

# Proportional-Integral Observers for Discrete time Systems\*

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## Abstract

This paper applies the proportional-integral (PI) observer in connection with LQG and LQG/LTR design for discrete time systems. Both the prediction and the filtering versions of the PI observer are considered. We show that a PI observer makes it possible to obtain time recovery, i.e., exact recovery for  $t \rightarrow \infty$ , under mild conditions. It is shown that LQG/LTR design is a special case of the cheap estimation problem. An analysis of the cheap estimation problem is derived. Based on these results, a systematic LTR design method for PI-observers, based on an extension of the LQG/LTR method for full-order, proportional (P) observers, is derived. Explicit expressions for the recovery error, when exact recovery is not achievable for all frequencies, are also given.

## 1 Introduction

Since the appearance of the papers by Doyle and Stein [2], [3] dealing with loop transfer recovery (LTR), many papers have been written on this topic for both continuous and discrete time systems. The most notable ones for continuous time systems are [1], [7], [8], [14], [9], [10].

Although there are certain similarities between the LTR of continuous and discrete time systems, there exist also fundamental differences. Without going into the details, it is well-known that an arbitrarily specified target loop transfer function is recoverable if the continuous time system is minimum phase and left invertible. However, this is not true for discrete time systems as discussed in [4], [5].

For discrete time systems there are two main types of observers; namely, prediction and filtering (current type) observers. They are used when computation time is either significant or negligible, respectively. The status

of the reported results in discrete time LTR indicates that the recovery of any arbitrarily specified target loop transfer function using filtering observer is possible for the strictly proper square minimum phase systems having only infinite zero of order one. On the other hand, it is impossible to have either exact or asymptotic LTR when the plant is nonminimum phase or prediction-type observer is used with a free target design. The fundamental difficulties are due to the facts that sampling usually introduces nonminimum phase zeros, that computation time is sometimes not negligible, and that practical systems contain time delays, and they are often non-strictly proper. Consequently, recent results [9], [15] were devoted to understand the behavior of LTR under these conditions.

If the applied system is non-minimum phase, it is not possible to achieve asymptotic recovery for a free target design. However, it is possible to overcome these problems by including an integral term in the full-order observer. By using this PI-observer in connection with LTR design, it is possible to obtain time recovery, i.e. recovery as  $t \rightarrow \infty$ , without using high gains. The continuous time case has been thoroughly investigated in [6], where it has been shown that it is also possible to obtain time recovery for non-minimum phase systems. In this paper we show explicitly that it is also possible to obtain time recovery in the discrete time case by using a PI-observer for both minimum phase as well as for non-minimum phase systems.

An alternative way to obtain good recovery at low frequencies, is to augment integrators to the plant before the target design is performed [1, 14]. By doing this, the target design is changed such it is easy to recover the target loop at low frequencies. However, this implies that in this approach, the target loop is no longer entirely free, because an integral effect needs to be included in the target loop. In contrast, when the PI-observer approach is used, the integral effect is included in the observer. Therefore, the target design is completely free.

The main goal of this paper is to give a complete description of discrete time PI-observers used in connection with LTR design.

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## 2 Discrete Time PI Observer

Consider a finite dimensional, linear, time-invariant discrete system described by a minimal state-space realization  $(A, B, C)$ :

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (1)$$

where  $x \in \mathcal{R}^n$ ,  $u \in \mathcal{R}^r$ , and  $y \in \mathcal{R}^m$  with  $m \geq r$ ,  $n > m$ ,  $(A, B)$  stabilizable,  $(C, A)$  detectable and  $C, B$  full rank.

Let the plant be controlled by an observer-based controller having the state feedback

$$u(t) = F\hat{z}(t) + r(t) = w(t) + r(t) \quad (2)$$

where  $F$  is the state feedback gain,  $\hat{z}$ , the state estimate and  $r(t)$  is external input. The states are estimated by using a proportional-integral (PI) observer. Analogous to the case of P-observers, it is possible to derive two versions of the PI observer for discrete time systems: a prediction PI observer and a filtering PI observer. The discrete time, prediction PI observer is equivalent to the continuous time version, [6]. Therefore, we can directly formulate a prediction PI observer as follows:

$$\begin{aligned} \hat{z}(t+1) &= A\hat{z}(t) + K_P(C\hat{z}(t) - y(t)) + Bu(t) + Bv(t) \\ v(t+1) &= v(t) + K_I(C\hat{z}(t) - y(t)) \end{aligned} \quad (3)$$

where  $K_P$  is the P observer gain and  $K_I$  is the I observer gain.

To derive a systematic design method, we let the PI observer-based controller be represented by an augmented state system given by

$$\begin{aligned} z(t+1) &= A_x z(t) + K_x (C_x z(t) - y(t)) + B_x u(t) \\ u(t) &= F_x z(t) \end{aligned} \quad (4)$$

where

$$\begin{aligned} A_x &= \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}, & B_x &= \begin{bmatrix} B \\ 0 \end{bmatrix}, \\ C_x &= [C \ 0], & K_x &= \begin{bmatrix} K_P \\ K_I \end{bmatrix}, \end{aligned} \quad (5)$$

and

$$F_x = [F \ 0]. \quad (6)$$

The feedback signal  $u(t)$  in a prediction observer is based on measurements up to time  $t-1$ ; on the other hand, in a filtering observer  $u(t)$  is based on measurements up to time  $t$ . The time delay due to calculation of the feedback signal  $u(t)$ , therefore, must be negligible.

A filtering PI observer can be derived from the full-order, filtering P observer by including an integral term. The resulting state-space description is equivalent to (3); however, the feedback signal is given by  $u(t) = F_f A \hat{z}(t) + F_f K_P (C \hat{z}(t) - y(t))$  where  $F = F_f A$  (it is assumed that  $A$  is invertible).

The compact form of the filtering PI observer-based controller is equivalent to (4), (5) with the matrix  $F_x$  given by

$$F_x = [F_f (A + K_P C) \ 0]. \quad (7)$$

### 2.1 LTR with P and PI Observers

To design a controller for the system  $(A, B, C)$  by the LTR design methodology, we first determine a static state feedback, the target design, which satisfies our design specifications. The design specifications, such as robust stability and nominal performance conditions, are assumed to be reflected at the plant input point [13].

Based on the target (full-state feedback) design gain  $F$  for the system, the target sensitivity function is given by

$$S_{TFL}(z) = (I - L_{TFL}(z))^{-1}. \quad (8)$$

where  $L_{TFL}(z) = F(zI - A)^{-1}B$  represents the target (full-state feedback) loop transfer function. Next the LTR step is performed in which we attempt to recover the target design over a range of frequencies by a dynamic compensator  $C(z)$ . This step gives a full-loop sensitivity transfer function of the form

$$S_I(z) = (I - C(z)G(z))^{-1} \quad (9)$$

where  $G(z)$  represents the plant transfer function.

Assuming that  $C(z)$  is implemented via an observer (or Kalman filter) based controller, the resulting loop transfer function  $C(z)G(z)$ , in general, is not the same as the target loop transfer function  $L_{TFL}(z)$ . In the LTR step the required observer is designed so as to recover either exactly (perfectly) or as good as possible the target loop transfer function.

For a more careful analysis, we define the loop transfer recovery error as

$$E_S(z) = S_{TFL}(z) - S_I(z) \quad (10)$$

and say that exact loop transfer recovery at the input point (ELTRI) is achieved if the closed-loop system comprised of  $C(z)$  and  $G(z)$  is asymptotically stable and  $E_S(z) = 0$ . It is here important to note that it is only possible to design LTR controllers in discrete time which result in exact recovery or non exact recovery. The approximate or asymptotic LTR, known for continuous time does not exist in discrete time LTR design due to the fact that poles are assigned in a compact set in discrete time observers [5], [15].

Let the applied controller  $C(z)$  be a prediction or a filtering PI observer-based controller as described above. We then have the following result.

**Lemma 2.1** *Let the recovery matrix  $M_I(z)$  be given by*

$$M_I(z) = F_x (zI - A_x - K_x C_x)^{-1} B_x \quad (11)$$

where  $A_x$ ,  $B_x$ ,  $C_x$ , and  $K_x$  are given by (5) and  $F_x$  by (6) or (7). Then

$$E_S(z) = -S_{TFL}(z)M_I(z). \quad (12)$$

*Proof:* The proof is analogous to the proof of the continuous time case, see e.g. [7], [6].

Based on the discrete time LTR formulation, we now give necessary and sufficient conditions for both exact and time recovery.

**Lemma 2.2** *Let the sensitivity recovery error be given by (10). ELTRI is obtained if and only if one of the following equivalent conditions holds:*

$$E_S(z) = 0, \quad (13)$$

$$M_I(z) = 0. \quad (14)$$

*Proof:* By virtue of lemma 2.1 and similar steps as applied for the proof of the continuous time case [7] the above conditions follow immediately.

In some cases the step response of the recovery error  $E_S$  tends to zero as  $t \rightarrow \infty$  which happens exactly when  $\lim_{z \rightarrow 1} E_S(z) = 0$ . We can then define time recovery for discrete time PI observer-based systems as follows.

**Definition 2.1** *Let  $M_I(z)$  be the recovery matrix. Time recovery is obtained if and only if*

$$M_I(1) = 0. \quad (15)$$

Analogous to the continuous time case, the condition for achieving time recovery with a PI observer can now be derived for the discrete time case. With respect to the prediction PI observer we have the following result.

**Theorem 2.1** *Time recovery is obtained with a prediction PI observer if and only if the largest invariant subspace of the matrix  $(I - A - K_P C)^{-1} B K_I C$  contained in the controllable subspace of the pair  $((I - A - K_P C)^{-1}, (I - A - K_P C)^{-1} B)$  corresponding to the eigenvalue  $z = 1$  is itself contained in the unobservable subspace of the pair  $(F, (I - A - K_P C)^{-1} B K_I C)$ .*

*Proof:* See [6].

In connection to Theorem 2.1, the following corollary gives a simple matrix condition which can be checked to determine whether or not time recovery is obtained. We state this result without its straightforward, but lengthy proof.

**Corollary 2.1** *Let the Jordan normal form of the matrix  $(I - A - K_P C)^{-1} B K_I C$  be given by*

$$T^{-1} ((I - A - K_C)^{-1} B K_I C) T = \begin{bmatrix} J_0 & 0 \\ 0 & \bar{J} \end{bmatrix} \quad (16)$$

where  $J_0$  contains all the Jordan blocks associated with the eigenvalue  $z = 1$  according to the partitionings

$$T = [ T_1 \quad T_2 ], \quad T^{-1} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}. \quad (17)$$

Then time recovery is obtained if and only if

$$F T_1 (I, J_0, \dots, J_0^{n-1}) S_1 (I - A - K_P C)^{-1} B = 0.$$

With respect to the filtering PI observer, the only difference is that the target design gain  $F$  in Theorem 2.1 and Corollary 2.1 is replaced by  $F_f(A + K_P C)$ .

Again, the condition on  $K_I$  for time recovery will generically be satisfied if  $K_I C$  has full row rank. As in the continuous time case, however, this condition is neither necessary nor sufficient.

### 3 LQG/LTR Design of PI Observers

Derivation of an LQG/LTR design method for discrete time systems parallels the derivation for continuous time systems given in [6] with the exception that a design can be obtained with zero weighting on the measurement signals, i.e., the cheap estimation in the discrete time case.

First, we need some preliminary results from LQG design.

#### 3.1 LQG Design

Consider the extended state form of a PI observer-based controller, given by (4). An LQG design for the system  $(A, B, C)$  can be done as follows. Select weighting matrices  $\Gamma$  and  $\Sigma$  which satisfy

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [ L_1^T \quad L_2^T ] \geq 0 \quad (18)$$

$$\Sigma \geq 0, \quad (19)$$

respectively. Solve the algebraic Riccati equation

$$P = A_x P A_x^T - A_x P C_x^T (\Sigma + C_x P C_x^T)^{-1} C_x P A_x^T + \Gamma \quad (20)$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}. \quad (21)$$

Then compute  $K_x$  by

$$K_x = -A_x P C_x^T (\Sigma + C_x P C_x^T)^{-1} \quad (22)$$

$$= \begin{bmatrix} -A P_{11} C^T D^{-1} - B P_{12}^T C^T D^{-1} \\ -P_{12}^T C^T D^{-1} \end{bmatrix} \quad (23)$$

where  $D = \Sigma + C P_{11} C^T$ . The I observer gain  $K_I$  has full rank if and only if  $C P_{12}$  has full rank. Rewriting the Riccati equation (20) as four (effectively three) simultaneous equations leads to

$$0 = -P_{11} + A P_{11} A^T + B P_{12}^T A^T + A P_{12} B^T + B P_{22} B^T - A P_{11} C^T D^{-1} C P_{11} A^T - A P_{11} C^T D^{-1} C P_{12} B^T - B P_{12}^T C^T D^{-1} C P_{11} A^T - B P_{12}^T C^T D^{-1} C P_{12} B^T + \Gamma_{11}, \quad (24)$$

$$0 = -P_{12} + A P_{12} + B P_{22} - A P_{11} C^T D^{-1} C P_{12} - B P_{12}^T C^T D^{-1} C P_{12} + \Gamma_{12}, \quad (25)$$

$$0 = -P_{12}^T C^T D^{-1} C P_{12} + \Gamma_{22}. \quad (26)$$

From (26) we see that  $CP_{12}$  has full rank if and only if  $\Gamma_{22} = L_2 L_2^T$  is positive definite. Moreover,  $\Gamma_{22}$  is the only submatrix of  $\Gamma$  which, via  $P_{12}$ , influences  $K_I$ . Therefore, LQG design of a PI observer generically yields time recovery.

From Shaked, [12], we have the following result for the cheap estimation case:

**Lemma 3.1** *Assume that the system  $(A, L, C)$  satisfies:*

$$CA^i L = 0, i = 1, \dots, l-2 \quad (27)$$

$$\det[CA^{l-1}L] \neq 0 \quad (28)$$

The singular stationary Riccati equation ( $\Sigma = 0$ ) for the system  $(A, L, C)$  is given by:

$$P = APA^T + \Gamma - APC^T (CPC^T)^{-1} CPA^T. \quad (29)$$

With  $\Gamma = LL^T$  the observer gain  $K$  is given by

$$K = -A^l \hat{L} (CA^{l-1} \hat{L})^{-1} \quad (30)$$

where the system  $(A, \hat{L}, C)$  is minimum phase, (it is the minimum phase projection of  $(A, L, C)$ ). For minimum phase systems  $\hat{L} = L$ .

*Proof:* This lemma has been proved in [12] for the case when  $(A, L, C)$  is asymptotic stable and without poles or zeros at origin. In [15] it has been shown how these conditions can be removed in the dual case.

With this lemma, we have the observer gain given in an explicit form in the cheap estimation case. Using this observer gain in the recovery matrix for the full-order prediction observer given by, [7]:

$$M_{I,PO}(z) = F(zI - A - K_p C)^{-1} B \quad (31)$$

we have the following result:

**Theorem 3.1** *Let the full-order prediction observer gain  $K$  be given by (30). Then the recovery matrix  $M_{I,PO}(z)$  is given by:*

$$M_{I,PO}(z) = F(zI - A)^{-1} [B - z^{-l} A^l \hat{L} \times (C(zI - A)^{-1} \hat{L})^{-1} C(zI - A)^{-1} B] \quad (32)$$

*Proof:* See [11].

### 3.2 LQG/LTR Design

LQG/LTR design of a full-order P observer can be done by using  $\Gamma = BB^T$  and  $\Sigma = 0$ , [5], [15]. Similarly if we let  $\Gamma = B_x B_x^T$  and  $\Sigma = 0$  in the PI observer design, we obtain via (20) through (23) the following solution for the observer gain:

$$K_x = \begin{bmatrix} -AB(CB)^{-1} \\ 0 \end{bmatrix} \quad (33)$$

with the assumption that the system  $(A, B, C)$  is minimum phase and  $CB$  has full rank. As in the continuous time case, therefore, the integral effect of the PI observer vanishes when using a traditional LQG/LTR design method. Now by using the result from section 3.2, it is possible to derive an LQG/LTR design method for PI-observers which will give time recovery. Let's consider the non minimum phase case. The minimum phase case can be derived out from the non minimum phase case.

First, it is assumed that  $CB$  has maximal rank, which result in the following optimal LQG/LTR gain for the P-observer:

$$K = -A\hat{B}(C\hat{B})^{-1} \quad (34)$$

where  $(A, \hat{B}, C)$  is the minimum phase image of  $(A, B, C)$ . An algorithm for calculating  $(A, \hat{B}, C)$  can be found in [15]. The connection between these two systems is:

$$G(z) = G_m(z)B_a(z) = C(zI - A)^{-1} \hat{B}B_a(z) \quad (35)$$

where  $B_a$  is stable, has zeros coinciding with the non minimum phase zeros of  $G(z)$ , and satisfies  $B_a(z^{-1})^T B_a(z) = I$ . The transfer function  $G_m(z)$  is minimum phase and is termed the minimum phase counterpart of  $G(z)$ .

If  $CB$  does not have full rank and instead satisfies the conditions given in Lemma 3.1, we obtain the expressions for the recovery matrices given in Theorem 3.1:

**Lemma 3.2** *Let the system  $(A, B, C)$  be non-minimum phase and let the optimal LQG/LTR gain be given by:*

$$K = -A^l \hat{B} (CA^{l-1} \hat{B})^{-1} \quad (36)$$

Then the recovery matrices for the prediction and the filtering observer are given by:

$$M_{I,PO}(z) = F(zI - A)^{-1} (B - z^{-l} A^l \hat{B} B_a) \quad (37)$$

$$M_{I,FO}(z) = F(zI - A)^{-1} (B - z^{1-l} A^{l-1} \hat{B} B_a) \quad (38)$$

*Proof:* Lemma 3.2 follows directly from Theorem 3.1 by using  $\hat{L} = \hat{B}$ .

When the PI-observer is applied, we get  $\hat{L}$  directly from the minimum phase condition on  $(A_x, \hat{L}, C_x)$  as

$$\hat{L} = \begin{bmatrix} \hat{B}_l \\ L_2 \end{bmatrix}. \quad (39)$$

By using (39) in (30), the following PI-observer gain is derived for the non-minimum phase case:

$$K_x = -A_x^l \hat{L} (C_x A_x^{l-1} \hat{L})^{-1} \quad (40)$$

$$= \begin{bmatrix} -A^l \hat{B}_l - A^{l-1} B L_2 - \dots - B L_2 \\ -L_2 \end{bmatrix} \times (CA^{l-1} \hat{B}_l)^{-1}. \quad (41)$$

It is now reasonable to state the following result:

**Theorem 3.2** *The recovery matrix  $M_{I,P}(z)$  for the prediction PI-observer take the following form when the optimal LQG/LTR gain in (41) is used:*

$$M_{I,P}(z) = F(zI - A)^{-1} [B - z^{-l}(A^l \hat{B}_l + A^{l-1}BL_2 + \dots + BL_2 + (zI - I)^{-1}BL_2)(zI - I) \times (zI - I + L_2)^{-1}B_a(z)] \quad (42)$$

Moreover, the recovery matrix for the filtering PI-observer take the following form:

$$M_{I,F}(z) = F(zI - A)^{-1} [B - z^{1-l}(A^{l-1} \hat{B}_l + A^{l-2}BL_2 + \dots + BL_2 + (zI - I)^{-1}BL_2) \times (zI - I)(zI - I + L_2)^{-1}B_a(z)] \quad (43)$$

*Proof:* See [11].

**Lemma 3.3** *Let the recovery matrices for the prediction and the filtering PI-observer be given as in Theorem 3.2 where  $|I - L_2| < 1$ . Then for  $z = 1$ , the recovery matrices satisfies*

$$M_{I,P}(1) = M_{I,F}(1) = 0 \quad (44)$$

*Proof:* Lemma 3.3 follows directly from Theorem 3.2 by setting  $z = 1$ .

## 4 Examples

An example is considered in this section. We will consider a SISO system given by the following state space realization:

$$A = \begin{bmatrix} 1.1036 & 1 & 0 \\ -0.4060 & 0 & 1 \\ 0.0498 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0803 \\ 0.1544 \\ 0.0179 \end{bmatrix} \quad (45)$$

$$C = [ 1 \quad 0 \quad 0 ], \quad D = 0$$

The discrete time system is nonminimum phase with one zero at  $z = -1.7989$  and one at  $-0.1239$ . Note also that  $CB \neq 0$ . When the system is nonminimum phase, exact recovery cannot be obtained with a free target design. As target design we use [9]:

$$F = [ 7.1222 \quad 7.5293 \quad 2.7373 ] \quad (46)$$

Applying the optimal LQG/LTR gains for the four types of observers considered in the above section result in the recovery matrices and sensitivity functions shown in Figs. 1 - 4. The PI-observers have been calculated for different values of the gain  $L_2$ . As expected, the filtering observers are quite better than the equivalent prediction versions. Furthermore, it turns out very clearly that

the PI-observers result in time recovery, i.e. the recovery matrix has small gain at low frequencies. It is also important to point out that one can shape the gain of the recovery matrix for the PI-observers at low frequencies by the selection of  $L_2$ .

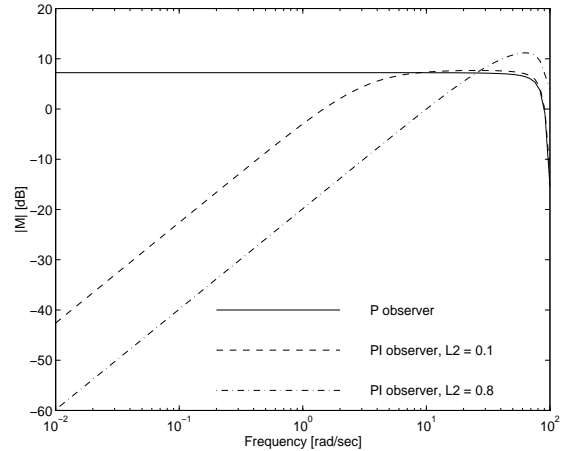


Figure 1: The recovery matrix for the prediction P and PI observer

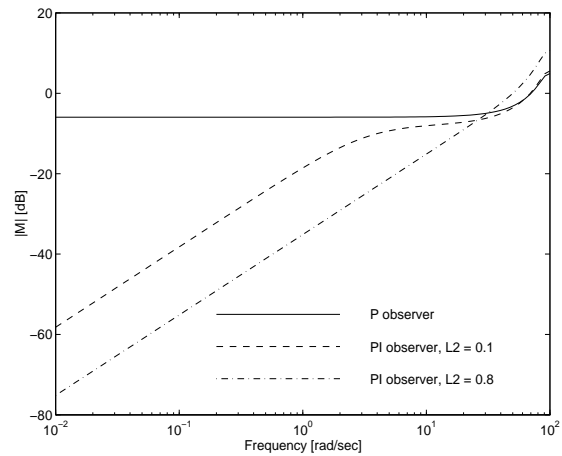


Figure 2: The recovery matrix for the filtering P and PI observer

## 5 Conclusion

This paper presented two versions of the discrete time PI observer, a prediction and a filtering PI-observer. Both LQG and LQG/LTR design methods were derived for each observer type with special attention to the time recovery effect of the PI observer. Necessary and sufficient conditions for achieving LTR and time recovery in PI observer-based systems are given.

Moreover, explicit expressions have been derived for the recovery matrices for both the P and the PI-observer in light of optimal LQG/LTR design. To this part, we have given the general explicit solution of the singular

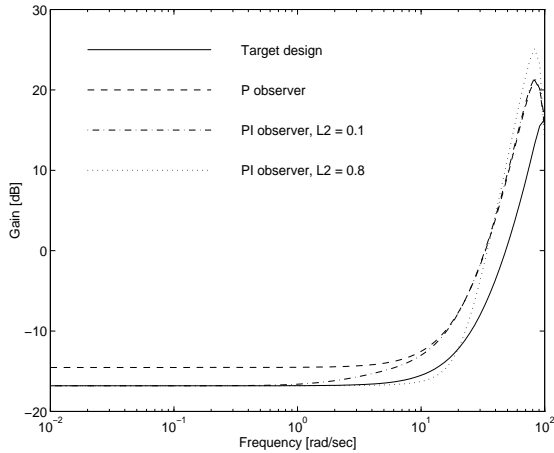


Figure 3: The sensitivity function for the target design and the prediction P and PI observer

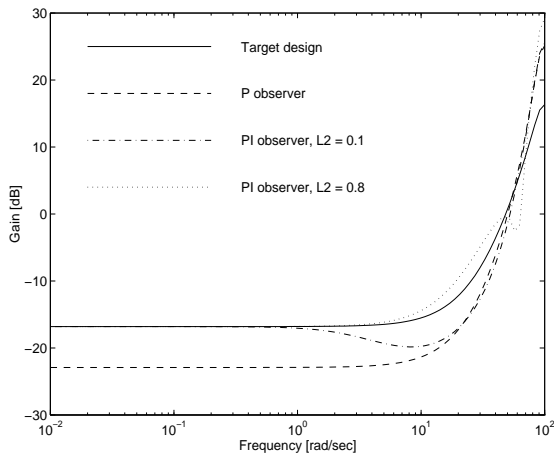


Figure 4: The sensitivity function for the target design and the filtering P and PI observer

discrete time Riccati equation. These explicit forms are derived for both minimum phase as well as for non minimum phase systems. As a direct consequence of these explicit forms for the recovery matrix, it turns out that it is in general always possible to obtain time recovery when PI-observers are applied. Furthermore, the LQG/LTR design method does not have to be employed for achieving time recovery.

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