A MULTI OBJECTIVE $\mathcal{H}_\infty$ SOLUTION TO THE RUDDER ROLL DAMPING PROBLEM *

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Abstract. Roll damping and simultaneous course steering by rudder control is a challenging problem where a key factor is roll damping performance in waves. Roll is a decisive factor for the operation of ships, both due to comfort of crew and passengers and due to requirements from cargo or on-board equipment. In the paper, roll damping and steering performance requirements are described and the controller design problem is formulated in an $\mathcal{H}_\infty$ framework. It is shown how this design problem can be handled by using a multi objective $\mathcal{H}_\infty$ approach. The design results are compared with an existing LQ design.

Keywords. Ship control, Multi objective design, $\mathcal{H}_\infty$ controller design, Youla parameterization

1 INTRODUCTION

A ship's rudder is primarily used to create torques to turn the ship - alter its course - but, at the same time, roll torques are created. This second effect from the rudder can be utillized to obtain damping of roll motion simultaneously with control of the ship course. When using the rudder for both tasks, some physical obstacles need to be considered. When a ship goes into a turn it always obtains a certain roll angle. If it is prevented to heel - the naval expression for steady roll angle - turning of the ship could not be obtained either. However, in the initial phase of a turn, the force from the rudder makes the ship roll opposite to the static state field. The nature of this problem is hence single input-multi output and a non minimum phase relation exists in the rudder to roll angle dynamics. Performance requirements to the control system includes that damping of roll is effective in the frequency range of natural and wave induced roll, but the disturbance this makes to the ship heading must be limited. For these reasons, roll damping by rudder control is not a straightforward control problem. Several design issues have been solved, and Rudder Roll Damping (RRD) systems have become increasingly popular in recent years. Commercial reasons include the cost-effectiveness of this approach compared with fin stabilizer solutions and the possibility of applying the RRD concept on existing vessels.

RRD design issues have been discussed in a number of papers. The first experiments were reported by Baitis et.al. Theoretic LQ results were derived (van Amerongen et.al.). Systems were designed and implemented (Blanke et.al. 1989, Källström and Schultz), $\mathcal{H}_\infty$ controllers were investigated (Katebi et.al.), and robustness properties of LQ based RRD were investigated (Blanke and Christensen).

Despite the progress, the effectiveness of RRD controls has been debated. Some results from full scale evaluation on vessels indicate very satisfactory results showing 50-70 % roll reduction Blanke et.al. Others indicate much less effectiveness in certain cases, and for some ships the physical properties have been such that traditional RRD designs could not be used at all. This has caused renewed research interest where robustness considerations and improvements in design methods are key issues.

In this paper we investigate the design of $\mathcal{H}_\infty$ controllers for the full single input-multi output RRD control problem. It turns out that the two performance specifications are related to two sensitivity transfer functions. This together with the fact that the model is marginal stable, make it possible to apply a sensitivity multi objective $\mathcal{H}_\infty$ design approach. A complete description of this design approach can be found in (Stoustrup and Niemann). The properties of the design are illustrated with theoretical data for a multipurpose naval vessel and the performance is compared with that of an existing LQ design.

*This work is supported by the Danish Technical Research Council under grant no. 26-1830.

The rest of this paper is organized as follows. In Section 2, the problem is formulated including a description of the performance specifications. Further, the design problem is formulated in the $\mathcal{H}_\infty$ framework. The multi objective $\mathcal{H}_\infty$ design approach is shortly reviewed in Section 3 and the applied $\mathcal{H}_\infty$ approach is given in Section 4. Section 5 include the design results followed by a conclusion in section 6.

2 PROBLEM FORMULATION

The mathematical model for the part of the system to be controlled is a 5th order state space equation for $x_s(t)$ with waves considered as an output disturbance.

$$y_{ship} = C_s x_s + y_w \tag{2.1}$$

A linear model of the ship is given by, (Blanke et.al., Blanke and Christensen):

$$\begin{array}{rcl} \dot{x}_s &=& A_s x_s + B_s u_s \\ y_s &=& C_s x_s \end{array} \tag{2.2}$$

or

$$G_s(s) = \left(\begin{array}{c|c} A_s & B_s \\ \hline C_s & 0 \end{array} \right)$$

where the state vector is $x_s = [\ v \quad r \quad \Psi \quad p \quad \phi\ ]^T$ (sway vel., turn rate, heading, roll rate, roll angle). The three matrices in (2.2) are given in Appendix A.

2.1 Disturbance Modeling

Wave disturbances cannot be modeled as a state space disturbance as forces - moments in (2.2). The reason is that wave forces act over the entire hull and the coefficients in a state space description would be frequency dependent. Calculation of wave induced motions is instead done as response functions from strip theory, or they may be measured. The result is that wave disturbances are characterized in a vector $y_w = [v, r, \Psi, p, \phi]_w$. The relation between wave height, ξ_w and hull motions in y_w are complex. They depend on wave length, λ, wave direction relative to the ship, χ, and encounter frequency, ω_e. To a first order approximation, wave motions are linear, and we can therefore obtain the motion of the hull as a superposition of the wave induced motion and that created by rudder activity.

The reduction ratio of a motion, i.e., the ratio between the uncontrolled and controlled response, is a key indicator for control quality in waves. For RRD, and the reduction function for roll damping is the crucial factor. The mean square of each component of the motion vector $y_{ship}(t)$ is determined by the powerspectrum of wave amplitude, $G_{\zeta\zeta}$ and the wave response operator, $WRO_{y_i\zeta_i}$ as

$$\begin{array}{l} E\left\{y^2_{ship,i}(t)\right\} = \\ \frac{1}{\pi}\int_0^\infty \left|\left[\frac{y_{ship,i}(\omega_e)}{y_{w,i}(\omega_e)}\right]_i\right|^2 |WRO_{y_i\zeta_i}(\omega)|^2 \\ G_{\zeta\zeta}(\omega)d\omega; i = 1, 2, ..., 5 \end{array} \tag{2.3}$$

The reduction ratio for each of the motions is

$$|T_{ii}(\omega_e)| = \left[\frac{y_{ship}(\omega_e)}{y_w(\omega_e)}\right]_i ; i = 1, 2, ..., 5 \tag{2.4}$$

Efficient roll damping is obtained when $|T_{55}(\omega_e)|$ is well below 1 over the range of frequencies, 0.7 to 1.1 rad/sec, where natural roll and wave induced motions occur. Requirements to roll damping performance are most convenient specified in terms of the shape of the $|T_{55}(\omega_e)|$ function at different values of ship speed. A maximum value of wave height needs also to be specified to check the linearity range for the rudder servo system.

Robust control is achieved if the required value of $|T_{55}(\omega_e)|$ is met regardless of changes in ship speed, loading conditions, hydrodynamic parameters or other coefficients in the equations of motion.

The basic performance problem is therefore, by nature, an $\mathcal{H}_\infty$ problem. The wave motion is an output disturbance and the roll reduction function is the sensitivity function of the closed loop control problem. The inverse of the $\mathcal{H}_\infty$ design weight function are shown as the dotted lines in figures 3 - 4.

2.2 Steering Performance

While there is a quite concise performance requirement to roll damping, steering properties are more vaguely expressed. There are two main requirements to steering performance. One is that wave motions in r and Ψ should not cause rudder fluctuation at wave frequencies. The reason is that noticeable propulsion losses occur if the rudder fluctuates too heavily and the rudder servo mechanics gets worn. A second is that the ship heading should be maintained despite steady state or low frequency disturbances, e.g., from wind. These performance requirements can be expressed in an $\mathcal{H}_\infty$ design weight function. The inverse of the selected weight function is shown as the dotted line in figure 3.

2.3 The $\mathcal{H}_\infty$ Design Setup

For the design of the robust controller, the design specifications for the roll angle and for the heading are given above by four weight matrices W_{rr}, W_{rh}, W_{hr}, and W_{hh}.

In terms of these weight matrices the design specifications take the following form.

Problem 1: *Consider the ship model (2.2). Let* γ_1 *and* γ_2 *be given positive numbers. Design, if possible, a controller such that the closed loop system satisfies*

$$\left\| \begin{matrix} W_{rr}S_{rr} \\ W_{hr}T_{hr} \end{matrix} \right\|_\infty < \gamma_1, \quad \left\| \begin{matrix} W_{rh}T_{rh} \\ W_{hh}S_{hh} \end{matrix} \right\|_\infty < \gamma_2 \quad (2.5)$$

where S_{rr} *and* S_{hh} *are the output sensitivities for the roll angle and the heading, respectively.* T_{hr} *is the transfer function from a disturbance placed at the roll angle measurement to the heading, and* T_{rh} *is the transfer function from a disturbance placed at the heading measurement to the roll angle.*

All weight functions are described in state space form:

$$G_w(s) = \left(\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array} \right) \quad (2.6)$$

In $\mathcal{H}_\infty$ control usually the weightings are introduced by lumping the physical model and the weights into a new, fictitious system which takes the form of a standard design problem. However, due to the fact, that our problem is formulated as a multi objective problem we shall instead need two such standard problems: one for a roll angle disturbance and one for a heading disturbance. Each system will take the form

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right) \quad (2.7)$$

where the state vectors for the two systems are respectively

$$x = \begin{cases} \begin{bmatrix} x_s \\ x_{wrr} \\ x_{wrh} \end{bmatrix} & \text{for a roll angle dist.} \\ \begin{bmatrix} x_s \\ x_{whr} \\ x_{whh} \end{bmatrix} & \text{for a heading dist.} \end{cases} \quad (2.8)$$

We shall not give the tedious details for the two systems which is straightforward to write down. They can also be generated automatically by software packages like SIMULINK in MATLAB$^{\text{TM}}$.

However, that it should be noted that in either case, the direct term D_{11} from w to z is not zero as required in the following $\mathcal{H}_\infty$ controller design. This is always the case when the design specification is an output sensitivity function. The direct term can, though, be removed very easily by using a loopshifting method from (Stoorvogel). The loopshifted system is given by:

$$G_{ls}(s) = \left(\begin{array}{c|cc} A_{ls} & B_{ls,1} & B_{ls,2} \\ \hline C_{ls,1} & 0 & D_{ls,12} \\ C_{ls,2} & D_{ls,21} & 0 \end{array} \right) \quad (2.9)$$

where the new matrices are given by:

$$\begin{aligned} A_{ls} &= A - \gamma^{-1} B_1 D_{11}^T C_1 \\ B_{ls,1} &= -B_1 (I - \gamma^{-2} D_{11}^T D_{11})^{-1/2} \\ B_{ls,2} &= B_2 - \gamma^{-2} B_1 D_{11}^T D_{12} \\ C_{ls,1} &= \gamma^{-1} (I - \gamma^{-2} D_{11} D_{11}^T)^{-1/2} C_1 \\ C_{ls,2} &= C_2 - \gamma^{-2} D_{21} D_{11}^T C_1 \\ D_{ls,12} &= \gamma^{-1} (I - \gamma^{-2} D_{11} D_{11}^T)^{-1/2} D_{12} \\ D_{ls,21} &= D_{21} (I - \gamma^{-2} D_{11}^T D_{11})^{-1/2} \end{aligned}$$

where γ is the selected $\mathcal{H}_\infty$ norm for the closed loop system. The connection between the two systems in (2.7) and (2.9) is given in the following lemma, based on (Stoorvogel):

Lemma 1: *Assume* $\|D_{11}\|_\infty < \gamma$. *Let a transfer function* K *of appropriate dimensions be given. Then the following two statements are equivalent*

1. *K is an internally stabilizing controller for the original system (2.7) which makes the closed loop $\mathcal{H}_\infty$ norm from w to z smaller than γ*
2. *K is an internally stabilizing controller for the loopshifted system (2.9) which makes the closed loop $\mathcal{H}_\infty$ norm from w to z smaller than γ*

3 MULTI OBJECTIVE SENSITIVITY CONTROL

In the following we shall study a multi output sensitivity problem formulated as a number of coupled $\mathcal{H}_\infty$ problems. The approach suggested can be applied to a huge number of variations on the multi output sensitivity problem, the complementarity sensitivity problem, and the control sensitivity problem, but for brevity we shall restrict attention to the output sensitivity problem in this section. A complete description of the multi objective $\mathcal{H}_\infty$ design approach can be found in (Stoustrup and Niemann).

Throughout the sequel we shall consider a finite dimensional, linear, time invariant system with a state space realization of the form

$$G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \quad (3.10)$$

and with transfer function $G(\cdot)$. We shall assume the plant to be square, with k inputs and outputs. For such a system, the multi objective sensitivity problem is depicted in Fig. 1.

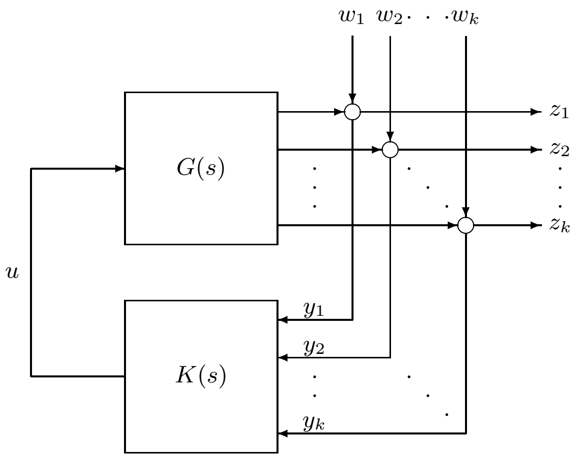

Fig. 1. Multi Objective Sensitivity Problem

The block diagram in Fig. 1 is described by the relations

$$\begin{aligned}\begin{pmatrix} z \\ y \end{pmatrix} &= \begin{pmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} \\ &= \begin{pmatrix} I & G \\ I & G \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} \\ u &= Ky \end{aligned}$$

with

$$\begin{aligned} w &= \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}, \\ z &= \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}. \end{aligned}$$

Writing the transfer function from w to z as a linear fractional transformation in K we get

$$\begin{aligned} T_{zw} &=: \begin{pmatrix} s_{11} & t_{12} & \cdots & t_{1k} \\ t_{21} & s_{22} & \cdots & t_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k1} & t_{k2} & \cdots & s_{kk} \end{pmatrix} \\ &= G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} \\ &= I + GK(I - GK)^{-1} \end{aligned}$$

where the functions s_{ii}, $i = 1 \dots k$, are the output sensitivities (Fig. 1). The functions t_{ij}, $i = 1 \dots k$, $j = 1 \dots k$, $i \neq j$, are crossover terms which indicate how much the i^{th} disturbance influences the j^{th} output.

Loopshaping one of the columns of T_{zw} by specifying upper bounds for the modulus of its entries can formulated as a standard $\mathcal{H}_\infty$ problem as follows.

Problem 2: *The* j^{th} SIMO problem *for the configuration in Fig. 1 is said to be solvable if and only if there exists a controller K which internally stabilizes the plant and such that*

$$\left\| \begin{matrix} W_{1j}t_{1j} \\ \vdots \\ W_{jj}s_{jj} \\ \vdots \\ W_{nj}t_{nj} \end{matrix} \right\|_\infty < 1$$

where

$$\begin{aligned} s_{jj}(\cdot) &= 1 + g_j(\cdot)K(\cdot)(I - G(\cdot)K(\cdot))^{-1}e_j \\ t_{ij}(\cdot) &= g_i(\cdot)K(\cdot)(I - G(\cdot)K(\cdot))^{-1}e_j, \quad i \neq j \end{aligned}$$

e_j is the (constant) vector

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}} \text{ position}$$

and $g_i(s)$ is the row of transfer functions from u to y_i.

In the sequel, we shall make extensive use of the following decoupling result for the above multi objective $\mathcal{H}_\infty$ problem for a stable plant, which in fact is extremely simple.

Theorem 1: *Consider the system (3.10). Assume that A is a stability matrix. Then, the following two statements are equivalent*

1. *There exists an internally stabilizing controller K such that*

$$\left\| \begin{matrix} W_{11}s_{11} \\ \vdots \\ W_{m1}t_{m1} \end{matrix} \right\|_\infty < 1 \quad , \quad \dots \quad ,$$
$$\left\| \begin{matrix} W_{1j}t_{1j} \\ \vdots \\ W_{jj}s_{jj} \\ \vdots \\ W_{mj}t_{mj} \end{matrix} \right\|_\infty < 1 \quad , \quad \dots \quad ,$$
$$\left\| \begin{matrix} W_{1m}t_{1m} \\ \vdots \\ W_{mm}s_{mm} \end{matrix} \right\|_\infty < 1$$

in the closed loop system simultaneously*,*

2. *Each of the m SIMO problems from Problem 2 is solvable* independently*.*

Remark 1: *The significance of Theorem 1 is that just as much can be achieved by a single controller which controls* all *the columns of T_{zw} simultaneously, as if the controller just had to control* one *of them. In fact, as shall be evident from the proof below, it is possible to design such a multi objective $\mathcal{H}_\infty$ controller, by designing an $\mathcal{H}_\infty$ controller for each of the SIMO problems from Problem 2.*

Proof: Let the plant G be row partitioned as

$$G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{pmatrix}$$

Since G is stable, the YJBK-parametrization (Youla - Jabr - Bongiorno - Kučera) of all stabilizing controllers is given by

$$K = Q(I + GQ)^{-1}, \quad Q \in \mathcal{RH}_\infty \tag{3.11}$$

where Q is given by

$$Q = K(I - GK)^{-1}$$

the transfer function from w to z becomes

$$\begin{aligned} T_{zw} &= I + GQ \\ &= \begin{pmatrix} s_{11} & t_{12} & \cdots & t_{1k} \\ t_{21} & s_{22} & \cdots & t_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k1} & t_{k2} & \cdots & s_{kk} \end{pmatrix} \end{aligned}$$

where $s_{ii} = 1 + g_i q_i$, $t_{ij} = g_i g_j$ and Q has the following column partition

$$Q = \begin{pmatrix} q_1 & q_2 & \cdots & q_k \end{pmatrix}$$

Now, the crucial observation is that since

$$\begin{pmatrix} W_{1j}t_{1j} \\ \vdots \\ W_{jj}s_{jj} \\ \vdots \\ W_{mj}t_{mj} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ W_{jj} \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} W_{1j}g_1 \\ \vdots \\ W_{jj}g_j \\ \vdots \\ W_{mj}g_m \end{pmatrix} q_j \tag{3.12}$$

the j^{th} SIMO problem depend on q_j only. Since the q_j's are free stable parameters, each optimization can be done completely independently, where after K is determined by (3.11). From this simple observation the claim becomes trivial.

□

From the proof of Theorem 1 it is apparent that an $\mathcal{H}_\infty$ controller K which satisfy the above multi objective problem can be found by determining the q_j's and then applying (3.11). Each of these k transfer matrices (columns) can be found by solving a single input standard $\mathcal{H}_\infty$ problem based on (3.12). For instance for the simple special case where the only nonzero weightings are for the sensitivity functions, each of the k associated standard problems based on (3.12) which in transfer function form is

$$\|w_j(1 + g_j q_j)\|_\infty < 1$$

has the following standard state space formulations

$$w_j s_{jj}(s) = \left(\begin{array}{cc|cc} A & 0 & 0 & B \\ 0 & Aw_j & Bw_j & 0 \\ \hline e_j'C & Cw_j & Dw_j & e_j'D \\ 0 & Cw_j & Dw_j & 0 \end{array}\right)$$

The multi objective $\mathcal{H}_\infty$ approach applied on unstable systems is considered in (Stoustrup and Niemann).

4 THE SINGULAR $\mathcal{H}_\infty$ DESIGN APPROACH

In the previous section we derived a model of the form

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right) \tag{4.13}$$

Unfortunately, the derived model does not satisfy the standard assumptions (Doyle et.al.). One assumption which is violated for the model obtained in Section 2.3 in the approach of (Doyle et.al.) are the regularity assumptions, i.e. that D_{12} and D_{21} must have full column and row ranks, respectively. To overcome this problem we shall take off from the approach of (Stoorvogel).

First, we need the following assumption:

Assumption 1: *It is assumed that the systems $(A,\ B_1,\ C_2,\ D_{21})$ and $(A,\ B_2,\ C_1,\ D_{12})$ have no invariant zeros at the imaginary axis.*

We have then the following result, (Stoorvogel):

Theorem 2: *Consider the system in (4.13) satisfying Assumption 1. Let $\gamma > 0$ be given. Then, there exist a FDLTI compensator $u = Q(s)y(s)$ for which the resulting closed loop system is internally stable, and for which the transfer function from w to z has an $\mathcal{H}_\infty$ norm smaller than γ, if and only if there exist positive semidefinite matrices P and Q such that*

1. $$\begin{aligned} F_\gamma(P) &:= \begin{pmatrix} A_P & C_P' \\ C_P & D_P \end{pmatrix} \\ &=: \begin{pmatrix} C_{1P}' \\ D_{12P}' \end{pmatrix} \begin{pmatrix} C_{1P} & D_{12P} \end{pmatrix} \geq 0 \end{aligned}$$

2. $G_\gamma(Q) := \begin{pmatrix} A_Q & B_Q \\ B_Q' & D_Q \end{pmatrix}$
$=: \begin{pmatrix} B_{1Q} \\ D_{21Q} \end{pmatrix} \begin{pmatrix} B_{1Q}' & D_{21Q}' \end{pmatrix} \geq 0$

3. $\mathrm{rank} \begin{pmatrix} C_{1P} & D_{12P} \end{pmatrix} = \mathrm{rank}_{\mathcal{R}(s)} \left[C_1(sI-A)^{-1}B_2 + D_{12}\right]$

4. $\mathrm{rank} \begin{pmatrix} B_{1Q} \\ D_{21Q} \end{pmatrix} = \mathrm{rank}_{\mathcal{R}(s)} \left[C_2(sI-A)^{-1}B_1 + D_{21}\right]$

5. $\mathrm{rank} \begin{pmatrix} A + \gamma^{-2}B_1B_1'P - s_0I & B_2 \\ C_{1P} & D_{12P} \end{pmatrix} = n + \mathrm{rank}_{\mathcal{R}(s)} \left[C_1(sI-A)^{-1}B_2 + D_{12}\right], \ \forall s_0 \in \mathbb{C}^+$

6. $\mathrm{rank} \begin{pmatrix} A + \gamma^{-2}QC_1'C_1 - s_0I & B_{1Q} \\ C_2 & D_{21Q} \end{pmatrix} = n + \mathrm{rank}_{\mathcal{R}(s)} \left[C_2(sI-A)^{-1}B_1 + D_{21}\right], \ \forall s_0 \in \mathbb{C}^+$

7. $\rho(PQ) < \gamma^2$

where

$$\begin{aligned} A_P &= A'P + PA + C_1'C_1 + \gamma^{-2}PB_1B_1'P \\ C_P &= PB_2 + C_1'D_{12} \\ D_P &= D_{12}'D_{12} \\ A_Q &= AQ + QA' + B_1B_1' + \gamma^{-2}QC_1'C_1Q \\ B_Q &= QC_2' + B_1D_{21}' \\ D_Q &= D_{21}D_{21}' \end{aligned}$$

By the method in (Doyle et.al.) an explicit controller formula can be given in terms of the two Riccati solutions. This is not the case in our more general setting where the Riccati equations are replaced by quadratic matrix inequalities. These can after a certain change of basis, however, be solved in terms of two reduced order Riccati equations.

To compute a controller, we first take C_{1P} and D_{12P} as given by Theorem 2(1), and B_{1Q} and D_{21Q} as given by Theorem 2(2). Moreover we define the matrices

$$\begin{aligned} A_{PQ} &= T(A + QA'P + \gamma^{-2}B_1B_1'P + \gamma^{-2}QC_1'C_1)T \\ B_{2PQ} &= T(B_2 + \gamma^{-2}QC_1'D_{12}) \\ C_{2PQ} &= (C_2 + \gamma^{-2}D_{21}B_1'P)T \\ B_{1PQ} &= TB_{1Q} \\ C_{1PQ} &= C_{1P}T \\ T &= (I - \gamma^{-2}QP)^{-1/2} \end{aligned}$$

Now, one possible controller is given by the following result (Stoorvogel):

Lemma 2: *Let A_{PQ}, B_{2PQ} and C_{2PQ} be as above. Let L be a state feedback, such that $A_{PQ} + B_{2PQ}L$ is stable, and such that:*

$$\|(C_{2PQ} + D_{12P})(sI - A_{PQ} - B_{2PQ}L)^{-1}\|_\infty < \gamma/(3\|B_{1PQ}\|)$$

Let M be an output injection, such that $A_{PQ} + MC_{2PQ}$ is stable and further:

$$\|(sI - A_{PQ} - MC_{2PQ})^{-1}(B_{1PQ} + MD_{21Q})\|_\infty < \min(\gamma/(3\|D_{12P}L\|), \|B_{1PQ}\|/\|B_{2PQ}L\|)$$

Then the controller:

$$u = -L(sI - A_{PQ} - B_{2PQ}L - MC_{2PQ})^{-1}My$$

makes the $\mathcal{H}_\infty$ norm of the resulting closed loop transfer function from w to z smaller than γ

In short, the above results demonstrates that for a singular $\mathcal{H}_\infty$ problem a controller can be found by solving two reduced order Riccati equations, and two disturbance decoupling problems, which for instance can be solved by pole placement methods, as was done in the design below.

5 DESIGN RESULTS

In both the LQ design and the $\mathcal{H}_\infty$ design, we have used gain scheduling, so the controller is optimal with respect to the ship speed.

5.1 An LQ Design

The results of a nominal design for a naval multirole vessel Blanke and Christensen are here used for comparing an LQ design, with the $\mathcal{H}_\infty$ approach described here. The controller is not a genuine LQ design, because sway velocity could not be estimated with sufficient accuracy. Instead, pole placement similar to that of LQ design was obtained using available state estimates. The details of the design can be found in Blanke et.al.

The LQ controller uses feedback from filtered turn rate and heading, i.e. the states r and Ψ not disturbed by wave motion, and measured roll rate and roll angle, i.e., p and Φ including wave motion. The LQ controller was speed scaled to obtain closed loop behaviour similar to that of the open loop system. Details can be found in the reference.

The LQ controller was:

$$\delta_{steering} = (0, -l_rU, -l_\Psi, 0, 0); \quad (5.14)$$

$$\delta_{roll} = (0, 0, 0, -l_pU^2, -l_\Phi U^2) \quad (5.15)$$

where $U = U_{design}/U_{actual}$.

In a seaway, waves will generate roll motion, and assessment of total performance will require the wave response operators for both p and Φ, and integration of the wave spectrum times the response operator and output disturbance sensitivity function of the closed loop RRD control. This requires fairly complex information about the ship and seaway. A simpler, yet sufficient performance indicator for our purpose is the $|rr_5|$ function that shows roll damping over frequency. The performance of the LQ controller is illustrated in figure 2. Roll damping is

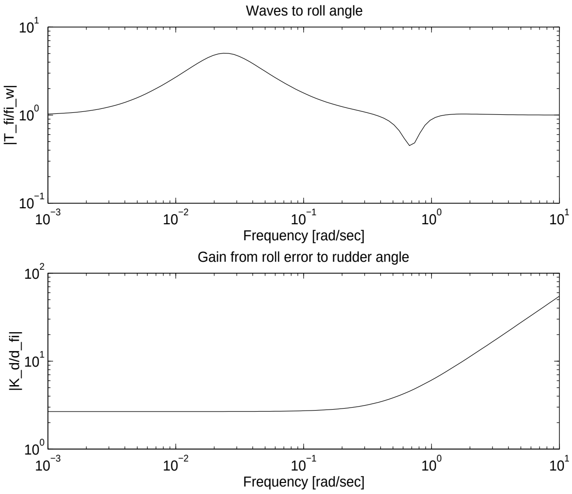

Fig. 2. LQ like design - Reduction ratio $|T_{\phi w}|$ and controller gain characteristics.

0.5 as required around 0.9 rad/sec. in the nominal design, but the interval where this is obtained is narrow. It is noted that (5.14) and (5.15) are approximations to a controller, used on a series of ships. The low frequency amplification does not exist in the real design. The actual design has also integral heading control.

5.2 The $\mathcal{H}_\infty$ Controller

Based on the formulated standard problem in Section 2.3 and the $\mathcal{H}_\infty$ results given in Section 4, we are able to design an internally stabilizing $\mathcal{H}_\infty$ controller which makes the $\mathcal{H}_\infty$ norm of the closed loop transfer function from w to z smaller than γ, where γ is a sufficiently large, positive number. In the following, γ has been selected to 1.1 times the optimal value of γ.

In figures 3-5, the result of the $\mathcal{H}_\infty$ design are shown for the ship speed $u = 9.0$ m/s. The solid lines in the figures are the closed-loop amplitudes and the dotted lines are the inverse of the respective weight function multiplied with γ. For satisfying the design specifications, the inverse of the weight function must be over the closed-loop transfer functions for all frequencies.

It can be seen directly from the figures, that the hard bound to satisfy is the specification for the roll angle. The reason is that the transfer function from control input to roll angle has a nonminimum phase zero at $z = 0.915$. Hence, the corresponding output sensitivity $S(\cdot)$ will satisfy a nontrivial Bode integral sensitivity bound. To obtain a reasonable design, the weight matrices has to satisfy the Bode bound themselves. In respect to space limitations we cannot survey the systematic procedures to take these interpolation constraints into account. Figures 3 to 5 show the results of a design with the following design constraints:

- Roll disturbance sensitivity is below 0.5 in a band around the natural roll frequency.
- Heading disturbance sensitivity is below 1.0 at all frequencies and goes towards zero below 0.1 rad/sec.
- Roll angle to heading crossover sensitivity is below 1.1 at all frequencies.
- Heading disturbance to roll crossover sensitivity is lower than one at all frequencies.

Roll Damping

The plots show that the required roll damping can be achieved over a frequency range which is somewhat broader than that obtained with the state feedback design. The controller gain used to obtain this is about 8 deg rudder/deg roll angle around 0.8 rad/sec whereas the state feedback design uses a gain of 2.7. The roll error gain is well below 1.0 at low frequencies. This is desired and necessary to obtain adequate turning capabilities for a vessel. The high frequency gain of the $\mathcal{H}_\infty$ controller increase more than 40 dB per decade, whereas the rate feedback term in the state feedback counterpart causes it to increase by 20 dB per decade. Such high frequency behavior is undesired above the primary wave frequency region, and in practical systems, the controller gain would need to be shaped and approach zero at high frequencies. Shaping can be implemented such that there are no significant penalty in roll damping performance.

Heading Control

Heading control is quite different for the two controllers. The state feedback controller of equation (5.14) has turn rate and heading angle feedback. In the actual implementation Blanke et.al., state feedback from turn rate and heading angle are taken from a Kalman filter that effectively suppress every wave induced motion from the feedback signals. The reason is that rudder activity due to wave motion in the lateral plane is undesired. Fluctuating rudder motion at these frequencies have literally no effect on the ship's heading, and significant propulsion losses may be generated.

Disregarding the filtering issue, the gain for the $\mathcal{H}_\infty$ controller is in the same order of magnitude as with the state feedback controller around natural roll frequency. At low frequencies, integral action in the $\mathcal{H}_\infty$ is achieved by shaping the heading weighting function. This is needed because wind load would otherwise cause large deviations in the ship's heading. Integral action is thus obtained by simple means.

Controller complexity

The model order increase with the degree of the weight function specifications is one of the practical obstacles with $\mathcal{H}_\infty$ designs. This is also the case

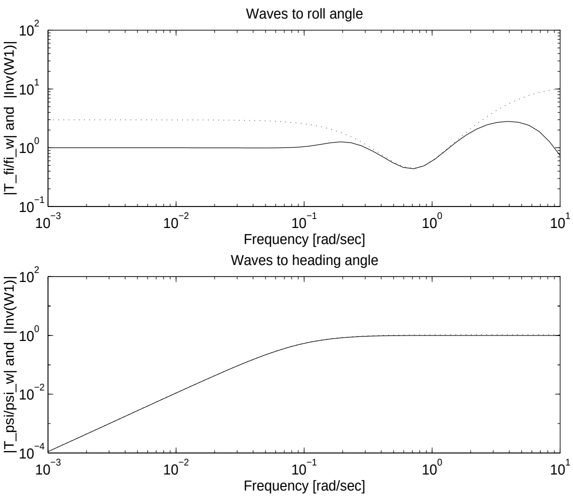

Fig. 3. $\mathcal{H}_\infty$ design - Sensitivity plots for roll $|T_{\Phi w}|$ and heading $|T_{\Psi w}|$. Dotted lines are specifications, solid lines are the design results.

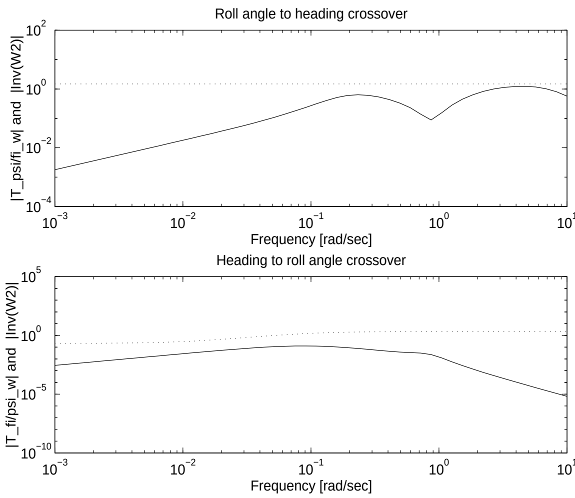

Fig. 4. $\mathcal{H}_\infty$ design - Crossover sensitivity plots for roll angle from heading disturbance and heading from roll disturbance. Dotted lines are specifications, solid lines are the results of the design.

here, where the controller order is 22 with the specifications used. Model reduction techniques can, however, fairly easily be applied, and a 7th order model can be used without any significant deviation from the specifications.

Model Uncertainties

Model uncertainty and rudder saturation in both slew rate and angle are (practical) major obstacles. The present design has attempted to present the results of a multi-objective design, whereas inevitable model uncertainty (Blanke and Christensen) and the nonlinear phenomena Blanke et.al. have not yet been included. These are issues of continued research.

6 CONCLUSION

A design problem for robust control of rudder-roll damping has been discussed.

Since the problem specifications were posed in frequency domain, an $\mathcal{H}_\infty$ design was a natural selection. An $\mathcal{H}_\infty$ controller was calculated by virtue of a new singular $\mathcal{H}_\infty$ approach and compared with a previous LQ like design.

As a design tool, the $\mathcal{H}_\infty$ method was fast and very direct, since no additional fine tuning was necessary on top of the weightings which were immediate from the specifications. It turned out that the hard bound to satisfy was the specification for the roll angle. The specifications could easily be met at the specified frequency range, but the transfer function need to blow up in some other frequency ranges for satisfying the Bode integral sensitivity bound. This trade off is the only part of the algorithm, where the designer might need some iterations.

In short, a comparison between the $\mathcal{H}_\infty$ and the LQ controller shows that the frequency fit of the $\mathcal{H}_\infty$ controller is significantly better, at the cost of complexity. The LQ controller amplifies some roll disturbances in the low frequency range, whereas the $\mathcal{H}_\infty$ controller does not.

A SHIP MODEL

The matrices for the linear ship model in (2.2) are given by:

$$\begin{aligned} A_s &= T^{-1}E^{-1}FT \\ B_s &= T^{-1}E^{-1}G \end{aligned}$$

where E, F and G are given by, (Blanke et.al., Blanke and Christensen):

$$E = \begin{bmatrix} & & & 0 & 0 \\ \bar{E}_1 & \bar{E}_2 & \bar{E}_3 & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{E}_1 = \begin{bmatrix} m - Y_{\dot{v}} \\ mx_G - N_{\dot{v}} \\ -mz_G - K_{\dot{v}} \end{bmatrix}$$

$$\bar{E}_2 = \begin{bmatrix} mx_G - Y_{\dot{r}} \\ I_{zz} - N_{\dot{r}} \\ -K_{\dot{r}} \end{bmatrix}$$

$$\bar{E}_3 = \begin{bmatrix} -mz_G - Y_{\dot{p}} \\ -N_{\dot{p}} \\ I_{xx} - K_{\dot{p}} \end{bmatrix}$$

$$F = \begin{bmatrix} & & & & 0 \\ F_1 & F_2 & F_3 & F_4 & 0 \\ & & & & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

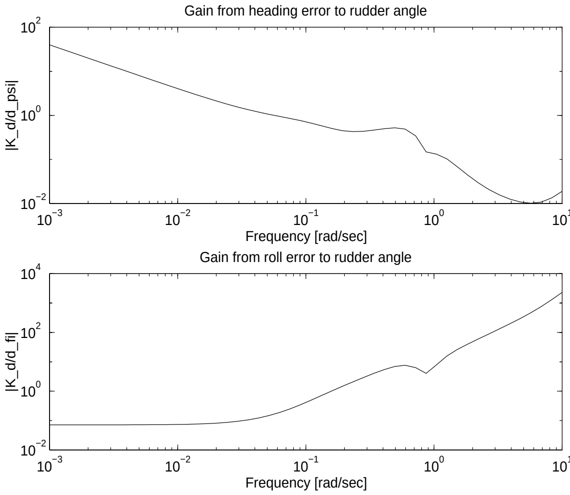

Fig. 5. $\mathcal{H}_\infty$ design - Controller gain from wave disturbance in heading angle (upper) and roll angle (lower) to rudder angle.

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} UY_{uv} & U(-m+Y_{ur}) \\ UN_{uv} & U(N_{ur}-mx_G) \\ UK_{uv} & U(K_{ur}+mz_G) \end{bmatrix}$$

$$\begin{bmatrix} F_3 & F_4 \end{bmatrix} = \begin{bmatrix} Y_p+UY_{up} & Y_\Phi+U^2Y_{\Phi uu} \\ N_p+UN_{up} & N_\Phi+U^2N_{\Phi uu} \\ K_p+UK_{up} & -gmGM+U^2K_{\Phi uu} \end{bmatrix}$$

$$G = \begin{bmatrix} U^2Y_{\delta uu} \\ l_{\delta x}U^2Y_{\delta uu} \\ -l_{\delta z}U^2Y_{\delta uu} \\ 0 \\ 0 \end{bmatrix}, C_s = \begin{bmatrix} 0 & I \end{bmatrix}$$

and T is given by

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

such that $x = Tx_s$ The values of the constants in the matrices can be found in (Blanke and Christensen).

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