Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

Robust Schur Stability and Robust $\mathcal{H}_2$ Performance for Uncertain Systems with Nonlinear Parametric Uncertainties

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This author's work is supported by the National Education Commission
and by the Shandong Provincial Science Foundation

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This author's work is supported by the Danish Technical Research Council under grant no. 26-1830

Abstract

The problems of robust stability and robust $\mathcal{H}_2$ performance for uncertain discrete time systems with nonlinear parametric uncertainties are addressed. Two fairly general families of state space models depending nonlinearly on one or two uncertain parameters are considered. For these two families explicit expressions for the Schur stability radius and for the $\mathcal{H}_2$ robust performance radius in the case of uncertainties with a single parameter are obtained, and a line search algorithm for the two problems in case of two parameters is provided. For both problems explicit necessary and sufficient conditions are derived. An illustrative example demonstrates the algorithms.

1 Introduction

In the dawn of robust control theory, most attention was paid to systems with unstructed uncertainty descriptions. It was soon realized, however, that in most applications the real uncertainties are better captured by structured uncertainty descriptions. This is definitely the case when the model applied is based on physical insight of the plant, such that the uncertainties are basically just imperfect determination of physically meaningful parameters. But even in the case, where the nominal model and the uncertainty are obtained entirely by identification methods, this still results in parametric uncertainty descriptions. The reason for this is that statistical methods always will have different preferences for different directions in s-plane, thus providing phase information. And uncertain phase information is only representable by structured uncertainty models.

Moreover, robust control theory has had far most emphasis on the *nominal performance/robust stability* paradigm, rather than the *robust performance* paradigm, which of course is the problem of ultimate importance. This is not because the significance of robust performance problems have been overlooked, but simply because the research has had little success in this field so far. One reason is that some of these problems are NP-hard.

A vast amount of papers have been devoted to the topic of robust stability bounds under structured perturbations. Let us mention a few which also have comprehensive lists of references: [1, 2, 15, 7, 8, 5].

For the $\mathcal{H}_\infty$ norm, robust performance bounds can be obtained by μ optimization, see [10] for a survey or [11] for an exposition in the line of this paper. A convex optimization approach for robust $\mathcal{H}_\infty$ analysis and synthesis for systems with parametric uncertainties is given in [14].

For linear time-invariant systems, the $\mathcal{H}_2$ performance metric arises naturally in a number of different physically meaningful situations, see [5, 4]. The $\mathcal{H}_2$ performance of a linear time-invariant system is measured via the $\mathcal{H}_2$ norm of its transfer matrix. As long as this $\mathcal{H}_2$ norm is less than a given upper bound, we can stop, and need not seek the minimal one due to robustness consideration. Suppose the $\mathcal{H}_2$ norm of a nominal (stable) system is less than a given upper bound. Is it still less than this upper bound after suffering parameter perturbation? or, how to find the "maximal domain" for perturbation parameters under stability and $\mathcal{H}_2$ norm constraints? This paper will consider this problem, and calculate the maximal (nonlinear) perturbation interval or radius in perturbation parameter space. The obtained results are not only sufficient, but also necessary. The paper is different from most of published papers which deal with a fixed parameter domain and affinely linear perturbations. For recent advances on robust $\mathcal{H}_2$ performance analysis for uncertain control systems, see the papers [6, 9] and references therein.

This paper deals with discrete time uncertain systems. The corresponding problems in continuous time has been addressed in [13]. The stability results are based on the paper [12].

We denote the real number set by $\mathbf{R}$. Let cs: $\mathbf{R}^{m\times n} \to \mathbf{R}^{mn}$ be the column stacking operator on a matrix, $\otimes$: $\mathbf{R}^{n\times n} \times \mathbf{R}^{m\times m} \to \mathbf{R}^{mn\times mn}$ the standard matrix Kronecker product (see [3]), and let $\lambda_k(\cdot)$ be the kth eigenvalue of a square matrix.

2 Problem formulation

Consider a linear time-invariant discrete-time system described by

$$G(z,q) = \left[\begin{array}{c|c} A(q) & B(q) \\ \hline C(q) & O \end{array}\right] \tag{1}$$

where $A(q)$, $B(q)$, and $C(q)$ with dimensions $n\times n$, $n\times m$, $p\times n$, $p\times m$, respectively, are continuous matrix functions of a perturbation parameter vector $q = [q_1, q_2, \cdots, q_l]^T \in \mathbf{R}^l$. A square constant matrix is called (Schur) stable if all of its eigenvalues lie in $\{z : |z| < 1\}$. We say $G(z,q)$ is (Schur) stable for a given q if $A(q)$ is stable. The $\mathcal{H}_2$ norm is defined by

$$\|G(z,q)\|_2 \doteq \left\{ \frac{1}{2\pi j} \oint_{|z|=1} \text{Trace}[G^*(z,q)G(z,q)]\frac{dz}{z} \right\}^{1/2} \tag{2}$$

where $G^*(z,q) \doteq G'(z^{-1},q)$ and $(\cdot)'$ denotes transpose.

Suppose for $q=0$, the nominal system of (1) satisfies

AS1. $A(0)$ is stable,

AS2. $\|G(z,0)\|_2^2 < \gamma$,

where γ is a known positive constant which reflects the tolerance of the system $\mathcal{H}_2$ performance (for instance, an acceptable output variance of (1) to a white noise signal). Our goal is to find "the maximal domain" in $\mathbf{R}^l$ so that $\|G(z,q)\|_2^2 < \gamma$ for every q in it. A prerequisite for this is that $A(q)$ is stable for every q in this domain. This problem will be solved in the two cases $l=1$ and $l=2$.

2.1 Single parameter case

Define

$$\begin{aligned}
r_s^- &\doteq \inf\{r<0 : A(q) \text{ is stable } \forall q \in (r,0)\}, \\
r_s^+ &\doteq \sup\{r>0 : A(q) \text{ is stable } \forall q \in (0,r)\}, \\
r_2^- &\doteq \inf\{r<0 : A(q) \text{ is stable and} \\
&\qquad \|G(z,q)\|_2^2 < \gamma\ \forall q \in (r,0)\}, \\
r_2^+ &\doteq \sup\{r>0 : A(q) \text{ is stable and} \\
&\qquad \|G(z,q)\|_2^2 < \gamma\ \forall q \in (0,r)\}
\end{aligned}$$

Then (r_s^-, r_s^+) is the maximal perturbation interval of q while keeping the stability of $A(q)$; and (r_2^-, r_2^+) the maximal perturbation interval of q while keeping $\|G(z,q)\|_2^2 < \gamma$.

Problem 1 *Suppose that system (1) satisfies AS1, AS2, and*

$$\begin{aligned}
A(q) &\doteq A_0 + qA_1 + \cdots + q^{m_1}A_{m_1}, \\
B(q) &\doteq B_0 + qB_1 + \cdots + q^{m_2}B_{m_2}, \\
C(q) &\doteq C_0 + qC_1 + \cdots + q^{m_3}C_{m_3},
\end{aligned}$$

where all of A_k, B_k, and C_k are given constant matrices.

(a). Find r_s^- and r_s^+.

(b). Find r_2^- and r_2^+.

Remark 1 Obviously, $(r_2^-, r_2^+) \subset (r_s^-, r_s^+)$.

2.2 Two parameter case

Denote by $U(r)$ and $\partial U(r)$ the circular disk $\{q = [q_1, q_2]' : \sqrt{q_1^2+q_2^2} < r,\} \subset \mathbf{R}^2$ and its boundary circle, respectively. Define

$$\begin{aligned}
r_s &\doteq \sup\{r : A(q) \text{ is stable } \forall q \in U(r)\}, \\
r_2 &\doteq \sup\{r : A(q) \text{ is stable and} \\
&\qquad \|G(z,q)\|_2^2 < \gamma\ \forall q \in U(r)\}.
\end{aligned}$$

Then $U(r_s)$ is the maximal perturbation circular disk for q while keeping the stability of $A(q)$; and $U(r_2)$ is the maximal perturbation circular disk for q while keeping $\|G(z,q)\|_2^2 < \gamma$.

Problem 2 *Suppose that system (1) satisfies AS1, AS2, and*

$$\begin{aligned}
A(q) &\doteq A_{00} + q_1A_{10} + q_2A_{01} + q_1^2A_{20} + q_1q_2A_{11} + q_2^2A_{02} \\
&\quad + \cdots + \textstyle\sum_{i+j=m_1} q_1^i q_2^j A_{i,j}, \\
B(q) &\doteq B_{00} + q_1B_{10} + q_2B_{01} + q_1^2B_{20} + q_1q_2B_{11} + q_2^2B_{02} \\
&\quad + \cdots + \textstyle\sum_{i+j=m_2} q_1^i q_2^j B_{i,j}, \\
C(q) &\doteq C_{00} + q_1C_{10} + q_2C_{01} + q_1^2C_{20} + q_1q_2C_{11} + q_2^2C_{02} \\
&\quad + \cdots + \textstyle\sum_{i+j=m_3} q_1^i q_2^j C_{i,j},
\end{aligned}$$

where $A_{i,j}$, $B_{i,j}$, and $C_{i,j}$ are given constant matrices for all i,j.

(a). *Find* r_s.

(b). *Find* r_2.

Remark 2 Obviously, $0 < r_2 \le r_s$.

Remark 3 The polynomial perturbation sets in Problems 1 and 2 are very general in the sense, that any nonlinear perturbation set that depends continuously on the parameters, defined on a compact set of parameters, can be approximated arbitrarily well by these types of uncertainties. The cost of a good approximation is that the computational requirements will be extensive, since the computational time involved with the solutions presented below, grows rapidly with increasing polynomial order.

3 Preliminaries

By doing simple operations on a matrix and its determinant (see [12]), we can get the maximal perturbation bounds for nonsingularity of matrices.

Lemma 4 *Let* $M(r) = M_0 + rM_1 + \cdots + r^m M_m$ *where all of* M_k *are* $n \times n$ *constant matrices, and* $|M_0| \neq 0$ ($|\cdot|$ denotes the determinant). *Define*

$$\begin{aligned} r^- &\doteq \sup\{r < 0 : |M(r)| = 0\}, && (3)\\ r^+ &\doteq \inf\{r > 0 : |M(r)| = 0\}. && (4) \end{aligned}$$

Then

$$r^- = \frac{1}{\lambda^-_{min}(\mathbf{M})}, \quad r^+ = \frac{1}{\lambda^+_{max}(\mathbf{M})} \tag{5}$$

where $\mathbf{M}$ *is an mnth order square matrix given by*

$$\mathbf{M} \doteq -\begin{pmatrix} \mathbf{O} & -\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\mathbf{I} \\ M_0^{-1}M_m & M_0^{-1}M_{m-1} & M_0^{-1}M_{m-2} & \cdots & M_0^{-1}M_1 \end{pmatrix}$$

where $\mathbf{O}$*, and* $\mathbf{I}$ *are the nth order zero matrix and identity matrix, respectively, and* $\lambda^-_{min}(\cdot)$ *stands for the minimal value of the negative real eigenvalues (let* $\lambda^-_{min}(\cdot) = 0^-$ *if there exist no negative real eigenvalues), and* $\lambda^+_{max}(\cdot)$ *the maximal value of the positive real eigenvalues (let* $\lambda^+_{max}(\cdot) = 0^+$ *if no positive real eigenvalues), respectively.*

The following lemma helps us to transform Problem 1a and 2a into that of the maximal perturbation bounds for nonsingularity of matrices.

Lemma 5 *Suppose that*
(i) Q *is a single connected domain in* $\mathbf{R}^l$*, and* $0 \in Q$*,*
(ii) $A(0)$ *is stable.*
Then $A(q)$ *are stable for all* $q \in Q$ *if and only if* $|A(q) \otimes A(q) - I \otimes I| \neq 0$ *for all* $q \in Q$*, where* I *is the nth-order identity matrix.*

Proof: The result is immediate if we recall the continuity of $A(q)$ in q, that the eigenvalues of a matrix are continuous functions of its entries, and that

$$\begin{aligned} \lambda_k(A(q) \otimes A(q)) &= \lambda_i(A(q))\lambda_j(A(q)), \\ & k = 1, \cdots, n^2;\ i, j = 1, \cdots, n. \square \end{aligned}$$

By using Lemma 5 we can show that

$$\begin{aligned} r_s^- &= \sup\{q < 0 : |A(q) \otimes A(q) - I \otimes I| = 0\} && (6)\\ r_s^+ &= \inf\{q > 0 : |A(q) \otimes A(q) - I \otimes I| = 0\} && (7)\\ r_s &= \inf\{r : |A(q) \otimes A(q) - I \otimes I| = 0 \\ &\qquad \text{for some } q \in \partial U(r)\}. && (8) \end{aligned}$$

Instead of (2) in the frequency domain, we here use the state space approach to compute

$$\|G(z,q)\|_2^2 = Trace\{C'(q)C(q)Q(q)\}$$

where $Q(q) = Q(q)'$ satisfies

$$A(q)Q(q)A'(q) - Q(q) + B(q)B(q)' = 0$$

By using the column stacking operation we can give a more compact formula

$$\begin{aligned} &\|G(z,q)\|_2^2 = -cs[C'(q)C(q)]' \\ &\cdot(A(q) \otimes A(q) - I \otimes I)^{-1} \cdot cs[B(q)B'(q)] \end{aligned} \tag{9}$$

Going one step from (9), we get the following result which helps us to transform Problem 1b and 2b into that of the maximal perturbation bounds for nonsingularity of matrices.

Lemma 6 *Suppose that*
(i) Q *is a single connected domain in* $\mathbf{R}^l$*, and* $0 \in Q$*,*
(ii) $A(q)$ *are Schur-stable* $\forall q \in Q$*,*
(iii) $\|G(z,0)\|_2^2 < \gamma$.
Then $\|G(z,q)\|_2^2 < \gamma\ \forall q \in Q$ *if and only if* $|M_\gamma(q)| \neq 0$ *for all* $q \in Q$*, where*

$$\begin{aligned} M_\gamma(q) \doteq\ & (A(q) \otimes A(q) - I \otimes I) \\ & + \tfrac{1}{\gamma} cs[B(q)B'(q)] \cdot cs[(C'(q)C(q)]' \end{aligned} \tag{10}$$

Proof: $\|G(z,q)\|_2^2 < \gamma$ for all $q \in Q$
$\Leftrightarrow \gamma + cs[C'(q)C(q)]' \cdot (A(q) \otimes A(q) - I \otimes I)^{-1} \cdot cs[B(q)B'(q)] > 0\ \forall q \in Q.$ (from (9))
$\Leftrightarrow |\gamma I + (A(q) \otimes A(q) - I \otimes I)^{-1} \cdot cs[B(q)B'(q)] \cdot cs[C'(q)C(q)]'| > 0\ \forall q \in Q.$

(use equality $|\gamma I + XY| = |\gamma I + YX|$)

$\Leftrightarrow |\gamma(A(q) \otimes A(q) - I \otimes I)^{-1} \cdot \mathbf{M}_\gamma(q)| > 0 \ \forall q \in Q$ (from (10))

$\Leftrightarrow |\mathbf{M}_\gamma(q)| \neq 0$ for all $q \in Q$ (due to the continuity of $A(q)$, $B(q)$, $C(q)$ to q, and Lemma 5)

The remaining part of the proof is obvious. □

By using Lemma 6 we obtain the following formulae which are suitable for calculations.

$$
\begin{aligned}
r_2^- &= \sup\{q \in (r_s^-, 0) : |M_\gamma(q)| = 0\} && (11)\\
r_2^+ &= \inf\{q \in (0, r_s^+) : |M_\gamma(q)| = 0\} && (12)\\
r_2 &= \inf\{r : r < r_s \text{ and } |M_\gamma(q)| = 0 \\
&\qquad \text{for some } q \in \partial U(r)\}. && (13)
\end{aligned}
$$

In Section 2 we presented two types of problems. One is the maximal perturbation bounds for system stability; the other is the maximal perturbation bounds for system performance. Lemmas 5 and 6 help us to transform both into the maximal perturbation bounds for nonsingularity of matrices.

4 Main results

In this section we shall combine the preliminary results in order to provide answers to Problems 1 and 2.

4.1 Single parameter case

By using matrix multiplication and the expressions of $A(q)$, $B(q)$, $C(q)$ in problem 1, then we have

$$
\begin{aligned}
&(A(q) \otimes A(q) - I \otimes I) \\
&\qquad = \mathbf{A}_0 + q\mathbf{A}_1 + \cdots + q^{2m_1}\mathbf{A}_{2m_1} \\
&cs[B(q)B'(q)] = \mathbf{b}_0 + q\mathbf{b}_1 + \cdots + q^{2m_2}\mathbf{b}_{2m_2} \\
&cs[C'(q)C(q)] = \mathbf{c}_0 + q\mathbf{c}_1 + \cdots + q^{2m_3}\mathbf{c}_{2m_3}
\end{aligned}
\qquad (14)
$$

where

$$
\begin{aligned}
&\mathbf{A}_0 = (A_0 \otimes A_0 - I \otimes I), \cdots, \\
&\mathbf{A}_i = \textstyle\sum_{j+k=i} A_j \otimes A_k, \cdots, \mathbf{A}_{2m_1} = A_{m_1} \otimes A_{m_1}, \\
&\mathbf{b}_0 = cs\left[B_0B_0'\right], \cdots, \mathbf{b}_i = cs\left[\sum_{j+k=i} B_jB_k'\right], \cdots, \\
&\mathbf{b}_{2m_2} = cs\left[B_{m_2}B_{m_2}'\right], \\
&\mathbf{c}_0 = cs\left[C_0'C_0\right], \cdots, \mathbf{c}_i = cs\left[\sum_{j+k=i} C_j'C_k\right], \cdots, \\
&\mathbf{c}_{2m_3} = cs\left[C_{m_3}'C_{m_3}\right].
\end{aligned}
$$

Substituting the above expressions for $A(q)$, $B(q)$, $C(q)$ in (10), then it can be rewritten as

$$
M_\gamma(q) = \mathbf{M}_{0\gamma} + q\mathbf{M}_{1\gamma} + \cdots + q^m\mathbf{M}_{m\gamma} \qquad (15)
$$

where $m = max\{2m_1, 2(m_2 + m_3)\}$, and

$$
M_{0\gamma} = (A_0 \otimes A_0 - I \otimes I) + \frac{1}{\gamma} cs[B_0B_0'] \cdot cs[C_0'C_0]', \quad (16)
$$

and all of other $M_{k\gamma}$ depend on $\mathbf{A}_i$, $\mathbf{b}_j$, and $\mathbf{c}_k$ (the detailed expressions are omitted here).

By recalling Lemma 4, and using (6), (7) and (14), then we can formulate

Theorem 7 (Max. pert. bounds for Prob. 1a) *Splitting $A(q) \otimes A(q) - I \otimes I$ as (14) gives us the coefficient matrices $\mathbf{A}_k$, $k = 0, \cdots, 2m_1$. Define the following $2m_1n$th order square matrix*

$$
\mathcal{A} \doteq - \begin{pmatrix}
\mathbf{O} & -\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & -\mathbf{I} & \cdots & \mathbf{O} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\mathbf{I} \\
\mathbf{A}_0^{-1}\mathbf{A}_m & \mathbf{A}_0^{-1}\mathbf{A}_{m-1} & \mathbf{A}_0^{-1}\mathbf{A}_{m-2} & \cdots & \mathbf{A}_0^{-1}\mathbf{A}_1
\end{pmatrix}
$$

where $\mathbf{O}$, and $\mathbf{I}$ are the nth order zero matrix and identity matrix, respectively. Then

$$
r_s^- = \frac{1}{\lambda_{min}^-(\mathcal{A})}, \quad \text{and} \quad r_s^+ = \frac{1}{\lambda_{max}^+(\mathcal{A})} \qquad (17)
$$

where $\lambda_{min}^-(\cdot)$ stands for the minimal value of the negative real eigenvalues (let $\lambda_{min}^-(\cdot) = 0^-$ if there exist no negative real eigenvalues), and $\lambda_{max}^+(\cdot)$ the maximal value of the positive real eigenvalues (let $\lambda_{max}^+(\cdot) = 0^+$ if no positive real eigenvalues).

From AS2, Lemma 6, and (16), it can be shown that $|M_{0\gamma}| \neq 0$. By recalling Lemma 4, and using (11), (12) and (15), then we can formulate

Theorem 8 (Max. pert. bounds for Prob. 1b) *Splitting $M_\gamma(q)$ as (15) gives us the coefficient matrices $M_{k\gamma}$, $k = 0, \cdots, m$ where $m = \max\{2m_1, 2(m_2 + m_3)\}$. Define the following $2mn$-order square matrix*

$$
\mathcal{M}_\gamma \doteq - \begin{pmatrix}
\mathbf{O} & -\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & -\mathbf{I} & \cdots & \mathbf{O} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\mathbf{I} \\
\mathbf{M}_{0\gamma}^{-1}\mathbf{M}_{m\gamma} & \mathbf{M}_{0\gamma}^{-1}\mathbf{M}_{(m-1)\gamma} & \mathbf{M}_{0\gamma}^{-1}\mathbf{M}_{(m-2)\gamma} & \cdots & \mathbf{M}_{0\gamma}^{-1}\mathbf{M}_{1\gamma}
\end{pmatrix}
$$

where $\mathbf{O}$, and $\mathbf{I}$ are n-order zero matrix and identity matrix, respectively. Then

$$
\begin{aligned}
r_2^- &= \max\left\{r_s^-, \frac{1}{\lambda_{min}^-(\mathcal{M}_\gamma)}\right\}, && (18)\\
r_2^+ &= \min\left\{r_s^+, \frac{1}{\lambda_{max}^+(\mathcal{M}_\gamma)}\right\}, && (19)
\end{aligned}
$$

where $\lambda_{min}^-(\cdot)$ stands for the minimal value of the negative real eigenvalues (let $\lambda_{min}^-(\cdot) = 0^-$ if there *exist no negative real eigenvalues), and $\lambda^+_{max}(\cdot)$ the maximal value of the positive real eigenvalues (let $\lambda^+_{max}(\cdot) = 0^+$ if no positive real eigenvalues), respectively.*

Remark 9 The algorithms corresponding to Theorem 7 and 8 do not need any iteration. Ref. [1] first gives the maximal perturbation bounds for Problem 1a in the simplest case (affinely linear perturbation of a single parameter).

4.2 Two parameter case

In order to solve Problem 2, we need to introduce polar coordinates, namely, $q_1 = r\cos\theta$, $q_2 = r\cos\theta$, thus

$$\begin{aligned} A(q) &= A(r,\theta) = A_0 + rA_1(\theta) + \cdots + r^{m_1}A_{m_1}(\theta) \\ B(q) &= B(r,\theta) = B_0 + rB_1(\theta) + \cdots + r^{m_2}B_{m_2}(\theta) \\ C(q) &= C(r,\theta) = C_0 + rC_1(\theta) + \cdots + r^{m_3}C_{m_3}(\theta) \end{aligned}$$

where

$$\begin{aligned} A_k(\theta) &\doteq \textstyle\sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j A_{ij},\ k = 1,\cdots,m_1 \\ B_k(\theta) &\doteq \textstyle\sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j B_{ij},\ k = 1,\cdots,m_2 \\ C_k(\theta) &\doteq \textstyle\sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j C_{ij},\ k = 1,\cdots,m_3 \end{aligned}$$

Obviously, for a fixed θ, Problem 2 is fully transformed into Problem 1. But now we need a grid for the interval $[0, 2\pi)$, finally

$$\begin{aligned} r_s &= \inf\{r_s^+(\theta), \theta \in [0, 2\pi)\}, \\ r_2 &= \inf\{r_2^+(\theta), \theta \in [0, 2\pi)\}. \end{aligned}$$

The algorithms corresponding to Problem 2a and 2b are briefly listed below

Algorithm 1 (Max. stab. rad. for Prob. 2a)

Step 1. Select a large natural number p, and let $\theta_j = 2j\pi/p$, $j = 0, 1, \cdots, p-1$;

Step 2. Let $A_k = A_k(\theta_j)$, repeatedly recall Theorem 7 to get r^+_{sj}, $j = 0, 1, \cdots, p-1$;

Step 3. Find $r_s = min\{r^+_{sj}, j = 0, 1, \cdots, p-1\}$, then output it.

Algorithm 2 (Max. stab. rad. for Prob. 2b)

Step 1. Select a large natural number p, and let $\theta_j = 2j\pi/p$, $j = 0, 1, \cdots, p-1$;

Step 2. Let $A_k = A_k(\theta_j)$, $B_k = C_k(\theta_j)$, and $A_k = C_k(\theta_j)$, repeatedly recall Theorem 8 to get r^+_{2j}, $j = 0, 1, \cdots, p-1$;

Step 3. Find $r_2 = min\{r^+_{2j}, j = 0, 1, \cdots, p-1\}$, then output it.

Remark 10 Solving Problem 2 involves a one dimensional search in contrast to Problem 1 which can be solved noniteratively.

5 Example

An example with a single perturbation parameter is cited below. Let

$$A(q) = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.5 \end{bmatrix} + q\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + q^2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$B(q) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},\ C(q) = [1\,1]$$

It is easy to show that

$$A_0 = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.5 \end{bmatrix}$$

is Schur stable, and

$$\begin{aligned} &A(q) \otimes A(q) - I \otimes I = \\ &\begin{bmatrix} -0.9900 & 0.1000 & 0.1000 & 1.000 \\ 0 & -0.9500 & 0 & 0.5000 \\ 0 & 0 & -0.9500 & 0.5000 \\ 0 & 0 & 0 & -0.7500 \end{bmatrix} \\ &+q\begin{bmatrix} 0 & 0.1000 & 0.1000 & 2.0000 \\ 0 & 0 & 0 & 0.5000 \\ 0 & 0 & 0 & 0.5000 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &+q^2\begin{bmatrix} 0 & 0 & 0 & 1.0000 \\ 0.1000 & 0 & 1.0000 & 0 \\ 0.1000 & 1.0000 & 0 & 0 \\ 0 & 0.5000 & 0.5000 & 0 \end{bmatrix} \\ &+q^3\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + q^4\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

after calculating $\mathcal{A}$ and all its eigenvalues, we get $(r_s^-, r_s^+) = (-1.6711, 0.7683)$. In this example we can show $G(z,0) = [\frac{1}{z-0.1}\ \frac{z+0.9}{(z-0.1)(z-0.5)}]$, and $\|G(z,0)\|_2^2 \approx 2.0162$. Now we select the upper bound of $\mathcal{H}_2$ performance as $\gamma = 2.1$.

$$cs[B(q)B'(q)] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + q\begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \end{bmatrix} + q^2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix},$$

and $cs[C'(q)C(q)]' = [1\ 1\ 1\ 1]$, furthermore,

$$
\begin{aligned}
&M_\gamma(q) = \\
&(A(q) \otimes A(q) - I \otimes I) + \tfrac{1}{\gamma} cs[B(q)B'(q)] \cdot cs[(C'(q)C(q)]' \\
&= \begin{bmatrix} -0.5138 & 0.5762 & 0.5762 & 1.4762 \\ 0 & -0.9500 & 0 & 0.5000 \\ 0 & 0 & -0.9500 & 0.5000 \\ 0.4762 & 0.4762 & 0.4762 & -0.2738 \end{bmatrix} \\
&+q \begin{bmatrix} 0.9624 & 1.0524 & 1.0524 & 2.9524 \\ 0.4762 & 0.4762 & 0.4762 & 0.9762 \\ 0.4762 & 0.4762 & 0.4762 & 0.9762 \\ 1.9048 & 1.9048 & 1.9048 & 1.9048 \end{bmatrix} \\
&+q^2 \begin{bmatrix} 0.4762 & 0.4762 & 0.4762 & 1.4762 \\ 0.5762 & 0.4762 & 1.4762 & 0.4762 \\ 0.5762 & 1.4762 & 0.4762 & 0.4762 \\ 2.3810 & 2.8810 & 2.8810 & 2.3810 \end{bmatrix} \\
&+q^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + q^4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

After calculating $\mathcal{M}_\gamma$ and all its eigenvalues, finally we get $(r_2^-, r_2^+) = (-1.6711, 0.0433)$.

6 Conclusion

In this paper we have investigated stability robustness and $\mathcal{H}_2$ performance robustness of discrete time systems with nonlinear parametric uncertainties.

We restricted ourselves to the class of polynomial uncertainty descriptions, since this class is dense in the set of continuous matrix valued functions defined on compact sets of parameters equipped with the topology of pointwise convergence.

For this class we obtained explicit formulae both for the stability robustness perturbation radius and for the $\mathcal{H}_2$ performance robustness perturbation radius in the case of a single parameter.

In the two parameter cases, we described line search algorithms as the natural extensions of the explicit formulae for the one parameter cases. More parameters could easily be included in the framework, but the computational cost involved would be quite considerable.

Further research could address $\mathcal{H}_\infty$ performance robustness, and possibly mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems under structured perturbations.

Acknowledgements

This paper was written while both authors were visiting the Industrial Control Centre, University of Strathclyde, Scotland. The authors would like to thank their hosts for providing this opportunity.

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