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Transmission Zero Matrix and Reduced Order LTR Controller

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Abstract- In this paper, we define a new matrix corresponding to the transmission zeros of a linear time invariant system and design an LTR controller based on a functional observer having the same order as the number of transmission zeros. It is shown that under certain conditions ELTR can be achieved by a single functional observer eventhough CB is not full rank.

1 Introduction

To design an LTR controller $C(s)$ for the system Σ: $\{A, B, C\}$ having the transfer function $G(s)$, we first determine the desired target feedback loop with the transfer function

$$L_{TFL}(s) = F(sI - A)^{-1}B \tag{1}$$

where F represents the state feedback gain. Next the LTR step is performed in which we attempt to recover the target design over a range of frequencies by a dynamic compensator $C(s)$. Assuming that $C(s)$ is implemented via an observer-based controller, the resulting loop transfer function $C(s)G(s)$, in general, is not the same as the target loop transfer function $L_{TFL}(s)$. We define the loop transfer recovery error as

$$E_L(s) = L_{TFL}(s) - C(s)G(s) \tag{2}$$

and say that exact loop transfer recovery at the input point (ELTRI) is achieved if the closed-loop system comprised of $C(s)$ and $G(s)$ is asymptotically stable and $E_L(s) = 0$. To define asymptotic LTR at the input point (ALTRI), we parametrize the family of controllers as $C(s,q)$, where q is a positive scalar, and say that ALTRI is achieved if the closed-loop system is asymptotically stable and $C(s,q)G(s) \to L_{TFL}(s)$ pointwise in s as $q \to \infty$, i.e., $E_L(s,q) \to 0$ pointwise in s as $q \to \infty$.

As an equivalent measure of the quality of the recovery, we usually define the so-called recovery matrix $M_I(s)$, which can be related to $E_L(s)$. This matix is constructed according to the defined observer structure [8].

Consider the full-order P observer based controller having the transfer function

$$C(s) = F(sI - A - KC - BF)^{-1}K \tag{3}$$

where F and K are the regulator and observer gains, respectively. Then ELTRI is achieved if and only if $E_L(s) = 0$ or equivalently $M_I(s) = 0$ where

$$M_I(s) = F(sI - A - KC)^{-1}B. \tag{4}$$

In practice, the condition $M_I(s) = 0$ can not always be satisfied exactly. Consequently, the size of $M_I(s)$ should be made small in some sense.

Let the controller be parametrized in terms of the observer gain by $K(q)$. Then to obtain ALTRI we seek a $K(q)$ such that

$$M_I(s) = F(sI - A - K(q)C)^{-1}B \to 0 \quad \text{as} \quad q \to \infty. \tag{5}$$

The literature reports several methods for ELTR and ALTR [1], [17], [12]. These references also explore the trade-off between robust stability and performance, and the level of recovery which is related to the singular values of $M_I(j\omega)$. Consequently, recent results [13], [9], [16], consider LTR design methods which use $\mathcal{H}_\infty$ control theory. However, a major concern in $\mathcal{H}_\infty/LTR$ design is the high dimensionality of the controller. It has been recognized that the order of such a controller can be higher than the system order, usually $r = 2n$, and one should perform frequency weighted model reduction to obtain an n-th order controller.

Other approaches consider observer-based controllers having structural changes so that either ELTR or ALTR is achieved without large filter or regulator gains [14], [15]. However, the disadvantage of high order LTR controller occurs also when we use PI observers [15] to achieve time and frequency recovery. In this case, $r = n + p$, where p is the number of outputs. Similarily, Okada *et al.* [10] faced with several conflicting goals in their optimization technique and noticed that improving robust recovery level by using a precompensation or extended perfect model following methods leads to high order controllers. One immediate solution to this problem is to apply model reduction techniques. However, the degree of approximation manifest itself a degradation in the recovery performance.

The loss of robustness in observer-based systems is due to the path from the control signal u to the observer via the control distribution matrix B (or H in the reduced order observer case). Based on this observation Chen *et al.* [2], [3] removed this path at the outset of controller design. This technique leads to a new compensator design philosophy which is outside the realm of observer theory and, hence, the separation principle. Consequently, one must prove that closed-loop stability and LTR are simultaneously achieved. Chen *et al.* [2], [3] and Saberi *et al.* [12] established necessary and sufficient conditions for the existence of a recoverable target loop for observer-based and general compensator structures. These authors showed that, corresponding to full-order and reduced-order observers, one can design full-order and reduced-order compensators of orders n and $n-p$; respectively, to achieve either ELTR or ALTR.

In spite of the deep studies of the observer and LTR theories [11], [12], there are still several unresolved issues, which should be investigated. This paper deals with one important issue, namely, LTR controllers with low dimensions.

2 Main Results

The above discussion motivates one to look into the possibilities of designing low order LTR controllers. Unlike [2], we remain within the framework of observer theory, and attempt to define alternative observers of order $r < n-p$ to achieve ELTR or ALTR. In particular, we concentrate on functional observers, which are capable of realizing this requirement and purposely define the order exactly the same as the number of transmission zeros.

Let us assume that the stabilizible and detectable system

$$\Sigma: \quad \begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \tag{6}$$

has an equal number of inputs and outputs (i.e., $m = p$). Recall [11] that a reduced-order functional observer-based controller for Σ has the form

$$\Sigma_{RFC}: \quad \begin{cases} \dot{z} &= \Phi z + Gy + Hu \\ w &= Mz + Ny \end{cases} \tag{7}$$

under the following constraints:

$$Re[\lambda(\Phi)] < 0 \tag{8}$$
$$TA - \Phi T = GC \tag{9}$$
$$H = TB \tag{10}$$
$$MT + NC = F \tag{11}$$

where z is an r vector and w is an m vector, which estimates the control law $u = Fx$. Various methods exist in the literature to design minimal order functional observers [4], [6], [5], [11]. However, we should emphasize that the main idea here is not the minimal order, rather, to establish a relationship between functional observer of a particular order and the LTR theory. This order is dictated by the number of transmission zeros.

Theorem 1: *Let the system Σ be left invertible, minimum phase and have all of its infinite zeros of order one (i.e., let CB have full rank). Then the reduced-order functional observer-based controller Σ_{RFC} achieves both asymptotic stability of the closed-loop system and ELTRI if and only if $M_I(s) = M(sI - \Phi)^{-1}H = 0$ or equivalently $H = TB = 0$. Furthermore, Σ_{RFC} achieves both asymptotic stability of the closed-loop system and ALTRI iff the system Σ is left invertible and minimum phase.*

The above exact recovery condition implies that the reduced-order functional observer-based controller transfer function

$$C(s) = M(sI - \Phi)^{-1}G + N \tag{12}$$

has r poles identical to LTR observer poles, which are the r transmission zeros of the system Σ. For reduced-order P observer-based controller Σ_{RPC},

$r = n - m$, w becomes $\hat{x}$, $u = F\hat{x}$ and F in the constraint equation (11) is replaced by I. Also, Comparing Σ_{RPC} with a full-order P observer-based controller Σ_{PC} yields $\hat{x} = z$, $u = F\hat{x}$, $A + KC = \Phi$, $-K = G$, $B = H$, and $T = I$. Under the assumptions of the above theorem, it can be shown that ELTR with Σ_{PC} is possible if and only if $FB = 0$.

In this paper we concentrate on single-input single-output system and without loss of generality assume that (6) is in observable cannonical form i.e.

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}, \quad C = [0 \quad 0 \quad \cdots \quad 0 \quad 1] \tag{13}$$

Note that the system has the transfer function $G(s) = \frac{n(s)}{d(s)}$, where $n(s) = b_{n-1}s^{n-1} + \cdots + b_1 s + b_0$ and $d(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ is the characteristic polynomial of A. In this case, writing the $r \times n$ matrix T in terms of its columns $T = [t_1 \quad t_2 \quad \cdots \quad t_n]$ and substituting $\{A, C, T\}$ in (9) yields

$$t_j = \Phi^{j-1} t_1, \quad for \quad j = 1, \ldots, n \tag{14}$$

$$G = -d(\Phi) t_1 \tag{15}$$

Thus T has the general form of

$$T = [I_r \quad \Phi \quad \Phi^2 \quad \cdots \quad \Phi^{n-1}] t_1 \tag{16}$$

and we make use of the following result [5], which has not been tied to LTR theory before.

Lemma 1: *Suppose for some r and t_1 the matrix $[T' \quad C']'$ in (11) has full rank $r+1$. Then (11) has a solution for the pair $\{M, N\}$ if and only if*

$$\Gamma F' = 0 \tag{17}$$

where Γ is the $(n - r - 1) \times n$ matrix defined by

$$\Gamma = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{r-1} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \gamma_0 & \gamma_1 & \cdots & \gamma_{r-2} & \gamma_{r-1} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & & & & & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{r-1} & 1 & 0 \end{bmatrix} \tag{18}$$

for $r = 1, 2, \ldots, n-2$, where $\gamma(s) = s^r + \gamma_{r-1}s^{r-1} + \cdots + \gamma_1 + \gamma_0$ is the characteristic polynomial of Φ. Also, for $r = 0$ and $r = n-1$ the matrix Γ becomes $[I_{n-1} \quad 0]$ and $[0]$, respectively.

So, the search for an observer has been reduced to finding the pair $\{\Phi, t_1\}$ such that (11) has a solution for the pair $\{M, N\}$. It turns out that the controllable pair

$$\Phi = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\gamma_0 \\ 1 & 0 & \cdots & 0 & -\gamma_1 \\ 0 & 1 & \cdots & 0 & -\gamma_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\gamma_{r-1} \end{bmatrix}, \quad t_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{19}$$

guarantees the existence of the solution to (11).

Now, we are taking an LTR approach based on the above observer, which assumes r to be equal to the number of transmission zeros. The corresponding LTR controller constructed by this observer is called an *r-th order approximate transmission zero LTR controller*. To achieve $ELTR$ when the system is minimum phase, γ_i's are replaced by b_i's and we call Γ the *transmission zero matrix*. The corresponding controller is called an *r-th order exact transmission zero LTR controller*.

The forgoing results leads us to the following theorems.

Theorem 2: *Assume that the single-input single-output system is controllable, observable and not necessarily minimum phase. Then there exists an r-th order approximate transmission zero LTR controller for the system (6) with the control law $u = Fx$ if and only if the following optimization problem*

$$Min \parallel M_I(s) \parallel \quad subject \ \ to \quad \Gamma F' = 0 \tag{20}$$

has a stable solution $\gamma(s)$, where $M_I(s) = M(sI - \Phi)^{-1}H$ and $\parallel . \parallel$ is a suitable norm.

In view of Theorem 1, it is easy to see that the objective function can be replaced by $min \parallel TB \parallel$.

Theorem 3: *Assume that the single-input single-output system (6) is controllable, observable and minimum phase, and let Γ be the transmission zero matrix. Then there exists an r-th order exact transmission zero LTR controller, which achieves $ELTR$ if and only if $\Gamma F' = 0$ has a stable solution F.*

We developed two algorithms corresponding to the above theorems. The first algorithm assumes that the target feedback loop is prescribed in terms of the state feedback gain F, as it is usually assumed in the standard LTR theory. In this case, the functional observer poles are reflected in the matrix Γ and they will get close to the stable transmission zeros of the system such that $\Gamma F' = 0$. Note that for nonminimum phase systems, the mirror image of nonminimum phase zeros should be encountered. We called this algorithm "approximate zero assignment LTR algorithm", which basically solves the optimization problem (20) or equivalently $min \parallel TB \parallel$ subject to $\Gamma F' = 0$.

The second algorithm considers the possibility of designing a functional observer to achieve ELTR by defining the matrix Γ as the transmission zero matrix. In this case, one is seeking an $F' \in null(\Gamma)$ such that a satisfactory target feedback loop is realized. Note that the feedback gain, which satisfies (17) and guarantees the asymptotic stability of the closed-loop system, is constructed in a different fashion as compared to the first step of the conventional LQG/LTR approach. In LQG/LTR the designer is tweaking the quadratic weights to provide a satisfoctory target feedback loop. Here, one computes the singular value decomposition of transmission zero matrix and tweaks certain parameters such that the desired target feedback loop is realized.

3 Examples

The following two examples illustrate the interesting case of ELTR achievable by an *r-th* order functional observer.

Example 1: Consider the following system

$$A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -8 \\ 0 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = [0 \quad 0 \quad 1]$$

with its associated transfer function $G(s) = \frac{s+1}{s^3+5s^2+8s+6}$ which has a transmission zero at -1.

The transmission zero matrix is constructed as $\Gamma = [1 \quad 1 \quad 0]$ and the target feedback loop is realized by $F = [1 \quad -1 \quad 2]$.

Note that the feedback gain satisfies the condition $FB = 0$. However, since CB is not full rank, neither a full-order nor a reduced-order observer can achieve $ELTR$. Using the procedure outlined in this paper, the first order observer with the following parameters achieves $ELTR$.

$$T = [1 \quad -1 \quad 1], \quad \Phi = -1, \quad G = -2, \quad M = 1, \quad N = 1$$

The transfer function of this LTR controller is $C(s) = \frac{-2}{s+1} + 1$, which satisfies $G(s)C(s) = F(sI - A)^{-1}B$; and the closed loop system $G_t(s) = \frac{G(s)}{1 - G(s)C(s)}$ is stable with characteristic polynomial $\Delta_t(s) = s^3 + 5s^2 + 7s + 7$.

Example 2: Consider the following unstable system

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 10 \\ 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \quad 0 \quad 0 \quad 0 \quad 1]$$

with its associated transfer function $G(s) = \frac{s^2+3s+2}{s^5+5s^4+9s^3+3s^2-8s-10}$ which has two transmission zeros at -1 and -2.

The transmission zero matrix is constructed as

$$\Gamma = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{bmatrix}$$

and the target feedback loop is realized by

$$F = [-0.0716 \quad 0.0713 \quad -0.0706 \quad 0.0693 \quad -6.0001]$$

Note that the feedback gain satisfies the condition $FB = 0$. However, since $CB = 0$, neither a full-order nor a reduced-order observer can achieve $ELTR$. Applying the proposed method, the pair $\{\Phi, t_1\}$ is constructed as

$$\Phi = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad t_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $\{T, G\}$ are obtained by $T = [\,I_r \quad \Phi \quad \Phi^2 \quad \Phi^3 \quad \Phi^4\,]\,t_1$ and $G = -d(\Phi)t_1$ as

$$T = \begin{bmatrix} 1 & 0 & -2 & 6 & -14 \\ 0 & 1 & -3 & 7 & -15 \end{bmatrix}, \quad G = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

and $M = [-0.0716 \quad 0.0713]$, $N = -5.9334$. The transfer function of this LTR controller is

$$C(s) = \frac{-5.933s^2 - 18.0860s - 12.4402}{s^2 + 3s + 2}$$

which satisfies $G(s)C(s) = F(sI - A)^{-1}B$ and the closed loop system $G_t(s) = \frac{G(s)}{1 - G(s)C(s)}$ is stable.

To conclude this section, we provide a third example to illustrate the recovery procedure for nonminimum phase systems.

Example 3: Consider the system $G(s) = \frac{s-0.2}{s^3+1.9s^2+1.8s+0.2}$ which has a transmission zero at 0.2. By applying the factored plant model approach we decompose the plant $G(s)$ into all-pass and minimum phase factors $G_{ap} = \frac{s-0.2}{s+0.2}$ and $G_{mp} = \frac{s-0.2}{s^3+1.9s^2+1.8s-0.2}$ with the following state space realizations

$$A_{ap} = [-0.2], \quad B_{ap} = [-0.4], \quad C_{ap} = [1], \quad D_{ap} = [1]$$

$$A_{mp} = \begin{bmatrix} 0 & 0 & 0.2 \\ 1 & 0 & -1.8 \\ 0 & 1 & -1.9 \end{bmatrix}, \quad B_{mp} = \begin{bmatrix} 0.2 \\ 1 \\ 0 \end{bmatrix}, \quad C_{mp} = [0 \quad 0 \quad 1]$$

The target feedback loop is realized by $F = [10 \quad -2 \quad 1 \quad 10]$. The transmission zero matrix for minimum phase system is constructed as $\Gamma = [0.2 \quad 1 \quad 0]$. Note that $F_{mp}B_{mp} = 0$ but $C_{mp}B_{mp}$ is not full rank, so neither a full-order nor a reduced-order observer can achieve ELTR for the minimum phase part of the system. However, with a functional observer of order one we can achieve ELTR for G_{mp}. Using transmission zero matrix algorithm, we obtain

$$T = [1 \quad -0.2 \quad 0.04], \quad \Phi = -0.2, \quad G = 0.492, \quad M = 10, \quad N = 0.6$$

The open loop transfer function of the compensated system has the transfer function $[I - F_{ap}(sI - A_{ap})^{-1}B_{ap}]^{-1}[M(sI - \Phi)^{-1}G + N]G(s) = C(s)G(s)$. Thus, the transfer function of the controller is $C(s) = \frac{0.6s+5.04}{s+4.2}$ and the closed loop system is stable with characteristic polynomial $\Delta_t(s) = s^4 + 6.1s^3 + 9.18s^2 + 2.44s + 0.168$.

4 Conclusions

This paper considered the problem of loop transfer recovery based on functional observer. It was shown that one can design LTR controller of order equal to the number of transmission zeros and achieve exact recovery. An interesting connection of this result and the parametrization reported in [18] is currently under investigation. This will be particularly important for the discrete-time systems.

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