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An *LMI* Approach to Fixed Order *LTR* Controller

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Abstract

In this paper the problem of optimal $\mathcal{H}_\infty$ norm approximation of an LTR controller with specified poles is considered. A structurally predefined fixed order LTR controller is introduced which has the same dimension as the number of transmission zeros. It is shown that the design problem can be reduced into an equivalent convex optimization problem involving LMI which can be solved efficiently. We demonstrate the performance of this LTR controller by comparing it to a functional observer-based LTR controller of the same dimension.

1 Introduction

The problem of loop transfer recovery (LTR) of linear systems has been a well studied subject. The literature reports several methods for exact loop transfer recovery (ELTR) and asymptotic loop transfer recovery (ALTR) (see [1] and references therein). To improve the recovery performance, recent results [2], [3] consider LTR design methods which use $\mathcal{H}_\infty$ control theory. However, a major concern in $\mathcal{H}_\infty/LTR$ design is the high dimensionality of the controller. It has been recognized that the order of such a controller can be higher than the system order. One immediate solution to this problem is to apply model reduction techniques. However, the degree of approximation manifest itself a degradation in the recovery performance.

This paper deals with one important issue, namely, LTR controllers with low dimensions. An initial attempt on this issue was performed in [4] based on functional observer theory. Here we further investigate the problem by using $\mathcal{H}_\infty$ theory and balanced model reduction technique. More specifically, we design a structurally predefined fixed order LTR controller based on LMI and compare its performance to that achieved by a direct functional observer-based LTR controller of order r, where r is the number of transmission zeros.

2 Problem Formulation

To design an LTR controller $C(s)$ for the system Σ: $\{A, B, C\}$ having the transfer function $G(s)$, we first determine the desired target feedback loop with the transfer function

$$L_{TFL}(s) = F(sI - A)^{-1}B \tag{1}$$

and say that exact loop transfer recovery at the input point ($ELTRI$) is achieved if the closed-loop system is asymptotically stable and $E_L(s) = L_{TFL} - C(s)G(s) = 0$. To define asymptotic LTR at the input point ($ALTRI$), we parameterize the family of controllers as $C(s,q)$, and say that $ALTRI$ is achieved if the closed-loop system is asymptotically stable and $C(s,q)G(s) \to L_{TFL}(s)$ pointwise in s as $q \to \infty$, i.e., $E_L(s,q) \to 0$ pointwise in s as $q \to \infty$.

An equivalent measure of the recovery, is the so-called recovery matrix $M_I(s)$, which can be related to $E_L(s)$. This matrix is constructed according to the defined observer structure. For example, consider the full-order observer-based controller having the transfer function

$$C(s) = F\left(sI - A + KC - BF\right)^{-1} K \tag{2}$$

where F and K are the regulator and observer gains, respectively. Then

$$M_I(s) = F\left(sI - A + KC\right)^{-1} B. \tag{3}$$

And for a functional observer-based controller

$$\Sigma_F: \quad \left\{ \begin{array}{lcl} \dot{z} & = & \Phi z + Gy + Hu \\ w & = & Mz + Ny \end{array} \right. \tag{4}$$

under the following constraints:

$$\begin{aligned} Re[\lambda(\Phi)] &< 0 \quad &(5)\\ TA - \Phi T &= GC \quad &(6)\\ H &= TB \quad &(7)\\ MT + NC &= F \quad &(8) \end{aligned}$$

we have

$$C_F(s) = [I - M(sI - \Phi)^{-1}H]^{-1}[M(sI - \Phi)^{-1}G + N] \tag{9}$$

$$M_I(s) = M(sI - \Phi)^{-1}H \tag{10}$$

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Problem 1 *Given a functional observer-based LTR controller with transfer function $C_F(s)$ of order r and the LTR controller $C_R(s)$ obtained by model reduction of high order LTR controller $C(s)$ with minimal realization $C_R(s) = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$. Compare the performance of $C_F(s)$ and $C_R(s)$ with $E_L(s) = L_{TFL}(s) - \hat{C}(s)G(s)$ where $L_{TFL}(s)$ is given by (1) and $\hat{C}$ represents either $C_F(s)$ or $C_R(s)$.*

Problem 2 *Given LTR controller $C(s)$ of order n with the minimal realization $C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$. Find the best fixed order LTR controller $C_R(s)$ of order r with minimal realization $C_R(s) = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$ and prescribed A_r and C_r such that $\| C(s) - C_R(s) \|_\infty$ is minimized.*

Note that there is a distinct difference between $C_R(s)$ defined in each of the above two problems. While the poles in problem 1 are not specified, the ones for problem 2 are given by the predefined structure of the matrix A_r. This fact shows a direct correlation between two types of reduced order controllers considered here and the functional observer with free and fixed poles known in observer theory.

3 Preliminaries

The interested readers should review [4] before continuing this section. Here we review the basic concepts of LMI and $\mathcal{H}_\infty$ norm approximation, which are well documented in [5] - [7].

One of the standard LMI based convex optimization problem is the following *prototype problem*, which plays an important role in the solution of problem2.

$$\underset{\alpha, P=P^T}{\text{Minimize}} \quad \alpha \tag{11}$$

subject to

$$A_cP + PA_c^T + B_cB_c^T \leq 0 \tag{12}$$

$$\begin{pmatrix} A_cP + PA_c^T & PC_c^T \\ C_cP & -\alpha I \end{pmatrix} \leq 0 \tag{13}$$

Definition 1 *The transfer function $C_R(s) \in R\mathcal{H}_\infty$ with r stable poles is called an rth order $\mathcal{H}_\infty$ norm approximation of a given transfer function $C(s) \in R\mathcal{H}_\infty$ with n stable poles, if $\| C(s) - C_R(s)\|_\infty$ is minimized*

Definition 2 *An $(n \times n)$ transfer function $C(s)$ is called γ − allpass if $C^\sharp(s)C(s) = C(s)C^\sharp(s) = \gamma^2 I$, where $C^\sharp(s) = C^T(-s)$.*

Theorem 1 *An $(n \times n)$ transfer function $C(s)$ with a minimal realization $C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_o \end{array}\right]$ is γ−all-pass if and only if*

$$A_cL_c + L_cA_c^T + B_cB_c^T = 0 \tag{14}$$
$$A_c^TL_o + L_oA_c + C_c^TC_c = 0 \tag{15}$$
$$L_cL_o = \gamma^2 I \tag{16}$$
$$D_oB_c^T + C_cL_c = 0 \tag{17}$$
$$D_o^TC_c + B_c^TL_o = 0 \tag{18}$$
$$D_oD_o^T = D_o^TD_o \tag{19}$$
$$= \gamma^2 I \tag{20}$$

Lemma 1 *Let $C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \in R\mathcal{H}_\infty$ be a minimal realization such that $(14 - 16)$ are all satisfied. Assume that $rank(B_c) = rank(C_c) = n$, where n is the number of rows of the square matrix $C(s)$. Then the unique D_o that satisfies the equations $(17-20)$ is given by*

$$D_o = -C_cL_c(B_c^T)^\dagger = -(C_c^T)^\dagger L_oB_c \tag{21}$$

where $(B_c)^\dagger = lim_{\epsilon \to 0}(\epsilon^2 I + B_cB_c^T)^{-1}B_c^T$.

The above lemma pertains to the case where B_c and C_c have full ranks. If $rank(B_c) = rank(C_c) = n1 < n$, then certain modifications are required to obtain D_o.

The Following theorem gives the characterization of the optimal solution for the zeroth order $\mathcal{H}_\infty$ norm approximation problem (see definition 1 when $r = 0$).

Theorem 2 *Given $C(s)$ of order n with minimal realization $C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$, then there exist a D_o such that $\| C - D_o \|_\infty = \gamma$, where γ is the minimum achievable error, if and only if there exist B_o, C_o such that*

$$A_cP + PA_c^T + B_cB_c^T + B_oB_o^T = 0 \tag{22}$$
$$A_c^TQ + QA_c + C_c^TC_c + C_o^TC_o = 0 \tag{23}$$

are satisfied, with

$$PQ = QP = \gamma^2 I \tag{24}$$

It can be seen that the above problem can easily be converted into the prototype problem, where $\alpha = \sqrt{\gamma}$ and $Q = \gamma^2 P^{-1}$. The above theorem provides a way of obtaining an optimal solution of zeroth order optimal $\mathcal{H}_\infty$ model reduction problem, but does not give an explicit scheme to compute the solution. An explicit computational scheme will be provided in the next section, in connection to the design of the structurally predefined LTR controller.

4 Structurally Predefined LTR Controller

In this section, we outline a procedure to solve the problem of finding the best B_r and D_r in $E(s) := C(s) - C_R(s) = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & -D_r \end{array}\right] = \left[\begin{array}{cc|c} A_c & 0 & B_c \\ 0 & A_r & B_r \\ \hline C_c & -C_r & -D_r \end{array}\right]$ such that $\| E \|_\infty$ is minimum. Results of the last section shows that the minimum achievable approximation is solely determined by A_c , B_c and C_c and then zeroth order approximation D_o is calculated. So, in this case, the minimum achievable error depends solely on B_r. After the optimal B_r is obtained, the problem of finding optimal D_r is the standard zeroth order approximation problem. The following theorem is based on this observation.

Theorem 3 *Given an LTR controller $C(s) \in R\mathcal{H}_\infty$ with a minimal realization $C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$. Let $C_R(s) = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$ be an optimal approximation of $C(s)$ with prescribed A_r and C_r. Then B_r and D_r are determined from the following steps.*

(1) Solve the following LMI eigenvalue problem

$$\underset{\alpha, P=P^T, (B_r)_{r \times n}}{\text{Minimize}} \quad \alpha \tag{25}$$

subject to

$$\begin{pmatrix} A_e P + P A_e^T & B_e \\ B_e^T & -I \end{pmatrix} \leq 0 \tag{26}$$

$$\begin{pmatrix} A_e P + P A_e^T & P C_e^T \\ C_e P & -\alpha I \end{pmatrix} \leq 0 \tag{27}$$

The minimum achievable norm is $\gamma = \sqrt{\alpha}$.

(2) Take $Q = \gamma^2 P^{-1}$ and calculate B_o, C_o from

$$\begin{aligned} B_o B_o^T &= -(A_e P + P A_e^T + B_e B_e^T) & (28) \\ C_o^T C_o &= -(A_e^T Q + Q A_e + C_e^T C_e) & (29) \end{aligned}$$

(3) Form $H_e(s)$

$$H_e(s) = \left[\begin{array}{c|cc} A_e & B_e & B_o \\ \hline C_e & 0 & 0 \\ C_o & 0 & 0 \end{array}\right] \tag{30}$$

(4) Use Lemma1 to compute D_H, the optimal zeroth order approximation of $H_e(s)$.

(5) Partition D_H as $D_H = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ such that D_{11} is $n \times n$.

(6) Finally D_r is given as $D_r = D_{11}$ and $C_R(s)$ is fully determined.

Example: Consider the following system $G(s) = \frac{s+4}{s^3+5s^2+8s+6}$ which has a transmission zero at -4. Let the target feedback loop be realized by $F = [1 \quad -4 \quad 2]$.

Application of the Transmission Zero Matrix Algorithm [4] yields the following parameters for first order functional observer-based LTR controller $C_F(s)$.

$$\Phi = [-4], \quad G = [10], \quad M = [1], \quad N = [-14], \quad H = [0]$$

Since $H = 0$, the controller $C_F(s)$ achieves $ELTR$.

Now the full order observer-based controller $C(s)$ with $K = [0 \quad 3 \quad 1]'$ is designed. Applying the proposed LMI based $\mathcal{H}_\infty$ model reduction, the structurally predefined fixed order controller becomes

$$A_r = [-4], \; B_r = [-1.2594], \; C_r = [1], \; D_r = [-1.9453]$$

Figure below shows that the structurally predefined LTR controller achieves better recovery than the full order observer based LTR controller and both obviously cannot outperform the functional observer based LTR controller (FO/LTR).

Several other examples have been constructed for the case of $ALTR$. We observed comparable performances between two approaches. These examples will be illustrated in the conference.

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