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The Filtered $\mathcal{H}_\infty$ State Feedback Problem[1]

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Abstract

The filtered $\mathcal{H}_\infty$ state feedback problem is considered in this paper. It turns out that it is possible to solve the filtered $\mathcal{H}_\infty$ state feedback problem by using a dynamic controller, which only requires a solution of one algebraic Riccati equation. The controller is given in an explicit state space form. Moreover, the order of the controller is equal to the order of the weight function only for the $\mathcal{H}_\infty$ state feedback problem.

1. Introduction

It is a known fact that if we have full state information in an $\mathcal{H}_\infty$ standard problem, we only need to use a static state feedback controller for satisfying the $\mathcal{H}_\infty$ norm condition, [DGKF89].

However, an important fact which is not always emphasized is that although complete physical state information might be available, we might still have to introduce dynamics in the controller if we wish to incorporate disturbance models, noise models, or modeling errors. Also if the design criterion is a frequency weighted function of states and inputs, a dynamic controller is required.

In the $\mathcal{H}_\infty$ literature [DGKF89], the design of dynamic controllers requires two Riccati equations together with a coupling conditions, whereas the original unweighted $\mathcal{H}_\infty$ state feedback problem only requires one Riccati equation. So including weight matrices in a pure $\mathcal{H}_\infty$ state feedback design problem, which is normally needed, result directly in a more complicated controller design. Moreover, the order of controllers obtained in this fashion will be the sum of the order of the plant itself and the order of the weightings. This is not very tractable in view that the plant states are available already. Hence, model order reduction, or special observer based controllers [SSC94] must be applied. The key result in this paper is to give a dynamic controller of low order in an explicit form, which involves solving one Riccati equation only (and hence no coupling condition).

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2. Problem Formulation

Let us consider the following two continuous-time systems described by:

$$G(s):\left\{\begin{array}{rclcl} \dot{x} &=& Ax &+& B_1 w + B_2 u \\ \xi &=& C_1 x &+& D_{12} u \\ y &=& Ix && \end{array}\right. \qquad (1)$$

and

$$W(s):\left\{\begin{array}{rcl} \dot{x}_w &=& A_w x_w + B_w \xi \\ z &=& C_w x_w + D_w \xi \end{array}\right. \qquad (2)$$

where $x \in \mathcal{R}^n$, $x_w \in \mathcal{R}^{n_w}$, $u \in \mathcal{R}^p$, $w \in \mathcal{R}^r$, $\xi \in \mathcal{R}^q$ and $z \in \mathcal{R}^s$. Further, (A, B_2) is assumed to be stabilizable and (A, B_2, C_1, D_{12}) is assumed to have no invariant zeros on the imaginary axis.

The system $G(s)$ in (1) describes the real system to be controlled and $W(s)$ in (2) is the associated weight matrix for the $\mathcal{H}_\infty$ design problem given by:

Problem 1 *Consider the system given in (1) and the weight function in (2). Let $\gamma > 0$ be given. Design an internally stabilizing dynamic controller $F(s)$, if such exist, such that*

$$\| W(s)\mathcal{F}_l\left(G(s), F(s)\right) \|_\infty < \gamma$$

This problem can be solved by using standard $\mathcal{H}_\infty$ techniques [DGKF89], which involve solving two Riccati equations together with a coupling condition. The order of such a controller will be that of the plant itself plus the order of the weighting matrix.

3. Main Result

The state space realization of the system in (1) together with the weight function in (2) is given by:

$$\begin{array}{rclcl} \dot{\bar{x}} &=& \bar{A}\bar{x} &+& \bar{B}_1 w + \bar{B}_2 u \\ z &=& \bar{C}_1 \bar{x} &+& \bar{D}_{12} u \\ y &=& \bar{C}_2 \bar{x} && \end{array} \qquad (3)$$

where $\bar{x} = \begin{pmatrix} x \\ x_w \end{pmatrix}$ and the six matrices are as follows:

$$\begin{array}{rclcrcl} \bar{A} &=& \begin{pmatrix} A & 0 \\ B_w C_1 & A_w \end{pmatrix}, & \bar{B}_1 &=& \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \\ \bar{C}_1 &=& \begin{pmatrix} D_w C_1 & C_w \end{pmatrix}, & \bar{B}_2 &=& \begin{pmatrix} B_2 \\ B_w D_{12} \end{pmatrix} \\ \bar{C}_2 &=& \begin{pmatrix} I & 0 \end{pmatrix}, & \bar{D}_{12} &=& D_w D_{12} \end{array} \qquad (4)$$

At first, we need the following lemma:

Lemma 1 *Assume that neither of the two systems in (1) and (2) have any invariant zeros on the imaginary axis, and D_w has full column rank. Then also the system given in (3) has no invariant zeros on the imaginary axis.*

Proof of Lemma 1. The proof follows directly by calculating the rank of the Rosenbrock matrix of the system (3). □

Based on this state space realization of the filtered $\mathcal{H}_\infty$ state feedback problem, we are now able to give the main result.

Theorem 2 *Consider the system in (3). Assume that $\bar{D}_{12}$ is injective. Then the following statements are equivalent:*

1. *There exists a dynamic internally stabilizing controller $F(s)$ such that when applying the feedback law $u = F(s)y$, the resulting closed loop transfer function from w to z has an $\mathcal{H}_\infty$ norm smaller than γ.*
2. *There exist a positive semidefinite solution $\bar{P}$ to the algebraic Riccati equation:*

$$0 = \bar{A}^T\bar{P} + \bar{P}\bar{A} + \gamma^{-2}\bar{P}\bar{B}_1\bar{B}_1^T\bar{P} -$$
$$(\bar{P}\bar{B}_2 + \bar{C}_1^T\bar{D}_{12})(\bar{D}_{12}^T\bar{D}_{12})^{-1}(\bar{B}_2^T\bar{P} + \bar{D}_{12}^T\bar{C}_1)$$

Moreover, one such dynamic controller $F(s)$ is then given by:

$$F(s) = \left(\begin{array}{c|c} A_w + B_wD_{12}F_2 & B_w(C_1 + D_{12}F_1) \\ \hline F_2 & F_1 \end{array}\right) \tag{5}$$

where $\bar{F} = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$ is given by:

$$\bar{F} = -(\bar{D}_{12}^T\bar{D}_{12})^{-1}(\bar{D}_{12}^T\bar{C}_1 + \bar{B}_2^T\bar{P}) \tag{6}$$

which makes the closed loop $\mathcal{H}_\infty$ norm from w to z smaller than γ.

The proof of Theorem 2 is based upon the fact, that nothing more can be achieved for an $\mathcal{H}_\infty$ problem than what can be achieved by a static state feedback controller using *all* (real or fictitious) states as stated in the following well known fact.

Lemma 3 *Assume that there exists an internally stabilizing control law $u = Ky$ for the system*

$$\begin{array}{rcl} \dot{x} &=& Ax + B_1w + B_2u \\ z &=& C_1x + D_{12}u \\ y &=& C_2x \end{array} \tag{7}$$

making the closed loop $\mathcal{H}_\infty$ norm from w to z smaller than γ.
Then there exists an internally stabilizing static *state feedback controller $u = Fy = Fx$ for the system*

$$\begin{array}{rcl} \dot{x} &=& Ax + B_1w + B_2u \\ z &=& C_1x + D_{12}u \\ y &=& x \end{array} \tag{8}$$

To establish the proof of Theorem 2 it can be verified that for the system (3) the reverse of Lemma 3 holds, i.e., that the existence of a static state feedback controller obtaining a certain $\mathcal{H}_\infty$ norm γ implies the existence of a dynamic measurement based controller obtaining the same $\mathcal{H}_\infty$ norm.

4. Conclusion

A design method for a low order dynamic controller which satisfies the filtered $\mathcal{H}_\infty$ state feedback problem has been derived in this paper. Only one Riccati equation is required for this controller design. Moreover, in Theorem 2 it was assumed that D_wD_{12} has full column rank. Hence, if D_wD_{12} is singular, $\bar{F}$ can be found by means of *singular* $\mathcal{H}_\infty$ theory, see e.g. [Sto92]. The controller expression (5) derived above still holds for such $\bar{F}$.
The controller order is minimal in the sense that for near optimal solutions to a $\mathcal{H}_\infty$ problem of the type described in this paper with *generic* data, the number of states required in any strictly proper controller solving the problem will be no less than for the one given above. Obviously, however, introducing direct terms satisfying well-posedness might reduce the controller order somewhat. Moreover, for design problems which are not near optimal, model reduction might be applied for the controller.

References

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