievable complex stability radius for a controlled system (using linear state feedback) by a modified Riccati equation (see [19] for more details):

$$(A+BF)^T P + P(A+BF) - PDD^T P - \rho^2 E^T E - \epsilon^2 F^T F = 0 \tag{26}$$

where $\epsilon, \rho > 0$. Starting with $F = \epsilon^{-2} B^T P_0$ one solves iteratively (26) to yield an increasing solution sequence P_k. The supremum solution P leads to $r_C^* = \lim_{\epsilon \to 0^+} \rho(\epsilon)$. Corresponding to this modified Riccati equation one can also form a Hamiltonian matrix and its associated Hermitian frequency function $H_\epsilon(j\omega)$

$$H_\epsilon(j\omega) = G_D(j\omega)G_D(j\omega)^* - \epsilon^{-2} G_B(j\omega)G_B(j\omega)^* \tag{27}$$

where $G_D(s) = E(sI - A)^{-1}D$ and $G_B(s) = E(sI - A)^{-1}B$. Denoting $\lambda(\epsilon) = \sup_{\omega \in R} \lambda_{max}(H_\epsilon(j\omega))$ and $\lambda_0 = \lim_{\epsilon \to 0} \max\{\lambda(\epsilon), 0\}$ we have the following result

Theorem 3.1 *Suppose that $\lambda(A) \subset C_g$. Then*

$$(i) \quad r_C^*(A, D, E, B) \leq \lambda_0^{-\frac{1}{2}}$$
$$(ii) \quad \delta(A, D, E, B) = \lambda_0^{-\frac{1}{2}}$$

where $\delta(A, D, E, B)$ is the complex stabilizability radius of the system $\dot{x} = Ax + Bu$ defined by

$$\delta(A, D, E, B) = \inf\{\|\Delta\|,\ \Delta \in C^{l \times q}, (A + D\Delta E, B) \textit{ is nonstabilizable}\}$$

As a consequence of Theorem 3.1, we have a characterization of those systems whose complex stability radius can not be improved by static linear state feedback.

Fact 2 The complex stability radius of a stable system is optimal with respect to state feedback if and only if its stability radius is equal to its stabilizability radius. Hence, if $r_C^*(A, D, E, B) < \delta(A, D, E, B)$ then there does not exist a feedback matrix maximizing the stability radius. In other words, this gap causes high gain feedback.

Obviously the above result is also valid for the class of metzlerian system and no further discussion is necessary. For the real stability radius maximization, there is no general result available. A recent result shows that the real stability radius can be computed by means of a two parameter optimization [24]. Consequently, due to its computational complexity, it is not yet apparent how to tackle its maximization problem. Also there is no counterpart of the real stability radius in H_∞ theory and there is no obvious connection to any type of Riccati equation. This needs further investigation and is a subject of future research.

Concentrating on the class of metzlerian systems, the problem of real stability radius maximization can be formulated and solved effectively. This is due to fact that closed form expression has been derived for real stability radius of this class of systems as we discussed in previous section. Let us formulate the problem formally.

Problem Formulation: Given the system (22) find if possible a feedback control law $u = Fx$ such that the closed-loop system is metzlerian stable and the associated real stability radius

$$r(F) = \frac{1}{\bar{\sigma}(-E(A+BF)^{-1}D)} \tag{28}$$

is maximized.

Note that we considered only (28) for simplicity. However, all stability radii expressions derived in Section 2 for the case of metzlerian can be used as well.

Theorem 3.2 *The matrix F solves the above problem if and only if the following optimization problem has a solution*

$$\max \ r(F) \quad \textit{subject to} \quad F \in \tilde{M}(F) \tag{29}$$

where $\tilde{M}(F) = \{F \in R^{m \times n} : (A+BF)_{ii} < 0,\ (A+BF)_{ij} \geq 0,\ (-|A+BF|)_\alpha > 0\}$ with α denoting the order of the leading principal minor.

The optimization problem (29) is termed stability radius optimization (SRO). It is clear that similar to LQG/LTR design methodology we may add an LTR observer structure to SRO (whenever an observer is necessary) in order to preserve the robustness properties. This leads to SRO/LTR design methodology which will be explored further in our illustrative example. Before doing this, let us consider an important special case, which makes (29) linear.

Corollary 3.1 *The system (22) with single input control law $u = Fx$ realizing metzlerian stabilization, and rank one perturbations $l = q = 1$ has*

$$r(F) = \left| \frac{\det(A+BF)}{\sum_i \det(A+BF)_i} \right| \tag{30}$$

where $(A+BF)_i$ is a matrix constructed from $A+BF$ by replacing its ith row (column) with the ith row (column) of the matrix product DE. Furthermore, the associated optimization problem (29) becomes a linear programming problem provided that $\sum_i \det(A+BF)_i$ is constant.

Note that one possiblity for this sum to be constant is $B = D$. A reliable method for solving the general optimization problem (29) with various $r(F)$ is currently under investigation.

Example: Consider the unstable continuous system specified by the triple $\{A, B, C\}$ as

$$A = \begin{bmatrix} 7 & -6 & -1 \\ 11 & -12 & 0 \\ 12 & -7 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

with perturbation structure $D = [1\ 1\ 1]^T$ and $E = [1\ 1\ 1]$. A satisfactory target feedback loop has been realized for the above system using LQR with $Q = I$, $R = 1$. The resulting feedback gain is given by $F = [-4.9444\ 1.7871\ 0.5933]$ which yields the real stability radius $r = 0.66189$. Note that $FB \neq 0$ and exact LTR can not be achieved by full-order observer [2]. However, since the system is minimum phase and $CB = 1$, one can design a reduced-order observer to achieve exact LTR.

Now we apply our SRO technique without requiring $FB = 0$. To achieve metzlerian stabilization with $F = [f_1\ f_2\ f_3]$ and maximizing the stability radius, the following optimization problem should be solved

$$\begin{aligned} \max\ r &= \frac{1}{\bar{\sigma}(E(-A-BF)^{-1}D)} \\ &= -.3194 - .3472f_1 - .2917f_2 - .3611f_3 \end{aligned}$$

subject to

$$\begin{aligned} &-11 < f_1 < -7 \\ &7 < f_2 < 12 \\ &1 < f_3 < 5 \\ &-6f_1 - 4f_2 - 18 > 0 \\ &-(23 + 25f_1 + 21f_2 + 26f_3) > 0 \end{aligned}$$

The objective function and the constraints are all linear and the solution to the linear programming occurs at the corner point feasible solution $f_1 = -11$, $f_2 = 7$, $f_3 = 1$ with $r^* = 1.0972$. Note that the coefficients of the objective function are all rounded off. Hence, the value of $r(F)$ becomes 1.0968, which is slightly less than the optimal value r^*. In order to show the flexibility of SRO, we choose the suboptimal solution $F = [-10.996\ 7.1\ 1.1]$ which realizes a similar feedback loop as the one in optimal LQR with equal phase margin of about 71^0. The real stability radius is computed with the above F and is given by $r^* = 1.03056$, which is larger than the one obtained from LQR. To recover this robustness property when observer is in the loop, we design the following reduced-order observer (see [2] and [3] for more details)

$$\begin{aligned} \dot{z} &= \Phi z + Gy \\ \hat{x} &= Mz + Ny \end{aligned}$$

with

$$\Phi = \begin{bmatrix} -5 & 0 \\ 1 & -5 \end{bmatrix}, \quad G = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note that Φ has also metzlerian structure with real stability radius $r = 2.27275$. To check the above result, one should compute the stability radius of the closed-loop system matrix consisting of the feedback control law and the reduced order observer.

Next we show that the condition $FB = 0$ can be incorporated in the linear programming problem above to obtain the feedback control law and design a full-order observer in the LTR step.

The solution to the linear programming problem with the additional constraint $f_1 + f_2 + f_3 = 0$ yields the suboptimal solution $F = [-10.99\ 9.98\ 1.01]$ with $r = 0.22097$ and the phase margin of the target feedback loop equals to 45^0. In this case the stability radius is reduced and it is smaller than the one obtained in LQR. However, it is still possible to design a full-order observer in the recovery stage. This full-order observer as expressed by equation (3) is realized by $K = [1\ 0\ 1]^T$. The eigenvalues of $A - KC$ are at $-1, -5, -5$ and the stability radius of the closed-loop system matrix as defined by (4) is 0.22097 which confirms the robust recovery.

4 References

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