A NEW APPROACH TO μ-SYNTHESIS FOR MIXED PERTURBATION SETS*

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Abstract

In this paper a new approach for design of μ-optimal controllers is presented. The methodology, denoted μ-K iteration, can be applied for complex as well as mixed complex and real perturbation sets. It is thus more general than the well-known D-K iteration procedure that applies only for complex perturbation sets. The design methodology has been successfully applied to the double integrator example in [9] and to a laboratory centrifugal pump/induction motor configuration resembling a small domestic water supply system.

1 Introduction

Design of controllers with guarantied closed loop stability and performance for uncertain plants has been the focus of active research for almost 2 decades now. Most of the research on robust control has focused on $\mathcal{H}_\infty$ like problems. However it turns out that many practical problems do not readily fit the standard $\mathcal{H}_\infty$ problem setup since the involved model uncertainty is structured rather than unstructured. This causes any $\mathcal{H}_\infty$ controller design to be potentially conservative and thus limits the obtainable performance of the closed loop system. In [6] it is furthermore shown that estimated frequency domain model uncertainty ellipses cannot be represented accurately using an unstructured perturbation set.

Fortunately theory exists that non-conservatively handles these problems, namely the *structural singular value* or μ theory. In many practical applications μ theory is more appropriate for system analysis and controller synthesis. μ theory has not been as widely recognized as $\mathcal{H}_\infty$ theory, probably due to the small amount of literature on μ and to the computational difficulties associated with μ. Recently however algorithms for computing μ^* have become commercially available through the MatLab† *μ-Analysis and Synthesis Toolbox* [1]. Furthermore controller synthesis for structured complex perturbation sets can also be accomplished with the aid of the toolbox.

Unfortunately many practical application problem calls for the use of mixed real and complex perturbation sets. E.g. analysis of plant parameter variations which is an often encountered problem rely on the use of mixed or even purely real perturbation sets. Until recently controller synthesis under mixed perturbation sets was an unsolved problem. A solution to this problem has been given by Young [7, 8]. Unfortunately the synthesis procedure proposed by Young is quite involved. Even though it relies on the same principles it is certainly more mathematically complex than the procedure used for complex perturbation sets.

The main purpose of this paper is to present an alternative μ synthesis procedure for mixed perturbation sets. The presented approach is computationally much simpler than the procedure proposed by Young. The usefulness of the approach is illuminated by two examples. The rest of the paper is organized as follows. In Section 2 a short introduction to the basics of μ theory is given. In Section 3 existing approaches to μ controller synthesis are discussed and in Section 4 the new μ-K approach is presented. Section 5 is devoted to examples and finally a short discussion of the presented results is given in Section 6.

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*More accurately upper and lower bounds on μ.

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2 Robust Stability and Performance

A general framework for robustness analysis of linear systems is illustrated in Figure 1. Any linear interconnection of control inputs u, measured outputs y, disturbances d', controlled outputs (error signals) e', perturbations $w = \Delta z$ and a controller K can be expressed within this framework. The robust control problem

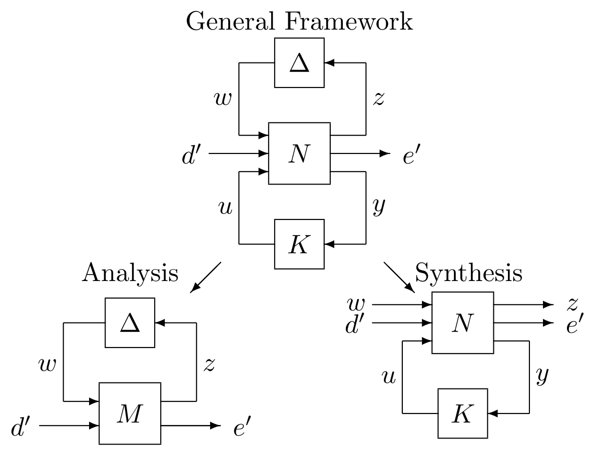

Figure 1: *The general framework with emphasis on analysis and synthesis.*

can then be loosely formulated as to design a controller K such that the perturbed closed loop system is stable and such that the error signal e' is kept "small" in the presence of disturbances d' and perturbations w.

Within the general framework analysis and synthesis constitutes two special cases as illustrated in Figure 1. Conventionally scalings and weights are absorbed into the transfer function N in order to normalize d', e' and Δ to norm 1. For robust analysis the transfer function F_u from d' to e' may be partitioned as a linear fractional transformation:

$$\begin{aligned} e' &= F_u(M,\Delta)d' \\ &= \left[M_{22} + M_{21}\Delta\left(I - M_{11}\Delta\right)^{-1} M_{12}\right] d' \end{aligned} \quad (1)$$

Here Δ is a member of the bounded subset:

$$\mathbf{B\Delta} = \{\Delta \in \mathbf{\Delta} \,|\bar{\sigma}(\Delta) < 1\} \quad (2)$$

where $\bar{\sigma}$ denotes largest singular value and $\mathbf{\Delta}$ is defined by:

$$\begin{aligned} \mathbf{\Delta} = \Big\{ &\mathrm{diag}\left(\delta_1^r I_{r_1}, \cdots, \delta_{m_r}^r I_{r_{m_r}}, \delta_1^c I_{r_{m_r+1}}, \cdots, \right. \\ &\left. \delta_{m_c}^c I_{r_{m_r+m_c}}, \Delta_1, \cdots, \Delta_n\right) |\delta_i^r \in \mathbf{R}, \delta_i^c \in \mathbf{C}, \\ &\Delta_j \in \mathbf{C}^{r_{m_r+m_c+j} \times r_{m_r+m_c+j}} \Big\} \end{aligned} \quad (3)$$

Define also the *corresponding complex perturbation set* $\mathbf{\Delta_c}$ as:

$$\begin{aligned} \mathbf{\Delta_c} = \Big\{ &\mathrm{diag}\left(\delta_1^c I_{r_1}, \cdots, \delta_{m_r+m_c}^c I_{r_{m_r+m_c}}, \Delta_1, \cdots, \Delta_n\right) \\ &\Big|\delta_i^c \in \mathbf{C}, \Delta_j \in \mathbf{C}^{r_{m_r+m_c+j} \times r_{m_r+m_c+j}} \Big\} \end{aligned} \quad (4)$$

The positive real-valued function μ is then defined by:

$$\mu_{\mathbf{\Delta}}(M) \triangleq \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\}} \quad (5)$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) = 0$.

Unfortunately Equation (5) is not suitable for computing μ since the implied optimization problem may have multiple local maxima [2, 3]. However upper and lower bounds for μ may be effectively computed for both complex and mixed perturbations sets. Algorithms for computing these bounds have been documented in several papers, see e.g. [2, 10]. In this paper the algorithms provided in the MatLab *μ-Analysis and Synthesis Toolbox* [1] were used for computing μ-bounds.

The following two Theorems may now be used for assessing robust stability and robust performance [2, 5]:

Theorem 2.1: The controlled system is stable for all $\Delta \in \mathbf{B\Delta}$ iff:

$$\|\mu_{\mathbf{\Delta}}\left(M_{11}\right)\|_\infty \leq 1 \quad (6)$$

where:

$$\|\mu_{\mathbf{\Delta}}\left(M_{11}\right)\|_\infty \triangleq \sup_\omega \mu\left(M_{11}(j\omega)\right) \quad (7)$$

Theorem 2.2: Let an $\mathcal{H}_\infty$ performance specification be given on the transfer function from d' to e' — typically a weighted sensitivity specification — of the form:

$$\|F_u(M,\Delta)\|_\infty = \sup_\omega \bar{\sigma}\left(F_u(M,\Delta)\right) < 1 \quad (8)$$

Then $F_u(M,\Delta)$ is stable and $\|F_u(M,\Delta)\|_\infty < 1 \;\; \forall \Delta \in \mathbf{B\Delta}$ iff

$$\left\|\mu_{\tilde{\mathbf{\Delta}}}(M)\right\|_\infty \leq 1 \quad (9)$$

where the perturbation set is augmented with a full complex performance block:

$$\begin{aligned} \tilde{\mathbf{\Delta}} = \Big\{\mathrm{diag}\left(\Delta, \Delta_p\right) \Big| \Delta \in \mathbf{\Delta}, \Delta_p \in \mathbf{C}^{k \times k}, \\ \bar{\sigma}\left(\Delta_p\right) < 1\Big\} \end{aligned} \quad (10)$$

Theorem 2.2 is the real payoff for measuring performance in terms of the ∞-norm and bounding model uncertainty in the same manner. Using μ it is then possible to test for both robust stability and robust performance in a nonconservative manner. Indeed, if the uncertainty is modeled exactly by Δ — i.e., if all plants in the norm-bounded set can really occur in practice, then the μ condition for robust performance is necessary and sufficient.

3 μ Design - The Existing Approach

For robust synthesis the transfer function F_l from $[w\ d']^T$ to $[z\ e']^T$ may be partioned as the linear fractional transformation:

$$\begin{bmatrix} z \\ e' \end{bmatrix} = F_l(N,K) \begin{bmatrix} w \\ d' \end{bmatrix}$$
$$= \left[N_{11} + N_{12}K\left(I - N_{22}K\right)^{-1} N_{21}\right] \begin{bmatrix} w \\ d' \end{bmatrix} \quad (11)$$

Noticing that $F_l(N,K) = M$ and using Theorem 2.2 a stabilizing controller K achieves robust performance if and only if for each frequency $\omega \in [0,\infty]$, the structured singular value satisfies:

$$\mu_{\tilde{\mathbf{\Delta}}}\left(F_l(N,K)(j\omega)\right) < 1 \quad (12)$$

Consequently our control problem becomes one of synthesizing a controller K that minimizes $\mu_{\tilde{\mathbf{\Delta}}}\left(F_l(N,K)\right)$ across frequency:

$$\inf_{K_s(j\omega)} \sup_{\omega} \left\{\mu_{\tilde{\mathbf{\Delta}}}\left(F_l(N(j\omega),K(j\omega))\right)\right\} \quad (13)$$

where $K_s(s)$ denotes a stabilizing controller $K(s)$. Since the above problem is not tractable (μ can not be directly computed), one may pose a *direct upper bound problem* instead:

$$\inf_{K_s(j\omega)} \sup_{\omega} \inf_{D(\omega)\in\mathbf{D}, G(\omega)\in\mathbf{G}} \inf_{\beta(\omega)\in\mathbf{R}_+} \left\{\beta(\omega) \,|\Gamma \leq 1\right\} \quad (14)$$

where

$$\Gamma = \bar{\sigma}\left(\left(\frac{D(\omega)F_l(N(j\omega),K(j\omega))D^{-1}(\omega)}{\beta(\omega)} - jG(\omega)\right)\cdot \left(I + G^2(\omega)\right)^{-\frac{1}{2}}\right) \quad (15)$$

By the notation *direct* we emphasize that the problem is posed directly in line with the way the upper bound is computed. In fact for fixed K the problem of finding $D(\omega)$, $jG(\omega)$ and $\beta(\omega)$ is just the mixed μ upper bound problem. Having found these scalings for a set of frequencies we may fit transfer function matrices $\mathcal{D}(s)$, $\mathcal{G}(s)$ and $\beta(s)$ to them in such a way that the interconnection is stable. For given $\mathcal{D}(s)$, $\mathcal{G}(s)$ and $\beta(s)$ the problem of finding the controller $K(s)$ will be reduced to a standard $\mathcal{H}_\infty$ problem. The general upper bound problem is significantly reduced in complexity for purely complex perturbation sets in which case the problem can be stated as:

$$\inf_{K_s(j\omega)} \sup_{\omega} \inf_{D(\omega)\in\mathbf{D}} \left\{\bar{\sigma}\left(D(\omega)\cdot F_l\left(N(j\omega),K(j\omega)\right)D^{-1}(\omega)\right)\right\} \quad (16)$$

Here we may impose the extra constraint that $\mathcal{D}(s)$ should be minimum phase (so that $\mathcal{D}^{-1}(s)$ is stable too) since any phase in $\mathcal{D}(j\omega)$ is absorbed into the complex perturbations. It can be shown, see [7], that this applies for the diagonal elements of $D(\omega)$ in the general case (14) also. However for any off-diagonal elements, we must fit both in magnitude and phase. This applies also for the off-diagonal elements of $jG(\omega)$. For the diagonal purely imaginary elements of $jG(\omega)$ we unfortunately also must fit both in magnitude and phase since the phase is not absorbed into the (real) perturbations. We must consequently require the phase of the diagonal elements of $\mathcal{G}(j\omega)$ to be $90°$ *for all frequencies* ω. The fitting of these purely imaginary diagonal elements is probably the Achilles' heel of the general upper bound problem. This can only be obtained using high order all pass structures causing the controller order to explode.

The procedure of iteratively solving Equation (14) is usually referred to as *D-K* iteration in the case of purely complex perturbations and *D,G-K* iteration for mixed perturbations.

4 μ Design - A New Approach

In this section a different approach denoted μ-*K* iteration for mixed μ synthesis will be presented. The authors acknowledge that the term μ-*K* iteration has been used elsewhere [4] for different purposes. However it applies well here and will be used throughout this paper. Whereas the procedure by Young is a *direct* upper bound minimization we instead propose to use an *indirect* upper bound minimization. By *indirect* we mean that an augmented system matrix $P(s)$ is constructed which does not directly reflect the structure of the μ upper bound.

The main idea of the proposed μ-*K* iteration scheme is to perform a series of scaled D-K iterations where the difference between mixed and complex μ is taken into account through an additional scaling matrix $\Gamma(s)$. The iteration is performed as follows:

Procedure 4.1 (μ-K Iteration)
1. Given the augmented system $N(s)$, let $\gamma_0(s) = 1$, $P_0(s) = \Gamma_0(s)N(s)$ where

$$\Gamma_0(s) = \begin{bmatrix} \gamma_0(s)I_{n_{wd}} & 0 \\ 0 & I_{n_u} \end{bmatrix}. \quad (17)$$

$n_{wd} = dim\ \{[w;d']\}$ denotes the number of external inputs and $n_u = dim\ \{u\}$ denotes the number of controlled inputs. Let $i = 1$.
2. Compute the optimal μ controller for the corresponding complex problem using D-K-iteration:

$$K_{i,j} = \inf_{K\ stab.} \sup_{\omega} \left\{\bar{\sigma}\left(F_l(D_{i,j-1}P_iD_{i,j-1}^{-1},K)\right)\right\} \quad (18)$$
$$D_{i,j}^* = \inf_{D\in\mathbf{D}} \bar{\sigma}\left(F_l(DND^{-1},K_{i,j})\right), \ \ \forall\omega \geq 0. \quad (19)$$

Here $D_{i,j}(s)$ are stable minimum phase transfer function estimates of $D^*_{i,j}(\omega)$. The iteration is repeated until $K_{i,j} = K_{i,j-1} = K_i$, $D_{i,j} = D_{i,j-1} = D_i$ and the complex μ upper bound $\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(N, K_i))$ is flat across frequency.
3. Compute the mixed μ upper bound $\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N, K_i))$ at each frequency ω.
4. Choose some constant α_i satisfying $\alpha_i \in [0, 1]$ and compute at each frequency ω the scalar

$$\gamma_i^* = (1 - \alpha_i)|\gamma_{i-1}| + \alpha_i \frac{\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N, K_i))}{\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(N, K_i))} \quad (20)$$

Fit, in magnitude, a stable minimum phase SISO transfer function $\gamma_i(s)$ to $\gamma_i^*(\omega)$ across frequency ω.
5. Compute the optimal complex μ controller for the system $P_i(s) = \Gamma_i(s)N(s)$ using D-K-*iteration as in 2.*
6. Compute the mixed and corresponding complex μ upper bounds $\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K_{i+1}(s)))$ and $\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(N(s), K_{i+1}(s)))$ and check whether

$$\left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K_{i+1}(s)))\right\|_\infty \leq \left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(\Gamma_i(s)N(s), K_{i+1}(s)))\right\|_\infty \quad (21)$$

$$\left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K_{i+1}(s)))\right\|_\infty \leq \left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K_i(s)))\right\|_\infty \quad (22)$$

If both Equation (21) and (22) are fulfilled, let $i = i+1$. If any of the inequalities are not fulfilled, return to 4 and reduce α_i.
7. Repeat from 4 until no further reduction in $\left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K_{i+1}(s)))\right\|_\infty$ can be achieved.

■

We now have the following lemma:

Lemma 4.1: *The μ-K iteration procedure described above is monotonically non-increasing in $\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K(s)))\|_\infty$ given perfect realizations of the $D(s)$ and $\gamma(s)$ scalings.*

Notice that Lemma 4.1 does not necessarily imply convergence to a local minimum for $\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K(s)))$. However this is an inherent difficulty in complex *D-K* iteration also and the numerical evidence presented later proves in favor of the algorithm.

Proof of Lemma 4.1 Given the complex μ optimal controller $K_i(s)$ and perfect realizations of the γ-scalings it is clear from (21) that:

$$\left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K_i(s)))\right\|_\infty \leq \left\|\gamma_{i-1}(s)\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(N(s), K_i(s)))\right\|_\infty \quad (23)$$

From (23) it can be shown that

$$\left\|\gamma_i(s)\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(N(s), K_{i+1}(s)))\right\|_\infty \leq \left\|\gamma_{i-1}(s)\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(N(s), K_i(s)))\right\|_\infty \quad (24)$$

Furthermore

$$\alpha_i \to 0 \Rightarrow \begin{cases} \bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N, K_{i+1})) \to \bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N, K_i)) \\ \bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(\Gamma_i N, K_{i+1})) \to \bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(\Gamma_{i-1} N, K_i)) \end{cases} \quad (25)$$

Then due to continuity of mixed and complex μ in $K(s)$ there exists an $\alpha_i \geq 0$ such that the following 2 inequalities are both fulfilled

$$\left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K_{i+1}(s)))\right\|_\infty \leq \left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(\Gamma_i(s)N(s), K_{i+1}(s)))\right\|_\infty \quad (26)$$

$$\left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(j\omega), K_{i+1}(j\omega)))\right\|_\infty \leq \left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(j\omega), K_i(j\omega)))\right\|_\infty \quad (27)$$

If $\alpha_i > 0$ can be chosen, the mixed μ upper bound will be reduced during the i'th iteration. From (24) it is seen that $\left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(\Gamma_i(s)N(s), K_{i+1}(s)))\right\|_\infty$ is monotonically non-increasing. Furthermore when

$$|\gamma_i(j\omega)| \to \frac{\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(j\omega), K_i(j\omega)))}{\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(N(j\omega), K_i(j\omega)))} \quad (28)$$

it is clear that

$$\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(\Gamma_i(j\omega)N(j\omega), K_{i+1}(j\omega))) \to \bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(j\omega), K_{i+1}(j\omega))). \quad (29)$$

Through the constraint (26) it is then assured that $\left\|\bar{\mu}_{\tilde{\boldsymbol{\Delta}}_{\mathbf{c}}}(F_l(\Gamma_i(s)N(s), K_{i+1}(s)))\right\|_\infty$ does not decay below the global (local) infimum for $\bar{\mu}_{\tilde{\boldsymbol{\Delta}}}(F_l(N(s), K(s)))$ in which case the algorithm would not converge. Numerical experience has shown that the lengthy *D-K* iteration in step 5 of μ-*K* iteration may be replaced by a single $\mathcal{H}_\infty$ optimization like in (18) and complex μ-analysis like in (19) provided α is chosen properly.

The main advantage of μ-*K* iteration compared with *D,G-K* iteration is that one need only fitting $D(s)$ and $\gamma(s)$ in magnitude, whereas in *D,G-K* iteration you have to fit jG both in magnitude and phase. Hence, like in *D-K* iteration, the extra constraint may be imposed that the transfer function estimates must be minimum phase. Notice furthermore that the standard *D-K* iteration for complex perturbations is just a special case of the proposed scheme since then $\gamma_i^*(\omega) = 1$, $\forall \omega, i \geq 0$.

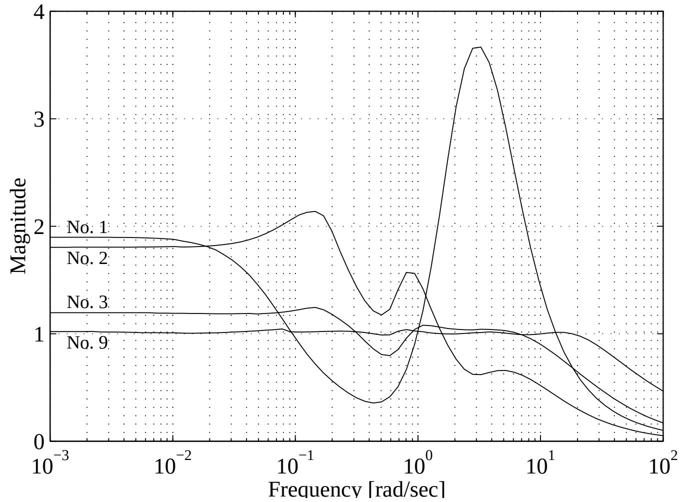

Figure 2: Upper μ bound for step 1 (the optimal $\mathcal{H}_\infty$ controller), 2, 3 and 9 in the μ-K iteration for the double integrator example.

Controller	$\left\|\bar{\mu}_{\tilde{\mathbf{\Delta}}}\left(F_l(N,K)\right)\right\|_\infty$
$\mathcal{H}_\infty$	3.67
1st μ	2.14
2nd μ	1.25
3rd μ	1.10
4th μ	1.08
5th μ	1.06
6th μ	1.05
7th μ	1.04
9th order reduced μ	1.05
9th order D,G-K [9]	1.25

Table 1: *Upper bound on mixed μ for each step in the μ-K iteration and for the D,G-K controller in [3].*

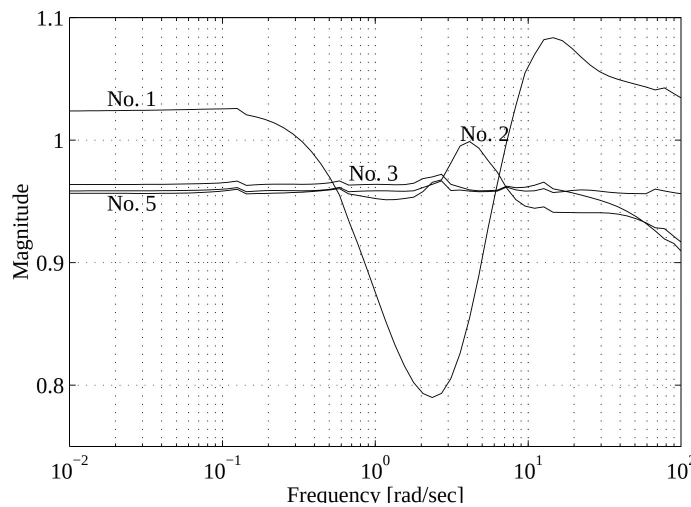

Figure 3: Upper μ bound for step 1 (the optimal $\mathcal{H}_\infty$ controller), 2, 3 and 5 in the μ-K iteration for the pump system example.

5 Two Illustrative Examples

In this section two design examples will be given. First a comparison between *D,G-K* and *μ-K* iteration will be given for the uncertain double integrator system introduced in [9]. Secondly it will be shown how estimated frequency domain uncertainty ellipses have been expressed in the general μ-framework using a mixed perturbation set and how a optimal μ controller has been designed using *μ-K* iteration.

5.1 The Double Integrator

The first example is a double integrator from the paper [9] by Young and Åström. The plant is given by:

$$P(s) = \frac{k_p}{s^2} \tag{30}$$

where the plant gain k_p is assumed uncertain, but confined to the interval:

$$0.1 \leq k_p \leq 10 \tag{31}$$

The plant is augmented with weighting matrices on sensitivity and complementary sensitivity. Please consult [9] for details. In [9] *D,G-K* iteration was performed on the augmented plant. The reported result with a 9th order controller was a peak value of $\bar{\mu}_{\tilde{\mathbf{\Delta}}}\left(F_l(N,K)\right)$ equal to 1.25. *μ-K* iteration with $\alpha = 0.5$ was performed on the exact same system with the results given in Table 1 and illustrated in Figure 2.

As seen in Table 1 an upper μ bound peak of 1.04 was achieved for the full order (24 states) μ controller. It was possible to reduce the order of the controller to 9 with very little performance degradation, see Table 1. Consequently the upper bound on μ was reduced with more than 15% in comparison with the results presented in [9]. Furthermore notice from Figure 2 that the mixed μ upper bound is flattened completely out indicating that we have reach a (local) minimum. For this particular example *μ-K* iteration consequently seems to perform better than *D,G-K* iteration. We believe that this is probably due to difficulties in fitting the scalings accurately.

5.2 Control of a Domestic Water Supply Unit

The *μ-K* iteration procedure has also been used in control of a laboratory centrifugal pump/induction motor configuration resembling a small domestic water supply system. In this example the discrete time coun-

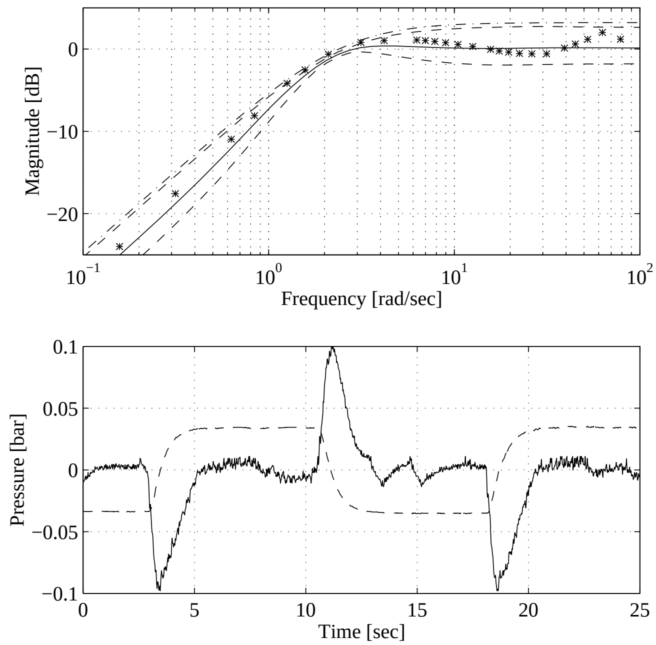

Figure 4: Results from applying μ-K iteration on the water supply system.

terpart of the μ-K iteration was used. Frequency domain uncertainty ellipses around a nominal Nyquist curve were estimated using the stochastic embedding approach. A mixed perturbation set was then used to approximate the uncertainty ellipses, for details please consult [6]. An upper bound on the closed loop sensitivity was given as performance specification and the μ-K synthesis procedure was used to find a controller K. This time α could be chosen as one. This example is documented in more details in [6]. In Figure 3 the mixed μ upper bound for each iteration is displayed. As seen, the algorithm again flattens out μ to achieve a local minimum. The final controller had 52 states, but was reduced to 6 states with virtually no increase in $\|\mu\|_\infty$. The reduced order controller was then implemented on the true system with the results illustrated in Figure 4. In the upper plot the nominal sensitivity (solid) with error bounds (dashed) is shown. The error bounds on the sensitivity were found by computing the smallest distance from the open loop uncertainty ellipses to the Nyquist point (-1,0). Also shown are the performance specification (dashed-dotted) and discrete sensitivity points ($*$) measured on the system in closed loop by applying pure sinusoids on the reference. Notice how smoothly the uncertain system stays below the performance specification and how the sensitivity measurements falls nicely within the estimated uncertainty limits. In the lower plot the pressure response (solid) of the closed loop system towards sudden changes in water consumption (dashed) is shown.

6 Discussion

A new method for designing μ optimal controllers for both complex and mixed perturbation sets was proposed. The method denoted μ-K iteration has the advantage that it is computational much more simple than the D,G-K iteration proposed by Young. Furthermore it was shown that μ-K iteration for a given example produced superior results. We believe that this is due to difficulties in D,G-K iteration when fitting pure imaginary scalings with stable transfer functions. This is avoided in μ-K iteration where the necessary scalings only have to be fitted in magnitude. Hence the additional constraint that the stable transfer function estimates shall be minimum phase could be enforced.

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