

THE H_∞ CONTROL PROBLEM USING STATIC OUTPUT FEEDBACK

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SUMMARY

In this paper we shall consider the H_∞ control problem using static output feedback. The approach uses some recent results from linear algebra. The main result shows that the H_∞ control problem is solvable by a static output feedback controller if and only if there exists a positive definite matrix satisfying two certain quadratic matrix inequalities. A parametrization of all static output feedback H_∞ controllers is given.

KEY WORDS H_∞ control Static output feedback Linear matrix inequality Covariance control

1. INTRODUCTION

Many linear controller design analysis problems can be reduced to certain matrix linear algebra problems of the type studied in the covariance control literature ([4–6] and references therein.) For further motivation see the linear matrix inequalities discussed in Reference 3. In fact, the parametrization of all stabilizing controllers (of order equal to or less than the order of the plant) for a large variety of situations (continuous and discrete systems, with or without measurement noise) all reduce to the solution of just *two* linear algebra problems. In this paper we shall apply these linear algebra results to solve the standard H_∞ control problem for static output feedback control [7–10]. Using the linear algebra results we obtain necessary and sufficient existence conditions and a parametrization of all stabilizing static H_∞ controllers.

P^+ will in the following denote the Moore–Penrose inverse of a matrix P . Everywhere, as usual, $Q > 0$ shall be taken to mean that Q is a positive definite, symmetric matrix. SVD stands for the singular value decomposition.

2. PRELIMINARIES

In the sequel we shall consider the following system:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x \end{aligned} \tag{1}$$

for which we wish to consider static output feedback laws

$$u = Ky$$

where K is a matrix of appropriate dimensions.

The following results can be easily obtained using a linear algebra result found in Reference 5. A proof will be given for completeness.

Theorem 1

Consider the system (1). Suppose $D_1^* D_{12} > 0$ and let a scalar $\gamma > 0$ be given. Then the following two statements are equivalent.

- (i) There exists a matrix $K \in \mathbb{R}^{m \times p}$, such that $A + B_2 K C_2$ is asymptotically stable, and such that $\|C_1 + D_{12} K C_2\|_\infty < \gamma$
- (ii) There exists $X > 0$ and $L \in \mathbb{R}^{n \times m}$ such that the following two are satisfied.

$$A'X + XA + C_1^* C_1 + \gamma^{-2} X B_1 B_1' X - (X B_2 + C_1^* D_{12}) R^{-1} (X B_2 + C_1^* D_{12})' + L L' < 0 \quad (2)$$

$$(I - C_2^* C_2) [(X B_2 + C_1^* D_{12}) R^{-1} (X B_2 + C_1^* D_{12})' - L L'] (I - C_2^* C_2) = 0 \quad (3)$$

where $R := D_1^* D_{12} > 0$.

In this case, all such static output feedback gains K are given by

$$K = [L V R^{-1/2} - (X B_2 + C_1^* D_{12}) R^{-1}]' C_2^* + Z (I - C_2 C_2^*), \text{ where } Z \text{ is arbitrary}$$

$$V = V_A \begin{pmatrix} I_r & 0 \\ 0 & U \end{pmatrix} V_B', \quad r = \text{rank}[(I - C_2^* C_2)L], \quad U U' = I \text{ arbitrary}$$

V_A can be found by the following singular value decomposition

$$(I - C_2^* C_2)L = U_A \Sigma_A V_A' \quad (4)$$

Similarly, V_B is given by the SVD

$$(I - C_2^* C_2)(X B_2 + C_1^* D_{12}) R^{-1/2} = U_A \Sigma_A V_B' \quad (5)$$

(where U_A and Σ_A are as in (4)).

Proof. We shall prove the case $\gamma = 1$ without loss of generality. From the strict bounded real lemma (e.g. Reference 12), statement (i) is equivalent to the following statement.

- (i) There exist $K, X > 0$ and $W > 0$ such that

$$(A + B_2 K C_2)' X + X(A + B_2 K C_2) + X B_1 B_1' X + (C_1 + D_{12} K C_2)' (C_1 + D_{12} K C_2) + W = 0 \quad (6)$$

Our strategy is the following: assuming that $X > 0$ and $W > 0$ are given, we shall solve the quadratic matrix equation (6) for K . In this case, solvability conditions are given in terms of X and W , which in turn characterize all matrix pairs (X, W) satisfying (6) for some K . The solution K will be given explicitly in terms of (X, W) .

To this end, define

$$T := X B_2 + C_1^* D_{12}, \quad R := D_1^* D_{12}$$

$$Q := A' X + X A + X B_1 B_1' X + C_1^* C_1 - T R^{-1} R'$$

Then, after expanding the terms in (6), completing the square with respect to $K C_2$ (6) becomes

$$(K C_2 + R^{-1} T)' R (K C_2 + R^{-1} T') = -Q - W$$

Since the left-hand side is nonnegative definite with rank $\leq m$, there exists $L \in \mathbb{R}^{n \times m}$ such that

$$\begin{aligned} Q + LL' &= -W < 0 \\ (KC_2 + R^{-1}T')' R(KC_2 + R^{-1}T') &= LL' \end{aligned} \quad (7)$$

Note that (7) is equivalent to the existence of an orthogonal matrix $V \in \mathbb{R}^{m \times m}$ such that

$$(KC_2 + R^{-1}T')' R^{1/2} = LV$$

or equivalently

$$C_2'K' = LVR^{-1/2} - TR^{-1}$$

The above linear equation is solvable for K if and only if

$$(I - C_2^+ C_2)(LV - TR^{-1/2}) = 0 \quad (8)$$

and all such solutions are given by

$$K' = C_2^{+'}(LVR^{-1/2} - TR^{-1}) + (I - C_2 C_2^+)Z$$

where $Z \in \mathbb{R}^{p \times m}$ is arbitrary. Using the result of Reference 5, there exists V such that $VV' = I$ and it satisfies (8) if and only if

$$(I - C_2^+ C_2)(LL' - TR^{-1}T')(I - C_2^+ C_2) = 0$$

and all such orthogonal matrices V are given by

$$V = V_A \begin{pmatrix} I_r & 0 \\ 0 & U \end{pmatrix} V_B', \quad r = \text{rank}[(I - C_2^+ C_2)L]$$

where U is an arbitrary orthogonal matrix, and V_A and V_B are given by the following SVDs;

$$(I - C_2^+ C_2)L = U_A \Sigma_A V_A', \quad (I - C_2^+ C_2)TR^{-1/2} = U_A \Sigma_A V_B'$$

This completes the proof. □

3. MAIN RESULTS

The main result of this paper is the following.

Theorem 2

Consider the system (1). The following two statements are equivalent.

- (i) There exists a matrix $K \in \mathbb{R}^{m \times p}$, such that when applying the static output feedback law $u = Ky$, the resulting closed loop system is internally stable, and the H_∞ norm from w to z is smaller than γ .
- (ii) There exists a positive definite solution X to the following two inequalities.

$$A'X + XA + C_1' C_1 + \gamma^{-2} X B_1 B_1' X - (X B_2 + C_1' D_{12}) R^{-1} (X B_2 + C_1' D_{12})' < 0 \quad (9)$$

$$V_2'(A'X + XA + C_1' C_1 + \gamma^{-2} X B_1 B_1' X) V_2 < 0 \quad (10)$$

where $R := D_{12}' D_{12}$ and

$$C_2 = (U_1 \quad U_2) \begin{pmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} (V_1 \quad V_2)' \quad (11)$$

is the singular value decomposition of C_2 .

Remark 1. Any matrix V_2 satisfying $\text{Im } V_2 = \text{Ker } C_2$ will work in (10). The explicit expression (11) is merely one possibility.

Proof. First, to prove necessity assume that there exists a matrix X satisfying (i). Applying Theorem 1 we immediately infer (9). Observing that $(I - C_2^+ C_2) = V_2 V_2'$ we get from (2) and (3) that

$$V_2'(A'X + XA + C_1' C_1 + \gamma^{-2} X B_1 B_1' X) V_2 < 0$$

proving necessity of (10).

To prove sufficiency, assume that $X > 0$ exists satisfying (9) and (10). We shall show the existence of L satisfying (2) and (3) constructively. Since $R > 0$ and $\text{rank}(R) = m$, it is possible to choose $T \in \mathbb{R}^{r \times m}$ as any matrix such that

$$TT' = V_2'(XB_2 + C_1'D_{12})R^{-1}(XB_2 + C_1'D_{12})' V_2 \tag{12}$$

Then with

$$\Omega := A'X + XA + C_1' C_1 + \gamma^{-2} X B_1 B_1' X - (XB_2 + C_1'D_{12})R^{-1}(XB_2 + C_1'D_{12})' < 0 \tag{13}$$

we have that

$$V_2'\Omega V_2 + TT' = V_2'(A'X + XA + C_1' C_1 + \gamma^{-2} X B_1 B_1' X) V_2 < 0 \tag{14}$$

Introducing

$$Q := V'\Omega V, \quad (\Omega = VQV')$$

or

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} := \begin{pmatrix} V_1'\Omega V_1 & V_1'\Omega V_2 \\ V_2'\Omega V_1 & V_2'\Omega V_2 \end{pmatrix}$$

we get from (13) and (14) that

$$Q < 0 \text{ and } Q_{22} + TT' < 0 \tag{15}$$

Applying Lemma 1 from Appendix A to (15) we obtain (constructively) the existence of $\Lambda \in \mathbb{R}^{(n-r) \times m}$ such that

$$Q + \begin{pmatrix} \Lambda \\ T \end{pmatrix} (\Lambda' \quad T') < 0$$

Finally, defining

$$L := V \begin{pmatrix} \Lambda \\ T \end{pmatrix}$$

we have

$$\Omega + LL' < 0$$

which is the inequality (2). The equality (3) follows directly from (12). This concludes the proof of Theorem 2. □

To emphasize the constructiveness of the proof presented above we can apply Theorem 1 to obtain the following parametrization of all static H_∞ controllers.

Corollary 1

Consider the system (1). Suppose there exists a static output feedback internally stabilizing controller $u = Ky$ such that the H_∞ norm of the transfer matrix from w to z is smaller than γ . Then all such controllers can be found by the following algorithm

1. Choose X satisfying (9) and (10)
2. Choose $Z \in \mathbb{R}^{m \times p}$ arbitrary
3. Choose $N \in \mathbb{R}^{(n-r) \times m}$ satisfying

$$\|S^{-1/2}N\| < 1, S = -Q_{11} + Q_{12}Q_{22}^{-1}Q_{12}'$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} = \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} [A'X + XA + C_1' C_1 + \gamma^{-2} X B_1 B_1' X - (X B_2 + C_1' D_{12}) R^{-1} (X B_2 + C_1' D_{12})'] \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

and V_1 and V_2 are given by the SVD of C_2 (11)

4. Compute T by

$$TT' = V_2'(X B_2 + C_1' D_{12}) R^{-1} (X B_2 + C_1' D_{12})' V_2, T \in \mathbb{R}^{r \times m}$$

5. Compute \bar{Q}_{22} and P by

$$\bar{Q}_{22} = Q_{22} + TT', P = I - T' \bar{Q}_{22}^{-1} T$$

6. Compute L by

$$L = \begin{pmatrix} \Lambda \\ T \end{pmatrix}, \Lambda = Q_{12} \bar{Q}_{22}^{-1} T P^{-1} + N P^{-1/2}$$

7. Compute the singular value decompositions

$$(I - C_2^+ C_2) L = U_A \Sigma_A V_A', (I - C_2^+ C_2) (X B_2 + C_1' D_{12}) R^{-1/2} = U_A \Sigma_A V_B'$$

(note that the first two matrices on the right-hand sides are identical)

8. Choose arbitrary U such that $UU' = I$ and compute V by

$$V = V_A \begin{pmatrix} I_r & 0 \\ 0 & U \end{pmatrix} V_B', r = \text{rank}[(I - C_2^+ C_2)L]$$

9. Compute the static H_∞ controller K by

$$K = [L V R^{-1/2} - (X B_2 + C_1' D_{12}) R^{-1}]' C_2^+ + Z(I - C_2 C_2^+)$$

Note that the freedom in Corollary 1 is given explicitly in terms of the parameters Z , N , U and the Riccati solution X .

Applying Theorem 2 and Theorem 1 we can get the state feedback result as a special case. Setting $C_2 = I$ we obtain the system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= x \end{aligned} \tag{16}$$

with $R := D_{12}' D_{12} > 0$, for which we have the following immediate result.

Corollary 2

There exists a state feedback $u = Fx$ such that the above closed-loop system is internally stable and has H_∞ norm from w to z smaller than 1 if and only if there exists an $X > 0$ such that

$$-\Phi := (A - B_2 R^{-1} D_{12} C_1)' X + X(A - B_2 R^{-1} D_{12} C_1) + C_1'(I - D_{12} R^{-1} D_{12}) C_1 + X(B_1 B_1' - B_2 R^{-1} B_2') X < 0$$

or equivalently if and only if there exists a $Y > 0$ such that

$$\begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} (A - B_2 R^{-1} D_{12} C_1)' & S \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A - B_2 R^{-1} D_{12} C_1 & 0 \\ S' & 0 \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} B_1 B_1' - B_2 R^{-1} B_2' & 0 \\ 0 & -I \end{pmatrix} < 0$$

where $SS' = C_1'(I - D_{12} R^{-1} D_{12}) C_1 \geq 0$.

Moreover, all possible state feedback gains are given by

$$F = -R^{-1}(D_{12} C_1 + B_2' X) + R^{-1/2} M \Phi^{1/2}$$

where M is an arbitrary matrix such that $\|M\| < 1$.

4. CONCLUSIONS

The existence of a static output feedback H_∞ controller is equivalent to the existence of a positive solution to two quadratic matrix inequalities. All static output feedback H_∞ controllers are parametrized explicitly (in closed form) in terms of this positive matrix and an additional free parameter.

The approach suggested in this paper is based on simultaneous solutions of two quadratic matrix inequalities. The first and the second inequalities in Theorem 2 define convex sets with respect to X^{-1} and X , respectively, but the set of all matrices X satisfying both inequalities simultaneously is not convex. Computational issues will be addressed for this nonconvex problem in subsequent papers. For convex problems (such as state feedback with convex sets of additional performance constraints specified by Y in Corollary 2), standard methods (e.g. References 11, 1 and 2) can be applied.

APPENDIX A: A MATRIX LEMMA

In this appendix we shall prove the following matrix result.

Lemma 1

Let $T \in \mathbb{R}^{r \times m}$ and

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

be given matrices with $Q < 0$, and assume that $\bar{Q}_{22} := Q_{22} + TT' < 0$. Then there exists a matrix $\Lambda \in \mathbb{R}^{(n-r) \times m}$ such that

$$Q + LL' < 0, \quad L := \begin{pmatrix} \Lambda \\ T \end{pmatrix} \quad (17)$$

and all such matrices Λ are given by

$$\Lambda = Q_{12}\bar{Q}_{22}^{-1}TP^{-1} + NP^{-1/2} \quad (18)$$

where $N \in \mathbb{R}^{(n-r) \times m}$ is any matrix such that

$$\|S^{-1/2}N\| < 1 \quad (19)$$

and

$$P := I - T'\bar{Q}_{22}^{-1}T, \quad S := -Q_{11} + Q_{12}\bar{Q}_{22}^{-1}Q_{12}' \quad (20)$$

Proof. Lemma 1 will be shown constructively. First note that

$$Q + LL' = \begin{pmatrix} Q_{11} + \Lambda\Lambda' & Q_{12} + \Lambda T' \\ Q_{12}' + T\Lambda' & \bar{Q}_{22} \end{pmatrix} < 0$$

is equivalent to

$$Q_{11} + \Lambda\Lambda' - (Q_{12} + \Lambda T')\bar{Q}_{22}^{-1}(Q_{12} + \Lambda T')' < 0 \quad (21)$$

since $\bar{Q}_{22} < 0$ by assumption. Noting that P defined by (20) is positive definite, we can complete the square with respect to Λ , then (21) becomes

$$(\Lambda - Q_{12}\bar{Q}_{22}^{-1}TP^{-1})P(\Lambda - Q_{12}\bar{Q}_{22}^{-1}TP^{-1})' < S \quad (22)$$

where $S > 0$ is defined in (20). Now, for any Λ satisfying (22), define

$$N := (\Lambda - Q_{12}\bar{Q}_{22}^{-1}TP^{-1})P^{1/2} \quad (23)$$

Then, from (22) N must satisfy

$$NN' < S \quad (24)$$

or equivalently, (19). Finally, solving (23) for Λ yields (18). Existence of Λ can be verified by choosing $N = 0$. This completes the proof. \square

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