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Sensitivity Synthesis for MIMO Systems: A Multi Objective $\mathcal{H}_\infty$ Approach[1]

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Abstract

A series of multi objective $\mathcal{H}_\infty$ design problems are considered in this paper. The problems are formulated as a number of coupled $\mathcal{H}_\infty$ design problems. These $\mathcal{H}_\infty$ problems can be formulated as sensitivity problems, complementary sensitivity problems, or control sensitivity problems for every output (or input) in the system. It turns out that these multi objective $\mathcal{H}_\infty$ design problems, based on a number of different types of sensitivity problems, can be exactly decoupled into k $\mathcal{H}_\infty$ sensitivity problems for stable systems, where k is the number of outputs (for unstable systems, independent stabilization is required).

1. Introduction

The area of robust control has received tremendous attention in the control literature recently. Especially, $\mathcal{H}_\infty$ theory has been in the focus since its breakthroughs during the 1980's.

In the main, $\mathcal{H}_\infty$ control is motivated by the following two applications. First, if the modeling errors are assumed to be bounded in $\mathcal{H}_\infty$ norm by a known bound, bounding a transfer function determined by the plant and the controller in $\mathcal{H}_\infty$ norm guarantees robust stability. Second, formulating optimality conditions as frequency domain bounds for a number of transfer functions, $\mathcal{H}_\infty$ theory can be applied as a loopshaping tool. In some cases, robust stability suffices, but in most applications it is required to satisfy some measure of optimality, and hence some kind of loopshaping technique has to be employed.

In the mainstream literature, it is suggested to use $\mathcal{H}_\infty$ theory for such purposes by *stacking* control objectives as, e.g., in the so called *mixed sensitivity approach*, where a design criterion of the form

$$\left\| \begin{pmatrix} W_S(\cdot)S(\cdot) \\ W_T(\cdot)T(\cdot) \end{pmatrix} \right\|_\infty < \gamma \qquad (1)$$

is considered, where $S(\cdot)$ and $T(\cdot)$ are the closed loop sensitivity and complementary sensitivity, and $W_S(\cdot)$ and $W_T(\cdot)$ are appropriate weightings. The motivation for the mixed sensitivity approach is that a controller satisfying (1) also satisfy that each entry of the matrices $W_S(\cdot)S(\cdot)$ and $W_T(\cdot)T(\cdot)$ is bounded by γ as well, which is usually the original goal.

The problem, however, which we shall address in this paper, is that an approach based on a criterion like (1) can be rather conservative since all possible cross-couplings are considered, which might not be motivated from physics. In effect it might not be possible to meet the performance specifications, although an admissible controller *does* exist, which bounds sufficiently each individual sensitivity.

In this paper we shall address design problems, which are based on criteria for individual entries in sensitivity functions, rather than criteria which equalize all directions.

2. Multiobjective Sensitivity Control

In the following we shall study a multi output sensitivity problem formulated as a number of coupled $\mathcal{H}_\infty$ problems. The approach suggested can be applied to a huge number of variations on the multi output sensitivity problem, the complementarity sensitivity problem, and the control sensitivity problem, but first we shall restrict attention to these three problems.

Throughout the sequel we shall consider a finite dimensional, linear, time invariant system with a state space realization of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (2)$$

and with transfer function $G(\cdot)$. We shall assume the plant to be square, with k inputs and k outputs.

For such a system, the multi objective sensitivity problem, the multi objective complementary sensitivity problem, and the control sensitivity problem is depicted in Figure 1, Figure 2, and Figure 3, respectively.

The block diagrams in Figure 1, 2, and 3 can all be described by the relations

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \quad , \quad u = Ky$$

with $w' = (w_1' \; w_2' \; \cdots \; w_k')$, $u' = (u_1' \; u_2' \; \cdots \; u_k')$, $z' =$

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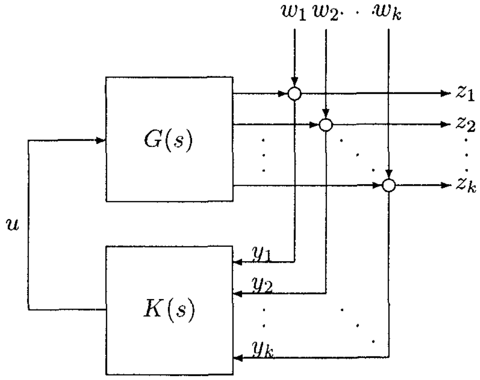

Figure 1: Multi Objective Sensitivity Problem

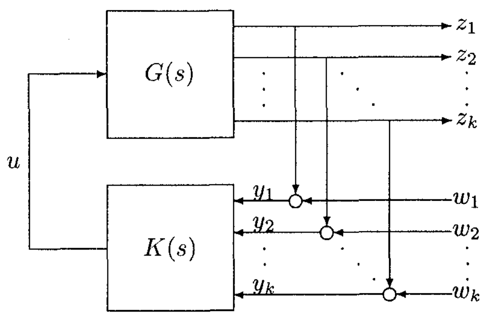

Figure 2: Multi Objective Complementary Sensitivity Problem

$(z_1' \; z_2' \; \cdots \; z_k')$ and $y' = (y_1' \; y_2' \; \cdots \; y_k')$ where

$$\begin{pmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{pmatrix} = \begin{cases} \begin{pmatrix} I & G \\ I & G \end{pmatrix} & \text{for Figure 1} \\ \begin{pmatrix} 0 & G \\ I & G \end{pmatrix} & \text{for Figure 2} \\ \begin{pmatrix} 0 & I \\ I & G \end{pmatrix} & \text{for Figure 3} \end{cases}$$

Writing the transfer function from w to z as a linear fractional transformation in K we get

$$\begin{aligned} T_{zw} &=: \begin{pmatrix} s_{11} & t_{12} & \cdots & t_{1k} \\ t_{21} & s_{22} & \cdots & t_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k1} & t_{k2} & \cdots & s_{kk} \end{pmatrix} \\ &= G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} \\ &= \begin{cases} I + GK(I - GK)^{-1} & \text{for Figure 1} \\ GK(I - GK)^{-1} & \text{for Figure 2} \\ K(I - GK)^{-1} & \text{for Figure 3} \end{cases} \end{aligned}$$

where the functions s_{ii}, $i = 1 \ldots k$, are the output sensitivities (Fig. 1), complementary sensitivities (Fig. 2), or control sensitivities (Fig. 3), respectively. The functions t_{ij}, $i = 1 \ldots k$, $j = 1 \ldots k$, $i \neq j$, are crossover terms

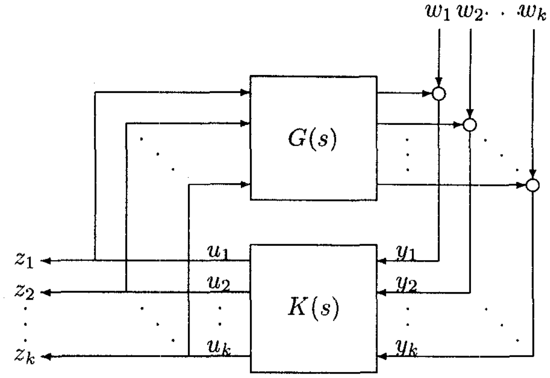

Figure 3: Multi Objective Control Sensitivity Problem

which indicate how much the i^{th} disturbance influences the j^{th} output.
Loopshaping just one of the sensitivities s_{ii} by specifying (the inverse of) an upper bound for the modulus of s_{ii} can be formulated as a standard $\mathcal{H}_\infty$ problem as follows.

Problem 1 *The* i^{th} SISO problem *for any of the configurations in Figure 1, Figure 2, or Figure 3 is said to be solvable if and only if there exists a controller K which internally stabilizes the plant and such that*

$$\|W_i s_{ii}\|_\infty < 1$$

where s_{ii} for Figure 1, 2 and 3 are given by

$$s_{ii}(\cdot) = \begin{cases} 1 + g_i(\cdot)K(\cdot)(I - G(\cdot)K(\cdot))^{-1}e_i \\ g_i(\cdot)K(\cdot)(I - G(\cdot)K(\cdot))^{-1}e_i \\ e_i'K(\cdot)(I - G(\cdot)K(\cdot))^{-1}e_i \end{cases}$$

e_i is the (constant) vector

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ position}$$

and $g_i(s)$ is the row of transfer functions from u to z_i, or equivalently, if there exists an internally stabilizing controller K for the system

$$G = \left(\begin{array}{cc|cc} A & 0 & 0 & 0 \\ B_{W_i}e_i'C_i & A_{W_i} & B_{W_i} & B_{W_i}e_i'D \\ \hline D_{w_i}e_i'C & C_{W_i} & D & D_{W_i}e_i'D \\ C & 0 & e_i & D \end{array}\right)$$

for Figure 1 or

$$G = \left(\begin{array}{cc|cc} A & 0 & 0 & B \\ B_{W_i}e_i'C & A_{W_i} & 0 & B_{W_i}e_i'D \\ \hline D_{W_i}e_i'C & C_{W_i} & 0 & D_{W_i}e_i'D \\ C & 0 & e_i & D \end{array}\right)$$

for Figure 2 or

$$G = \left(\begin{array}{cc|cc} A & 0 & 0 & B \\ 0 & A_{W_i} & 0 & B_{W_i} e_i' \\ \hline 0 & C_{W_i} & 0 & D_{W_i} e_i' \\ C & 0 & e_i & D \end{array} \right)$$

for Figure 3 such that when applying the control law $u = Ky$, the resulting $\mathcal{H}_\infty$ norm from w to z is less than 1. Here, W_i is assumed to have the following state space realization:

$$\begin{array}{rcccc} \dot{\xi} & = & A_{W_i}\xi & + & B_{W_i}u_i \\ y_i & = & C_{W_i}\xi & + & D_{W_i}u_i \end{array}$$

In the sequel, we shall give a number of decoupling results for the above multi objective $\mathcal{H}_\infty$ problems. First we shall give the results for a stable plant, which is extremely simple.

Theorem 1 *Consider the system (2). Assume that A is a stability matrix. Then, the following two statements are equivalent*

1. *There exists an internally stabilizing controller K such that for each s_{ii},*

$$\|W_i s_{ii}\|_\infty < 1$$

2. *For each s_{ii} there exists an internally stabilizing controller K such that*

$$\|W_i s_{ii}\|_\infty < 1$$

Remark 1 The significance of Theorem 1 is that just as much can be achieved by a single controller which controls *all* the s_{ii}'s, as if the controller just had to control *one* of the s_{ii}'s. In fact, as shall be evident from the proof below, it is possible to design such a multi objective $\mathcal{H}_\infty$ controller, by designing an $\mathcal{H}_\infty$ controller for each s_{ii}.

Proof. Let the plant G be row partitioned as

$$G' = \begin{pmatrix} g_1' & g_2' & \cdots & g_k' \end{pmatrix}$$

Since G is stable, the YJBK-parametrization of all stabilizing controllers [YJB71] is simply given by

$$K = Q(I + GQ)^{-1}, \quad Q \in \mathcal{RH}_\infty \tag{3}$$

where Q is given by

$$Q = K(I - GK)^{-1}$$

the transfer function from w to z becomes

$$T_{zw} = \begin{cases} I + GQ & \text{for Figure 1} \\ GQ & \text{for Figure 2} \\ Q & \text{for Figure 3} \end{cases}$$

$$= \begin{cases} \begin{pmatrix} 1 + g_1q_1 & g_1q_2 & \cdots & g_1q_k \\ g_2q_1 & 1 + g_2q_2 & \cdots & g_2q_k \\ \vdots & \vdots & \ddots & \vdots \\ g_kq_1 & g_kq_2 & \cdots & 1 + g_kq_k \end{pmatrix} \\ \begin{pmatrix} g_1q_1 & g_1q_2 & \cdots & g_1q_k \\ g_2q_1 & g_2q_2 & \cdots & g_2q_k \\ \vdots & \vdots & \ddots & \vdots \\ g_kq_1 & g_kq_2 & \cdots & g_kq_k \end{pmatrix} \\ \begin{pmatrix} q_1 & q_2 & \cdots & q_k \end{pmatrix} \end{cases}$$

$$= \begin{pmatrix} s_{11} & t_{12} & \cdots & t_{1k} \\ t_{21} & s_{22} & \cdots & t_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k1} & t_{k2} & \cdots & s_{kk} \end{pmatrix}$$

where Q has the following column partitioning

$$Q = \begin{pmatrix} q_1 & q_2 & \cdots & q_k \end{pmatrix}$$

Now, the crucial observation is that since

$$s_{ii} = \begin{cases} 1 + g_iq_i & \text{for Figure 1} \\ g_iq_i & \text{for Figure 2} \\ e_i'q_i & \text{for Figure 3} \end{cases} \tag{4}$$

each s_{ii} depend only on q_i. Since the q_i's are free stable parameters, the optimization of the s_{ii}'s can be done completely independently, where after K is determined by (3). From this simple observation the claim becomes obvious. □

An important observation, which can be made from the proof of Theorem 1, is that sensitivities, complementary sensitivities, and control sensitivities can be mixed arbitrarily. Pairs of corresponding w_i's and z_i's can be chosen for $\mathcal{H}_\infty$ specifications from each of the above configurations in such a way that no pairs with the same numbering are chosen from any configuration.
For a stable plant, it is trivial that selecting $K = 0$ satisfies the problems in Figs. 2&3. Hence, the corresponding optimization problems make sense only in combinations with sensitivity specifications following Fig. 1.
In the next section, we shall provide a more general result, which incorporates all three types of specifications. From the proof of Theorem 1 it is apparent that an $\mathcal{H}_\infty$ controller K which satisfy any of the above multi objective problems can be found by determining the q_i's and then applying (3). Each of these k transfer matrices (columns) can be found by solving a scalar standard $\mathcal{H}_\infty$ problem based on (4). For example for a sensitivity problem, each of the k associated standard problems based on (4) which in transfer function form is

$$\|W_i(1 + g_iq_i)\|_\infty < 1$$

has the following standard state space formulations

$$G = \left(\begin{array}{cc|cc} A & 0 & 0 & B \\ 0 & A_{W_i} & B_{W_i} & 0 \\ \hline e_i'C & C_{W_i} & D_{W_i} & e_i'D \\ 0 & C_{W_i} & D_{W_i} & 0 \end{array} \right)$$

3. Multi Objective Control with Simultaneous Specifications for every Transfer Function

In the previous section, we were concerned with the problem of shaping just the diagonal entries in the (complementary/control) sensitivities. However, in a series of control problems, it is reasonable to include

1. simultaneous specifications for sensitivities, complementary sensitivities, and control sensitivities
2. specifications for both diagonal and off-diagonal terms

In a disturbance rejection problem, for instance, considering the diagonal terms only indicates that any of the disturbances is assumed to influence one output only (in open *or* closed loop). This is not very realistic in most cases, and hence we have to specify the off-diagonal terms as well, which can be interpreted as the influence on one output from an output disturbance on another. Moreover, if sensitivities are considered isolated, disturbance rejection is achieved at the cost of robustness.
The approach taken below will use a technique similar to mixed sensitivity $\mathcal{H}_\infty$ design, where the design criteria are stacked. In similarity with mixed sensitivity we can avoid conservatism only by selecting weight matrices in a clever way. This conservatism, however, will be the only one introduced.
Loopshaping one of the columns of T_{zw} by specifying upper bounds for the modulus of its entries can be formulated as a standard $\mathcal{H}_\infty$ problem as follows.

Problem 2 *The* j^{th} SIMO problem *for the configuration in Fig. 1 is said to be solvable if and only if there exists a controller K which internally stabilizes the plant and such that*

$$\left\|\bar{W}_j\right\|_\infty = \left\| \begin{matrix} W^s_{1j}s_{1j} \\ \vdots \\ W^s_{kj}s_{kj} \\ W^t_{1j}t_{1j} \\ \vdots \\ W^t_{kj}t_{kj} \\ W^c_{1j}c_{1j} \\ \vdots \\ W^c_{kj}c_{kj} \end{matrix} \right\|_\infty < 1, \tag{5}$$

where

$$\begin{aligned} s_{ij}(\cdot) &= e_i'\left(I + G(\cdot)K(\cdot)(I - G(\cdot)K(\cdot))^{-1}\right)e_j \\ t_{ij}(\cdot) &= e_i'G(\cdot)K(\cdot)(I - G(\cdot)K(\cdot))^{-1}e_j \\ c_{ij}(\cdot) &= e_i'K(\cdot)(I - G(\cdot)K(\cdot))^{-1}e_j \end{aligned}$$

and $g_i(s)$ is the row of transfer functions from u to y_i. $W^s_{ij}(\cdot)$, $W^t_{ij}(\cdot)$, and $W^c_{ij}(\cdot)$ are the weighting matrices for the ij'th entry of the sensitivity, the complementary sensitivity, and the control sensitivity, respectively.

Remark 2 The three problems discussed in Section 2 can be obtained as special cases of Problem 2 by selecting the weights properly. For instance, by choosing $W^s_{ii}(\cdot)$ as weights for the sensitivities, $W^s_{ij}(\cdot) \equiv 0$, $i \neq j$, and $W^t_{ij}(\cdot) \equiv W^c_{ij}(\cdot) \equiv 0$, the sensitivity problem from Section 2 is re-obtained.

Remark 3 Note, that $W^s_{ij} = W^t_{ij}$, $i \neq j$. Hence, there is some redundancy in the setup, which should be removed in implementations.

As a generalization of Theorem 1 the multi variable multi objective problem will be solved by solving a series of SIMO problems, as demonstrated by the following result.

Theorem 2 *Consider the system (2). Assume that A is a stability matrix. Then, the following two statements are equivalent*

1. *There exists an internally stabilizing controller K such that*
$$\left\|\bar{W}_1\right\|_\infty < 1\ ,\ \left\|\bar{W}_j\right\|_\infty < 1\ ,\ \left\|\bar{W}_k\right\|_\infty < 1 \tag{6}$$
in the closed loop system simultaneously,
2. *Each of the m SIMO problems from Problem 2 is solvable* independently.

where $\bar{W}_j$ is defined in (5).

Proof. Following the line of proof of Theorem 1. □

The main significance of Theorem 2 is described in terms of the following corollary.

Corollary 3 *Let K be given, satisfying (6). Then*

$$\left\|W^s_{ij}(\cdot)\right\|_\infty < 1,\ \left\|W^t_{ij}(\cdot)\right\|_\infty < 1,\ \left\|W^c_{ij}(\cdot)\right\|_\infty < 1,\ \forall i,j$$

Remark 4 Corollary 3 shows that each transfer function is optimized entrywise. This entrywise optimization is without introduction of conservatism, except that originating from stacking which can be avoided by cleverly, possibly iteratively, selecting the weights.

Proof. The corollary is immediate from the theorem, upon noting that the $\mathcal{H}_\infty$ norm of a column of transfer functions being smaller than γ implies that each of its entries is smaller than γ. □

The design is done by finding an appropriate q_j for each SIMO problem, and then combining them all by (3).

4. Multi Objective Control of Unstable Plants

In general, the multi objective control problem is much harder for an unstable plant than for a stable one. Provided, however, that one output is available for stabilization only, the results from above can be applied directly for unstable plants also. To exemplify the procedure, let us consider a system described by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} u \tag{7}$$

where we apply the control law

$$u = \begin{pmatrix} K_1 & K_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Now, we have the following straightforward result.

Lemma 4 *Consider the system (7). Assume that K_2 stabilizes the plant, i.e., such that*

$$\tilde{G} = G_1(I - K_2G_2)^{-1}$$

is stable. Moreover, assume that $Q_1 \in \mathcal{RH}_\infty$ satisfies

$$\left\| W_1(\cdot) + W_1(\cdot)\tilde{G}(\cdot)Q_1(\cdot) \right\|_\infty < \gamma$$

Then one controller, satisfying

$$\|W_1(\cdot)S_1(\cdot)\|_\infty < \gamma$$

is given by

$$K = \begin{pmatrix} Q_1 \left(I - G_1(I - K_2G_2)^{-1}Q_1\right)^{-1} & K_2 \end{pmatrix}$$

Proof. The lemma follows by elementary algebra, and by applying Theorem 1. □

Obviously, the principle from Lemma 4 can be extended to any number of outputs or inputs, applying the results regarding stable systems. Although all results previously given in the paper applies in this manner, we shall not give the results explcitly due to space limitations, since they are straightforward. It should be pointed out, however, that there is some restriction in the fact, that one of the outputs is used for stabilization only, and it is not trivial to pose any specifications simultaneously. In practice, it might not be reasonable to introduce an additional sensor or actuator just for this purpose.

5. Conclusion

A series of multi objective $\mathcal{H}_\infty$ design problems have been considered in this paper. It has been shown how it is possible to exactly decouple a number of $\mathcal{H}_\infty$ design problems based on weighted output sensitivity functions, complementary sensitivity functions, and control sensitivity functions. Further, the derived design approach works equally well for continuous or discrete time systems and has also been extended to handle sampled-data systems, see [SN95]. At last, the multi objective $\mathcal{H}_\infty$ design approach has been applied for roll damping of a ship by rudder control, [SNB95].

As shown in section 4, the derived design approach can also in some cases handle unstable systems. In this case we need to use one or more outputs to stabilize the system. As a consequence of this, the number of allowable $\mathcal{H}_\infty$ design requirements is reduced.

Only sensitivity functions at the outputs have been considered in this paper. However, by duality, all methods given can be used also for input sensitivities without any modifications.

The coupled $\mathcal{H}_\infty$ design problems need not be based on different types of weighted sensitivity functions only. It is possible to make a minor generalization of the above $\mathcal{H}_\infty$ approach to handle non sensitivity problems, e.g. to handle explicitly actuator and sensor dynamics. This induces, however, certain rank and minimum phase conditions on some of the transfer functions in the resulting four block problem.

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