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Robust $\mathcal{H}_\infty$ Almost Disturbance Decoupling*

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Abstract

The robust $\mathcal{H}_\infty$ almost disturbance decoupling problem, i.e. an $\mathcal{H}_\infty$ almost disturbance decoupling problem along with an additional $\mathcal{H}_\infty$ side constraint, is considered. Necessary and sufficient conditions for solvability of this problem are given in terms of solvability of an algebraic equation and an $\mathcal{H}_\infty$ constrained problem. Explicit controller design algorithms are derived, utilizing the necessary and sufficient conditions.

1 Introduction

One of the most well known control problems is the problem of disturbance decoupling or disturbance attenuation. In most cases, as e.g. in the case of an output disturbance, disturbances can not be exactly decoupled but only asymptotically or "almost". The solution to almost disturbance decoupling problems lead to the design of high gain feedback control. However, introducing large gains in a control loop potentially implies severe robustness problems, as they require very good confidence in the model.

Therefore there is a need for a consistent way to introduce a notion of robustness in solving almost disturbance decoupling problems.

One possibility which we shall discuss in this paper is to introduce an $\mathcal{H}_\infty$ constraint in the almost disturbance decoupling problem taking care of the influence of modeling errors.

The above design problem cannot directly be solved by using a standard $\mathcal{H}_\infty$ technique. The design problem turns out to be a multi objective design problem which cannot be handled by standard $\mathcal{H}_\infty$ techniques without introducing conservatism. The conservatism appears from the off diagonal elements in the design setup. Scaling of the external input and/or output signals cannot reduce this conservatism. Instead we introduce an $\mathcal{H}_\infty$ constraint in the almost disturbance decoupling problem. The design procedure for this design problem given in this paper is based on that we first parameterize all controllers which solve a disturbance decoupling problem. Based on this parameterization, the $\mathcal{H}_\infty$ constraint can then directly be formulated as a standard $\mathcal{H}_\infty$ design problem.

The rest of this paper is organized as follows. A problem formulation is given in Section 2. Section 3 include the main result for the case where the system is open loop stable. The general case is described in Section 4 followed by a design procedure in section 5. An example is given in Section 6 and a conclusion in Section 7.

2 Problem Formulation

In Figure 1 a robust almost disturbance decoupling problem is depicted.

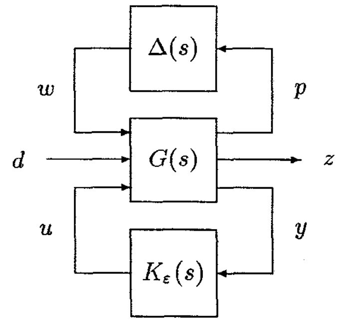

Figure 1: A Robust Almost Disturbance Decoupling Problem

The problem illustrated in Figure 1 is to find a sequence of controllers K_ε which make the $\mathcal{H}_\infty$ norm of the transfer function from d to z tend to zero, while maintaining robust stability with respect to the uncertainty Δ.

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To be more specific we introduce a state space model of the system shown in Figure 1 which has the form:

$$\begin{array}{rcccccccc} \dot{x} &=& Ax &+& B_1 w &+& B_2 d &+& B_3 u \\ z &=& C_1 x &+& D_{11} w &+& D_{12} d &+& D_{13} u \\ p &=& C_2 x &+& D_{21} w &+& D_{22} d &+& D_{23} u \\ y &=& C_3 x &+& D_{31} w &+& D_{32} d &+& D_{33} u \end{array} \quad (1)$$

Then the robust almost disturbance decoupling problem is defined in the following way.

Problem 1 *Let a positive number γ be given. The* robust $\mathcal{H}_\infty$ almost disturbance decoupling problem ($\mathcal{H}_\infty$/ADDP) *is said to be solvable for the system (1) if there exists a sequence of internally stabilizing controllers K_ε such that for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that the $\mathcal{H}_\infty$ norms of the closed loop transfer functions from w to p and from d to z are smaller than γ and δ, respectively, for all $\varepsilon < \varepsilon_0$.*

In terms of the systems parameters, each controller in the sequence K_ε, $\varepsilon < \varepsilon_0$, has to satisfy

$$\begin{array}{rcl} \|S_1 + S_2 K_\varepsilon (I - G_{33} K_\varepsilon)^{-1} S_3\|_\infty &<& \gamma \\ \|T_1 + T_2 K_\varepsilon (I - G_{33} K_\varepsilon)^{-1} T_3\|_\infty &<& \delta \end{array} \quad (2)$$

where

$$\begin{array}{rcl} S_1 &=& C_2(sI - A)^{-1} B_1 + D_{21} \\ S_2 &=& C_2(sI - A)^{-1} B_3 + D_{23} \\ S_3 &=& C_3(sI - A)^{-1} B_1 + D_{31} \\ T_1 &=& C_1(sI - A)^{-1} B_2 + D_{12} \\ T_2 &=& C_1(sI - A)^{-1} B_3 + D_{13} \\ T_3 &=& C_3(sI - A)^{-1} B_2 + D_{32} \\ G_{33} &=& C_3(sI - A)^{-1} B_3 + D_{33} \end{array} \quad (3)$$

3 Main Results

The results in this paper is based mainly on two results.

The first result from [Wil82, WW89] establishes equivalence between solvability of an $\mathcal{H}_\infty$ almost disturbance decoupling problem and the solvability of a certain rational matrix equation.

Lemma 1 *Assume that $T_1, T_2, T_3 \in \mathcal{RH}_\infty$. The inequality*

$$\|T_1 + T_2 Q_\varepsilon T_3\|_\infty < \delta$$

has a proper, stable, rational solution Q_ε for all $\delta > 0$ if and only if the equation

$$T_1 + T_2 Q T_3 = 0 \quad (4)$$

has a stable, rational solution Q (not necessarily proper).

In terms of disturbance decoupling this means that the almost disturbance decoupling problem is solvable by a usual proper controller if and only if the disturbance decoupling problem can be solved exactly by allowing differentiating controllers.

The second result to which we shall appeal is from [Sch90, SLS95] and establishes that suboptimal $\mathcal{H}_\infty$ control problems can be solved by rational controllers if and only if they can be solved by *proper* rational controllers.

Lemma 2 *Assume that $T_1, T_2, T_3 \in \mathcal{RH}_\infty$. Consider the suboptimal $\mathcal{H}_\infty$ model matching problem*

$$\|T_1 + T_2 Q T_3\|_\infty < \gamma \quad (5)$$

Then there exists a stable, rational $Q(s)$ such that $T_1 + T_2 Q T_3 \in \mathcal{RH}_\infty$ satisfying (5), if and only if there exists a proper *stable rational $Q(s)$ such that $T_1 + T_2 Q T_3 \in \mathcal{RH}_\infty$ satisfying (5).*

The following is the main result of the paper.

Theorem 3 *Consider the system (1) and assume that A is stable. Then the following two statements are equivalent*

1. *There exists a sequence of controllers K_ε solving the robust $\mathcal{H}_\infty$ almost disturbance decoupling problem*
2. *There exists a stable, rational solution Q to the equation*

$$T_1 + T_2 Q T_3 = 0 \quad (6)$$

such that

(a) $S_1 + S_2 Q S_3 \in \mathcal{RH}_\infty$

(b) $\|S_1 + S_2 Q S_3\|_\infty \leq \gamma$

Proof. (1) $\Leftarrow$ (2). Assume a stable, rational $Q(s)$ satisfies the conditions of Theorem 3(2). Now, using the YJBK-parameterization [YJB71] of all stabilizing controllers, from the proofs of Lemma 1 and Lemma 2 it follows that the series of controllers given by

$$\begin{array}{rcl} K_\varepsilon(s) &=& Q_\varepsilon(s)\,(I + G_{33}(s) Q_\varepsilon(s))^{-1} \quad \text{where} \\ Q_\varepsilon(s) &=& \dfrac{1}{(\varepsilon s + 1)^n} Q(s) \end{array}$$

are (strictly) proper and solve the robust $\mathcal{H}_\infty$ almost disturbance decoupling problem.

(1) $\Rightarrow$ (2). The details of this part are omitted, but the main ideas are the following. Assume that there

exist such a sequence of controllers $K_\varepsilon(s)$. Then it can be shown that there exists also a solution, i.e. a sequence $\tilde{K}_\varepsilon(s)$, with the additional property that

$$Q_\varepsilon(s) = \tilde{K}_\varepsilon(s)\left(I - G_{33}(s)\tilde{K}_\varepsilon(s)\right)^{-1}$$

tends to a (possibly) nonproper $Q(s)$ as $\varepsilon \to 0$ in the topology of pointwise convergence, where $Q(s)$ satisfies (6). Now, from

$$S_1 + S_2 Q_\varepsilon S_3 \in \mathcal{RH}_\infty \quad \text{and} \quad \|S_1 + S_2 Q_\varepsilon S_3\|_\infty < \gamma$$

Theorem3(2a) and Theorem3(2b) follows immediately because of the convergence, since otherwise for some ω_0, $\left\|\tilde{K}_\varepsilon(i\omega_0)\left(I - G_{33}(i\omega_0)\tilde{K}_\varepsilon(i\omega_0)\right)^{-1}\right\| > \gamma$ for ε sufficiently small. □

4 General Case

Until now, it has been assumed that A is stable. However, if A is not stable, it is still possible to apply the $\mathcal{H}_\infty$/ADDP design approach. This can be done by using the Q-parameterization via an observer based controller. Let's assume that (A, B_3) is stabilizable (C_3, A) is detectable and the direct term $D_{33} = 0$. Further, let F and L be selected such that $A + B_3F$ and $A + LC_3$ are stable. Then a stabilizing observer based controller for the system (1) is given by:

$$K(s) = \left[\begin{array}{c|c} A + B_3F + LC_3 & -L \\ \hline F & 0 \end{array}\right] \qquad (7)$$

Based on this controller, all stabilizing controllers for the system is then given by $\mathcal{F}_l(J, Q)$ where $Q \in \mathcal{RH}_\infty$ is free and

$$J = \left[\begin{array}{c|cc} A + B_3F + LC_3 & -L & B_3 \\ \hline F & 0 & I \\ -C_3 & I & 0 \end{array}\right] \qquad (8)$$

With this parameterization of all stabilizing controllers for the system in (1), the design conditions given by (2) take then the following form:

$$\begin{aligned} \|\bar{S}_1 + \bar{S}_2 Q_\varepsilon \bar{S}_3\|_\infty &< \gamma \\ \|\bar{T}_1 + \bar{T}_2 Q_\varepsilon \bar{T}_3\|_\infty &< \delta \end{aligned} \qquad (9)$$

where the six transfer functions are given by:

$$\begin{aligned} \bar{T} &= \begin{bmatrix} \bar{T}_1 & \bar{T}_2 \\ \bar{T}_3 & 0 \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} A_F & -B_3F & B_1 & B_3 \\ 0 & A_L & B_{1,L} & 0 \\ \hline C_{2,F} & -D_{23}F & D_{21} & D_{23} \\ 0 & C_3 & D_{31} & 0 \end{array}\right] \\ \bar{S} &= \begin{bmatrix} \bar{S}_1 & \bar{S}_2 \\ \bar{S}_3 & 0 \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} A_F & -B_3F & B_2 & B_3 \\ 0 & A_L & B_{2,L} & 0 \\ \hline C_{1,F} & -D_{13}F & D_{12} & D_{13} \\ 0 & C_3 & D_{32} & 0 \end{array}\right] \end{aligned} \qquad (10)$$

with

$$\begin{aligned} A_F &= A + B_3F \\ A_L &= A + LC_3 \\ B_{i,L} &= B_i + LD_{3i} \\ C_{j,F} &= C_j + D_{j3}F \end{aligned}$$

Based on this transformed system, the unstable situation can be dealt with in completely the same way as the stable situation.

5 Design Procedures

In this section we shall describe controller design algorithms based on Theorem 3. In the most general case, the design algorithms are complicated optimization based procedures, but with some reasonable additional assumptions, the design procedures become, in fact, very simple.

First of all, we shall describe the algorithm only for stable A. For unstable A, one just substitutes the matrices from Section 4. Second, we assume either left invertibility of $T_2(s)$ or (dually) right invertibility of $T_3(s)$. We shall give the algorithm in the left invertible case only. The right invertible case is straightforward from this.

Algorithm 1

Indata: The 7 transfer matrices in (3) and γ
Outdata: A series of controllers $K_\varepsilon(s)$

1. Find all (including nonproper) stable rational solutions $Q(s)$ to the equation

$$T_1(s) + T_2(s)Q(s)T_3(s) = 0$$

and parameterize $Q(s)$ as

$$Q(s) = Q_0(s) + \tilde{Q}(s)L(s)$$

where $Q_0(s)$ is an arbitrary stable solution, $\tilde{Q}(s)$ is a free stable parameter, and $L(s)$ satisfies $L(s)T_3(s) = 0$. (This step involves solving linear (polynomial) equations.)

2. Choose a stable invertible minimum phase matrix $R(s)$ (not necessarily proper) such that

$$\tilde{S}_3(s) = R(s)L(s)S_3(s)$$

is proper. (This step is trivial.)

3. Determine a stable rational matrix $\tilde{Q}_0(s)$ such that

$$\tilde{S}_1 = S_1 + S_2Q_0S_3 + S_2\tilde{Q}_0\tilde{S}_3$$

becomes proper. (This step involves solving linear (polynomial) equations.)

4. Determine a proper stable rational matrix Q_∞ such that

$$\left\|\tilde{S}_1 + S_2Q_\infty\tilde{S}_3\right\|_\infty < \gamma$$

(This step involves solving two algebraic Riccati equations.)

5. Determine that largest zero excess k of $Q_0(s) + \tilde{Q}_0(s)L(s)$. (This step is trivial.)

6. Compute $K_\varepsilon(s)$:

$$K_\varepsilon(s) = Q_\varepsilon(s)\left(I + G_{33}(s)Q_\varepsilon(s)\right)^{-1}$$

where

$$Q_\varepsilon(s) = \frac{1}{(\varepsilon s+1)^k}\left(Q_0(s) + \left(\tilde{Q}_0(s) + Q_\infty(s)\right)R(s)L(s)\right)$$

(This step is trivial.)

Proposition 4 *Assume that A is stable, and that the system described by the quadruple of matrices (A, B_3, C_1, D_{13}) is left invertible. Then $K_\varepsilon(s)$ computed from Algorithm 1 solves the robust $\mathcal{H}_\infty$ almost disturbance decoupling problem.*

Proof. The parameterization in Step 1 follows from left invertibility of T_2, since for the corresponding homogenous equation:

$$T_2(s)Q(s)T_3(s) = 0 \Rightarrow Q(s)T_3(s) = 0$$

Since $R(s)$ in Step 2 is invertible and minimum phase, we have

$$\left\{\tilde{S}_1 + S_2QLS_3 : Q \text{ is stable and rational}\right\}$$
$$= \left\{\tilde{S}_1 + S_2\tilde{Q}\tilde{S}_3 : \tilde{Q} \text{ is stable and rational}\right\}$$

By the same reasoning, according to Theorem 3, $\tilde{Q}_0(s)$ as in Step 3 can be found.

Now, since the $\mathcal{H}_\infty$ problem in Step 4 involves only proper matrices, we know from [Sch90, SLS95] that existence of a stable rational solution is equivalent to existence of a *proper* stable rational solution.

In Step 5 the number of poles at infinity which have to be approximated by large poles in the proper controllers are determined. (In an $\mathcal{H}_\infty$ almost disturbance decoupling problem, this approximation can be done basically in an arbitrary way.)

Finally, applying the YJBK-parameterization, the controller given in Step 6 solves the robust $\mathcal{H}_\infty$ almost disturbance decoupling problem according to Theorem 3 (and its proof), since the matrix $Q(s) = Q_0(s) + \left(\tilde{Q}_0(s) + Q_\infty(s)\right)L(s)$ satisfies the necessary and sufficient conditions. □

6 Example

We consider a system of the form 1 given by:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}, \quad D_{11} = D_{12} = D_{13} = 0$$

$$C_2 = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}, \quad D_{21} = D_{22} = D_{13} = 0$$

$$C_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$D_{31} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D_{32} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{33} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The system is open loop stable, so there is no need for a preliminary stabilizing controller. The system has equally many outputs and actuators, but more sensors than disturbances. Hence, we can apply Algorithm 1 directly.

1. One (nonproper) solution $Q_0(s)$ to (6) is:

$$Q_0(s) = \begin{bmatrix} \frac{-2s^4-17s^3-53s^2-71s-33}{6s^3+43s^2+94s+65} & 0 \end{bmatrix}$$

A left annihilator $L(s)$ for $T_3(s)$ is given by:

$$L(s) = \begin{bmatrix} -\frac{s^2+2s+1}{3s^2+11s+10} & 1 \end{bmatrix}$$

2. By these choices of $Q_0(s)$ and $L(s)$, the matrices $\tilde{S}_3 = L(s)S_3(s)$ and $\tilde{S}_1 = S_1 + S_2Q_0S_3$ are already proper, so we might make the trivial choices $R(s) = I$, and

3. $\tilde{Q}_0(s) = 0$.

4. The three matrices defining the $\mathcal{H}_\infty$ model matching problem are given by:

$$\tilde{S}_1(s) = \frac{8\,s^4 + 80\,s^3 + 266\,s^2 + 376\,s + 194}{6\,s^4 + 55\,s^3 + 180\,s^2 + 253\,s + 130}$$

$$S_2(s) = \frac{-4\,s^2 - 16\,s - 14}{s^3 + 6\,s^2 + 11\,s + 6}$$

$$\tilde{S}_3(s) = \frac{-s^2 + 4\,s + 9}{3\,s^2 + 11\,s + 10}$$

Note, that this is a *singular* $\mathcal{H}_\infty$ problem, so we have to use singular methods such as described in [Sto92] or, alternatively, cheap control methods with the two Riccati equation approach to compute a controller.

The function $\tilde{S}_3(s)$ contains a non-minimum phase zero. The corresponding interpolation constraint gives rise to a nonzero infimal γ of 1.4310. Since we are introducing large gains anyway, we might as well look for a near optimal solution, and hence, we choose $\gamma = 1.44$.

One possible suboptimal $\mathcal{H}_\infty$ controller is:

$$Q_\infty(s) = \frac{a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6}{b_0 s^6 + b_1 s^5 + b_2 s^4 + b_3 s^3 + b_4 s^2 + b_5 s + b_6}$$

where

$$\begin{array}{lllllll} a_0 & = & 33429.75792 & a_1 & = & 314144.1466 \\ a_2 & = & 1205573.783 & a_3 & = & 2469599.975 \\ a_4 & = & 2888871.335 & a_5 & = & 1830627.131 \\ a_6 & = & 486496.3764 & & & \\ b_0 & = & 22815.96423 & b_1 & = & 709703.1454 \\ b_2 & = & 6145793.974 & b_3 & = & 23766988.0 \\ b_4 & = & 46269726.20 & b_5 & = & 44366787.72 \\ b_6 & = & 16667651.22 & & & \end{array}$$

5. The zero excess is 1.

6. Computing $K_\varepsilon(s)$ from:

$$K_\varepsilon(s) = Q_\varepsilon(s)\,(I + G_{33}(s)Q_\varepsilon(s))^{-1}$$

where

$$Q_\varepsilon(s) = \frac{1}{\varepsilon s + 1}\,(Q_0(s) + Q_\infty(s)L(s))$$

for $\varepsilon = \{0.1, 0.01, 0.001, 0.0001\}$ results in the two sets of curves shown in Fig 2 and Fig 3.

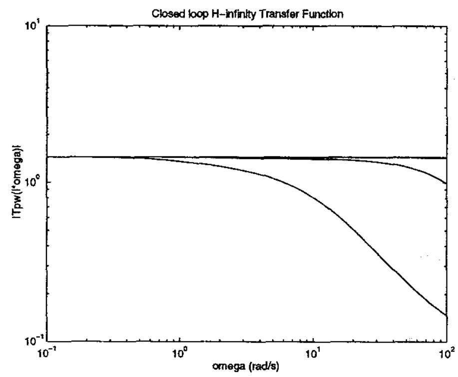

Figure 2: Closed loop transfer function from w to p

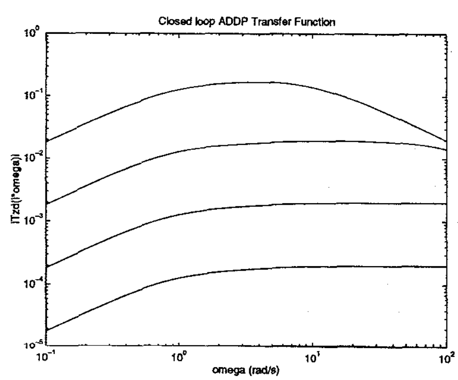

Figure 3: Closed loop transfer function from d to z

7 Conclusions

A robust $\mathcal{H}_\infty$ almost disturbance decoupling problem has been considered in this paper. Necessary and sufficient conditions for solvability of this problem are given in terms of solvability af an algebraic equation and an $\mathcal{H}_\infty$ constrainted problem. Further, an explicit controller design algorithm is derived. The algorithm is based on solving linear matrix equations together with a standard $\mathcal{H}_\infty$ design problem.

The design algorithm is based on that either $T_2(s)$ is left invertible or $T_3(s)$ is right invertible. This is equivalent to require that the number of actuatores must be greater than the number of outputs (i.e. dim. $u >$ dim. z) or that the number of measurements must be greater than the number of disturbance inputs (i.e. dim. $y >$ dim. w) is satisfied. A more complicated design algorithm is needed if both $T_2(s)$ and $T_3(s)$ is not invertible.

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