Linear Algebra in the Classroom — A Note on Similarity

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Abstract

In this note we provide a new lemma on similar matrices, which might be helpful in teaching undergraduate linear algebra classes.

1 Introduction

In any linear algebra curriculum the concept of *similarity* is inevitable. The pedagogical scope of this concept is of course to exercise with the students that different matrices can be representations of the same linear finite dimensional operator in different bases.

Hence, in practically any linear algebra course, students are presented with problems where they have to decide if two given matrices are similar.

This kind of problems occasionally gives rise to a certain question from the students, which can be hard to answer in a sufficiently elementary way. In this brief note we shall pinpoint the problem, and give an (easy) result which in part overcomes the problem.

2 A Lemma on Similar Matrices

As usual, two matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are said to be similar, if there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that

$$B = T^{-1}AT \tag{1}$$

From a mathematical point of view, the crucial result to determine whether two matrices are similar or not, is the following.

Theorem 1 *Two square matrices are similar if and only if they have the same Jordan normal form.*

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This result, in various versions, is the only necessary and sufficient condition for similarity of general matrices given in standard textbooks, as e.g. [HJ85, Gan59, Bel70].

However, many engineering or business school curricula does not include the Jordan normal form so exercises therefore rather appeal to simpler versions of the Jordan theorem, such as:

Corollary 2 *Two square matrices A and B are similar only if they have the same eigenvalues, and the algebraic multiplicity of any eigenvalue λ for A equals the algebraic multiplicity of λ for B.*

Corollary 3 *Two square matrices A and B are similar only if they have the same eigenvalues, and the geometric multiplicity of any eigenvalue λ for A equals the geometric multiplicity of λ for B.*

As usual the algebraic multiplicity of an eigenvalue λ denotes the multiplicity of λ as root in the characteristic polynomial $\det(\lambda I - A)$, and the geometric multiplicity of λ denotes the corank of the matrix $\lambda I - A$.

Most lecture notes also give necessary and sufficient conditions for special classes of matrices:

Corollary 4 *Two matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ which each have n linearly independent eigenvectors are similar if and only if they have the same characteristic polynomials.*

Corollary 5 *Two normal (e.g. symmetric) matrices are similar if and only if they have the same characteristic polynomials.*

Sample problems based on the necessary conditions in Corollaries 2 &3 or satisfying the assumption of Corollary 4 are:

Problem 1 *Determine whether the two matrices*

$$A = \begin{pmatrix} -3 & -5 & 5 \\ 2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -4 & 2 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$$

are similar.

Answer: A and B are not similar according to Corollary 2, since 1 is a simple eigenvalue for A but a double eigenvalue for B.

Problem 2 *Determine whether the two matrices*

$$A = \begin{pmatrix} -1 & 1 & -1 \\ -4 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 2 & 0 \\ 3 & -1 & 3 \end{pmatrix}$$

are similar.

Answer: A and B are not similar according to Corollary 3, since two linearly independent eigenvectors corresponding to the eigenvalue 1 can be chosen for A, but only one can be chosen for B.

Problem 3 *Determine whether the two matrices*

$$A = \begin{pmatrix} -3 & 2 & -2 \\ -5 & 4 & -2 \\ 5 & -2 & 4 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} -2 & -4 & 4 \\ 1 & 3 & -1 \\ -2 & -2 & 4 \end{pmatrix}$$

are similar.

Answer: A and B are similar according to Corollary 4, since they have the same characteristic polynomials and three linearly independent eigenvectors can be chosen for each matrix.

However, once in a while one of the better students might come up with the question, whether the condition in Corollary 3 is sufficient as well. As we know from Theorem 1, the answer is no. One example is the two matrices

$$A = \begin{pmatrix} 2 & 2 & -2 & 1 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \tag{2}$$

A and B share the same characteristic polynomial $(\lambda - 1)^4$ and at most two linearly independent eigenvectors can be chosen for each matrix. Yet, they are not similar, which can be seen easily by computing the Jordan normal forms, which turn out to be different.

But how does a teacher explain this phenomenon to an eager engineering student who does not have the Jordan normal form in his or her curriculum?

In this note, we shall suggest the following result, which provides a candidate answer.

Lemma 1 *Two matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are similar if and only if the rank of $(\lambda I - A)^p$ equals the rank of $(\lambda I - B)^p$ for any complex number λ and for any integer p, $1 \le p \le n$.*

Remark 1 *In order to check the condition, one just needs to verify that the two matrices have identical characteristic polynomials and then test the condition with λ being the (common) eigenvalues. Moreover, for any eigenvalue λ only values of p less than or equal to the algebraic multiplicity of λ need to be investigated. This can easily be done manually or by any matrix manipulation software package, e.g. MATLAB or $MATRIX_X$.*

The proof of this lemma is easy based on Theorem 1. In an elementary engineering course, this result can just be claimed, with the proof completely omitted, or it can be given with a proof of necessity but with omission of the sufficiency part, which requires the Jordan theorem. Necessity is straightforward, since

$$\begin{aligned} B = T^{-1}AT \quad \Longrightarrow \quad & \operatorname{rank}(\lambda I - B)^p = \operatorname{rank}(\lambda I - T^{-1}AT)^p \\ & = \operatorname{rank} T^{-1}(\lambda I - A)^p T = \operatorname{rank}(\lambda I - A)^p \end{aligned}$$

With this result at hand the two matrices in (2) are easily seen not to be similar, since $\operatorname{rank}(I - A)^2 = 0$ but $\operatorname{rank}(I - B)^2 = 1$.

References

[Bel70] R. Bellman. *Introduction to Matrix Analysis.* McGraw-Hill, New York, 2 edition, 1970.

[Gan59] F. Gantmacher. *The Theory of Matrices.* Chelsea, New York, 1959.

[HJ85] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, London, 1985.