IMPROVED RECOVERY IN H_∞/LTR DESIGN [1]

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Abstract. This paper shows the possibility of including weighting functions in $\mathcal{H}_\infty$/LTR design to improve the recovery in specific frequency ranges. It turns out that it is still possible to derive a solution by solving only one Riccati equation in both cases. The observer gain is given in explicit form. The weighted LTR design method can be applied to both minimum phase systems as well as to non-minimum phase systems.

Keywords. Loop Transfer Recovery, Observer based controllers, $\mathcal{H}_\infty$ design, Riccati equations.

1. INTRODUCTION

The motivation for this paper is the fact that Loop Transfer Recovery (LTR) design methods normally minimizes a suitable norm of the recovery matrix, Niemann *et al.* (1991), Niemann *et al.* (1993), Saberi *et al.* (1993). Only in few cases, the recovery error is minimized directly, Moore and Tay (1989), Stoustrup and Niemann (1993). However, by minimizing the recovery error directly, the controller order will be $2n$ or more. Another possibility is to apply alternative observer structures instead of the standard full-order observer for obtaining special recovery properties. By using a PI-observer, it is possible to obtain good recovery at low frequencies, i.e. obtaining time recovery, Niemann *et al.* (1995), Shafai *et al.* (1994).

The key result in this paper is to show that it is possible to minimize a suitable norm of the weighted recovery matrix by solving only one Riccati equation or a Quadratic Matrix Inequality. The advantage of multiplying the recovery matrix by a weight matrix, is that it is possible to obtain good recovery in a specific frequency range without using high observer gains for both minimum and nonminimum phase systems. The rest of the paper is organized as follows. In section 2, the LTR design methodology is briefly described. In section 3, the solution of the $\mathcal{H}_\infty$/LTR design problem is given followed by section 4 where the weighted $\mathcal{H}_\infty$/LTR design problem is considered. A solution is given in explicit forms. A non-minimum phase LTR design example is included in section 5 followed by a conclusion in section 6.

2. LTR DESIGN

Consider a finite dimensional, linear, time-invariant system described by a state-space realization (A, B, C, D):

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^r$, and $y \in \mathcal{R}^m$, (A, B) stabilizable, (C, A) detectable, and where C and B has full rank. In the following, there is no condition on the direct term D to have full rank, although it does sometimes simplify the solution, as we shall see below.

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Let the plant be controlled by an observer based controller having the state feedback

$$u = F\hat{x} + r = w + r \tag{2}$$

where F is the state feedback gain and $\hat{x}$, the state estimate and r is the reference input. F is required to be stabilizing, i.e. $A + BF$ having eigenvalues in the left half plane and otherwise free. The states are estimated by a Luenberger observer given by Luenberger (1971):

$$\begin{aligned} \dot{z} &= Ez + Gy + Hu \\ \hat{x} &= Mz + Ny \end{aligned} \tag{3}$$

with the following constraints:

$$\mathrm{Re}\,[\lambda(E)] < 0, \tag{4}$$

$$TA - ET = GC, \tag{5}$$

$$H = TB - GD, \tag{6}$$

$$MT + NC = F, \tag{7}$$

$$ND = 0 \tag{8}$$

where the matrix T relates the observer and the system through $z = Tx + e$, which in turn is related to the state reconstruction error by $\tilde{x} = \hat{x} - x = M(z - Tx)$.

In the following, we need the transfer function for the Luenberger observer $C(s)$ and the transfer function of the recovery matrix $M_{I,L}(s)$, Niemann *et al.* (1993), Stoustrup and Niemann (1993) given by:

$$\begin{aligned} C(s) &= M(sI - E - HM)^{-1}(G - HN) + N \\ M_{I,L}(s) &= M(sI - E)^{-1}H \end{aligned}$$

To design a controller for the system (1) by the LTR design methodology, we first determine a static state feedback, the target design, which satisfies our design specifications. The design specifications, such as robust stability and nominal performance conditions, are assumed to be reflected at the plant input point, Stein and Athans (1987).

Based on the target (full-state feedback) design gain F for the system (1), the target sensitivity function is given by

$$S_{TFL}(s) = \left(I - F(sI - A)^{-1}B\right)^{-1}. \tag{9}$$

Next the LTR step is performed in which we attempt to recover the target design over a range of frequencies by a dynamic compensator $C(s)$. This step gives a full-loop, sensitivity transfer function of the form

$$S_I(s) = (I - C(s)G(s))^{-1} \tag{10}$$

where $G(s)$ represents the plant transfer function.

Assuming that $C(s)$ is implemented via an observer (or Kalman filter) based controller, the resulting loop transfer function $C(s)G(s)$, in general, is not the same as the target loop transfer function $S_{TFL}(s)$. In the LTR step the required observer is designed so as to recover either exactly (perfectly) or asymptotically (approximately) the target loop transfer function.

For a more careful analysis, we define the sensitivity loop transfer recovery error as

$$E_S(s) = S_I(s) - S_{TFL}(s) \tag{11}$$

and say that exact loop transfer recovery at the input point (ELTRI) is achieved if the closed-loop system comprised of $C(s)$ and $G(s)$ is asymptotically stable and $E_S(s) = 0$. To define approximate or asymptotic LTR at the input point (ALTRI), see Doyle and Stein (1981), Stein and Athans (1987), we parameterize the family of controllers as $C(s, q)$, where q is a positive scalar, and say that ALTRI is achieved if the closed-loop system is asymptotically stable and $S_I(s) \to S_{TFL}(s)$ pointwise in s as $q \to \infty$, i.e., $E_S(s, q) \to 0$ pointwise in s as $q \to \infty$.

The sensitivity recovery error is related to the so-called recovery matrix $M_{I,L}(s)$ given in Niemann *et al.* (1991) by the equation

$$E_S(s) = S_{TFL}(s)M_{I,L}(s)\ . \tag{12}$$

With this background we are ready to discuss the LTR of full-order observers.

Consider the full-order observer-based controller having the transfer function

$$C(s) = -F\left(sI - (A + KC) - BF\right)^{-1}K \tag{13}$$

where F and K are the regulator and observer gains, respectively. Then ELTRI is achieved if and only if $E_S(s) = 0$ or equivalently $M_{I,fo}(s) = 0$ where $M_{I,fo}$ is the recovery matrix for the full order observer based controller given by

$$M_{I,fo}(s) = F\left(sI - A - KC\right)^{-1}B. \tag{14}$$

In practice, the condition $M_{I,fo}(s) = 0$ can not always be satisfied exactly. Consequently, the size of $M_{I,fo}(s)$ should be made small in some sense.

Let the controller be parameterized in terms of the observer gain by $K(q)$. Then to obtain ALTRI we seek a $K(q)$ such that for all ω

$$\begin{aligned} M_{I,fo}(i\omega) &= F\left(i\omega I - A - K(q)C\right)^{-1}B \\ &\to 0 \text{ as } q \to \infty. \end{aligned} \tag{15}$$

In general, this is possible only if the system is minimum phase, or if F is selected carefully. The literature reports several methods, Doyle and Stein (1979), Athans (1986), Stein and Athans (1987), to obtain such a $K(q)$. Hence, good recovery can be achieved only in the limit as $q \to \infty$ which implies that $\|K(q)\| \to \infty$, in case that exact recovery is not possible.

3. $\mathcal{H}_\infty$/LTR DESIGN

Based on the above section, an LTR design method using $\mathcal{H}_\infty$ optimization will shortly be presented in this section. The basic idea of the $\mathcal{H}_\infty$/LTR design method is to make the $\mathcal{H}_\infty$ norm of the recovery matrix smaller than a specified level γ. A more detailed description can be found in Niemann *et al.* (1991), Niemann *et al.* (1993), Saberi *et al.* (1993).

Let the recovery matrix for the full-order observer have the following state space description:

$$M_{I,fo}(s) = \left(\begin{array}{c|cc} A & B & I \\ \hline F & 0 & 0 \\ C & D & 0 \end{array}\right) \tag{16}$$

with the controller

$$u(s) = Ky(s) \tag{17}$$

where K is the full-order observer gain.

For designing an observer gain K such that the transfer function from w to z, i.e. the recovery matrix $M_{I,fo}$, has an $\mathcal{H}_\infty$ norm smaller than γ, we have the following result from Stoorvogel (1992)

Theorem 1. It is assumed that the system (A, B, C, D) has no invariant zeros on the imaginary axis. Then the following two statements are equivalent.

(1) There exists an observer gain K such that $A+KC$ is stable and such that
$$\|F(sI - A - KC)^{-1}(B + KD)\|_\infty < \gamma$$
(2) There exists $Q \geq 0$ such that the following three hold
- $G_\gamma(Q) := \begin{pmatrix} X & Y' \\ Y & Z \end{pmatrix} =: \begin{pmatrix} B_Q \\ D_Q \end{pmatrix} \begin{pmatrix} B_Q' & D_Q' \end{pmatrix} \geq 0$
- $\operatorname{rank} \begin{pmatrix} B_Q \\ D_Q \end{pmatrix} = \operatorname{rank}_{\mathcal{R}(s)} \left[F(sI - A)^{-1}B + D\right], \ \forall s \in \overline{\mathbb{C}^+}$
- $\operatorname{rank} \begin{pmatrix} A + \gamma^{-2}QC'C - sI & B_Q \\ F & D_Q \end{pmatrix} = n + \operatorname{rank}_{\mathcal{R}(s)} \left[F(sI - A)^{-1}B + D\right], \ \forall s \in \overline{\mathbb{C}^+}$

where $X = AQ + QA' + BB' + \gamma^{-2}QF'FQ$, $Y = CQ + DB'$ and $Z = DD'$.

Whenever $Q \geq 0$ exists satisfying the three conditions in item (2) of Theorem 1 such Q can be found by solving a reduced order Riccati equation. Moreover, it can be shown that Q is unique (see Stoorvogel (1992)). When the direct term D has full rank, we get directly the DGKF Riccati equation given by:

$$\begin{aligned} 0 &= AQ + QA' + BB' + \gamma^{-2}QF'FQ \\ &\quad -(QC' + BD')(DD')^{-1}(CQ + DB') \end{aligned} \tag{18}$$

Based on the solution Q to the Quadratic Matrix Inequality in Theorem 1 or the Riccati equation in (18), we can define the following transformed system $G_Q(s)$:

$$G_Q(s) = \left(\begin{array}{c|cc} A_Q & B_Q & I \\ \hline F & 0 & 0 \\ C & D_Q & 0 \end{array}\right) \tag{19}$$

where B_Q and D_Q are defined in Theorem 1 and A_Q is given by:

$$A_Q = A + \gamma^{-2}QC'C \tag{20}$$

The relation between the original system $G(s)$ and the transformed system $G_Q(s)$ is given by the following lemma from Stoorvogel (1992):

Lemma 2. The following two statements are equivalent:

(1) There exists an observer gain K such that $A+KC$ is stable and
$$\|F(sI - A - KC)^{-1}(B + KD)\|_\infty < \gamma$$
(2) There exists an observer gain K such that A_Q+KC is stable and
$$\|F(sI - A_Q - KC)^{-1}(B_Q + KD_Q)\|_\infty < \gamma$$

The significance of Lemma 2 is that the problem of finding an observer gain for the original problem can be reduced in finding a controller for the transformed problem. This is much easier, since this problem does not have zeros in the closed right half plane. The problem of designing the observer gain has been transformed to an Almost Disturbance Decoupling Problem (ADDP). Based on this fact, the observer gain has to be designed such that $A_Q + KC$ is stable and

$$\|F(sI - A_Q - KC)^{-1}(B_Q + KD_Q)\|_\infty < \gamma \tag{21}$$

This problem can always be solved, due to the fact that the transformed system is minimum phase. In the regular case, the above ADDP can be solved exactly (although this is not required). When D (and D_Q) has full row rank, the observer gain which solves the ADDP exactly is given by:

$$K = -B_Q D_Q^{\dagger} \tag{22}$$

By using the equations for the transformation of the original system, the above observer gain can be rewritten into:

$$K = -(QC' + BD')(DD')^{-1} \tag{23}$$

4. WEIGHTED $\mathcal{H}_\infty$/LTR DESIGN

The $\mathcal{H}_\infty$/LTR result given in the above section is based on an optimization of the $\mathcal{H}_\infty$ norm of the recovery matrix. In Stoustrup and Niemann (1993) the $\mathcal{H}_\infty$ design method has been applied to the $\mathcal{H}_\infty$ norm optimization of the sensitivity recovery error given by (11) or (12). However, it turns out that it require solving two Quadratic Matrix Inequalities (or two Riccati equations) for solving this problem. Moreover, the final controller will be of order $2n$.

In this paper, we will look at a related LTR problem, which can be solved by using only one Quadratic Matrix Inequality or one Riccati equation. Our weighted $\mathcal{H}_\infty$/LTR design problem is as follows.

Problem 1. Consider the recovery matrix for the full-order observer given by (14) and a weight function W(s) given by:

$$\begin{aligned} \dot{x}_W &= A_W x_W + B_W \xi \\ y_W &= C_W x_W + D_W \xi \end{aligned} \tag{24}$$

Let $\gamma > 0$ be given. Design an internally stabilizing dynamic controller $K(s)$, if such exist, such that

$$\|F(sI - A - K(s)C)^{-1}(B + K(s)D)W(s)\|_\infty < \gamma$$

Again, using standard $\mathcal{H}_\infty$ techniques, this problem can be solved by using two Quadratic Matrix Inequalities (or two Riccati equations in the regular case) together with the coupling condition. However, instead of using the standard $\mathcal{H}_\infty$ technique, the problem can be solved by using only one Quadratic Matrix Inequality (or one Riccati equation in the regular case) without the coupling condition. To that end, let us consider a state space description of the recovery matrix together with the weight function (24) is given by:

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}_1 w + \bar{B}_2 u \\ z &= \bar{C}_1 \bar{x} \\ y &= \bar{C}_2 \bar{x} + \bar{D}_{21} w \end{aligned} \tag{25}$$

where $\bar{x} = \begin{pmatrix} x \\ x_W \end{pmatrix}$ and the six matrices are as follows:

$$\begin{aligned} \bar{A} &= \begin{pmatrix} A & BC_W \\ 0 & A_W \end{pmatrix}, \quad \bar{B}_1 = \begin{pmatrix} BD_W \\ B_W \end{pmatrix} \\ \bar{B}_2 &= \begin{pmatrix} I \\ 0 \end{pmatrix} \\ \bar{C}_1 &= \begin{pmatrix} F & 0 \end{pmatrix} \\ \bar{C}_2 &= \begin{pmatrix} C & DC_W \end{pmatrix}, \quad \bar{D}_{21} = DD_W \end{aligned} \tag{26}$$

The controller is given by

$$u(s) = K(s)y(s)$$

Furthermore, we need the following lemma.

Lemma 3. Assume that neither of the two systems in (1) and (24) have any invariant zeros on the imaginary axis, and D_W has full row rank. Then the system given in (25) has no invariant zeros on the imaginary axis.

Proof of Lemma 3. The proof follows directly by calculating the rank of the Rosenbrock matrix of the system (25). □

Based on this state space realization of the weighted LTR problem and Lemma 3, we are now able to give the main result for the $\mathcal{H}_\infty$/LTR design problem.

Theorem 4. Consider the system in (25). Then the following statements are equivalent:

(1) There exists a dynamic internally stabilizing controller $K(s)$ such that when applying the feedback law $u = K(s)y$, the resulting closed loop transfer function from w to z has an $\mathcal{H}_\infty$ norm smaller than γ.
(2) There exists $\bar{Q} \geq 0$ such that the following three hold
   - $\bar{G}_\gamma(\bar{Q}) := \begin{pmatrix} \bar{X} & \bar{Y}' \\ \bar{Y} & \bar{Z} \end{pmatrix} =: \begin{pmatrix} \bar{B}_{1,Q} \\ \bar{D}_{21,Q} \end{pmatrix} \begin{pmatrix} \bar{B}'_{1,Q} & \bar{D}'_{21,Q} \end{pmatrix} \geq 0$
   - $\operatorname{rank} \begin{pmatrix} \bar{B}_{1,Q} \\ \bar{D}_{21,Q} \end{pmatrix} = \operatorname{rank}_{\mathcal{R}(s)} \bar{G}(s), \ \forall s \in \overline{\mathbb{C}^+}$
   - $\operatorname{rank} \begin{pmatrix} \bar{A} + \gamma^{-2}\bar{Q}\bar{C}'_2\bar{C} - sI & \bar{B}_{1,Q} \\ \bar{C}_1 & \bar{D}_{21,Q} \end{pmatrix} = n + n_W + \operatorname{rank}_{\mathcal{R}(s)} \bar{G}(s), \ \forall s \in \overline{\mathbb{C}^+}$

   where $\bar{X} = \bar{A}\bar{Q} + \bar{Q}\bar{A}' + \bar{B}_1\bar{B}'_1 + \gamma^{-2}\bar{Q}\bar{C}'_1\bar{C}_1\bar{Q}$, $\bar{Y} = \bar{C}_2\bar{Q} + \bar{D}_{21}\bar{B}'_1$, $Z = \bar{D}_{21}\bar{D}'_{21}$ and $\bar{G}(s) = \bar{C}_1(sI - \bar{A})^{-1}\bar{B}_1 + \bar{D}_{21}$. Moreover, one such dynamic controller $K(s)$ is then given by:

$$K(s) = \left(\begin{array}{c|c} A_W + K_2 D C_W & K_2 \\ \hline (K_1 D + B) C_W & K_1 \end{array}\right) = \left(\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right) \tag{27}$$

where $\bar{K} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ satisfies the norm inequality:

$$\|\bar{C}_1(sI - \bar{A}_Q - \bar{K}\bar{C}_2)^{-1}(\bar{B}_{1,Q} + \bar{K}\bar{D}_{21,Q})\|_\infty < \gamma$$

where $\bar{A}_Q$ is given by:

$$\bar{A}_Q = \bar{A} + \gamma^{-2}\bar{Q}\bar{C}_2'\bar{C}_2$$

Again, in the regular case, the Quadratic matrix Inequality can be replaced by a Riccati equation given by:

$$\begin{aligned} 0 &= \bar{A}\bar{Q} + \bar{Q}\bar{A}' + \bar{B}_1\bar{B}_1' + \gamma^{-2}\bar{Q}\bar{C}_1'\bar{C}_1\bar{Q} \\ &- (\bar{Q}\bar{C}_2' + \bar{B}_1\bar{D}_{21}')(\bar{D}_{21}\bar{D}_{21}')^{-1}(\bar{C}_1\bar{Q} + \bar{D}_{21}\bar{B}_1') \end{aligned} \tag{28}$$

and the observer gain $\bar{K}$ is given by:

$$\bar{K} = -(\bar{Q}\bar{C}_2' + \bar{B}_1\bar{D}_{21}')(\bar{D}_{21}\bar{D}_{21}')^{-1} \tag{29}$$

The proof of Theorem 4 is based upon the fact, that nothing more can be achieved for an $\mathcal{H}_\infty$ problem than what can be achieved by a static observer gain using *all* (real or fictitious) inputs as stated in the following well known fact.

Lemma 5. Assume that there exists an internally stabilizing control law $u = Hy$ for the system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \\ y &= C_2 x + D_{21} w \end{aligned} \tag{30}$$

making the closed loop $\mathcal{H}_\infty$ norm from w to z smaller than γ.

Then there exists an internally stabilizing *static* output injection controller $u = Ky$ for the system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + Iu \\ z &= C_1 x \\ y &= C_2 x + D_{21} w \end{aligned} \tag{31}$$

which makes the closed loop $\mathcal{H}_\infty$ norm from w to z smaller than γ.

Proof of Theorem 4. To establish the proof of Theorem 4 we shall verify that for the system (25) the reverse of Lemma 5 holds, i.e., that the existence of a static output injection controller obtaining a certain $\mathcal{H}_\infty$ norm γ implies the existence of a dynamic measurement based controller obtaining the same $\mathcal{H}_\infty$ norm. In fact, it can be proved the stronger result, that the same closed loop transfer function can be obtained. This derivation will not be given in this paper. □

Finally, let us write up the overall controller structure. When using the controller

$$K(s) = \left(\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right)$$

derived above, along with a full order observer we get the following controller:

$$C(s) = \left(\begin{array}{cc|c} A + D_K C + BF & C_K & -D_K \\ B_K C & A_K & -B_K \\ \hline F & 0 & 0 \end{array}\right)$$

which is of order $n + n_W$.

5. EXAMPLE

Let us consider a simple nonminimum phase system described by the following state space model

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} -2 & 1 \end{pmatrix} x \end{aligned}$$

This system has a nonminimum phase zero at $z = 2$. Given the target design

$$F = \begin{pmatrix} -50 & -10 \end{pmatrix}$$

the problem is to design a dynamic compensator, such that the recovery (in terms of the recovery matrix) is better than -20 dB at frequencies smaller than 10^{-2} and better than 20 dB for all frequencies. This is achieved by selecting a first order weight with a zero at -10^{-2}, and a pole two decades further. γ can be selected as 20 dB if small gains are essential, or it can be found by iteration for the quadratic matrix inequality in Theorem 4 if worst case recovery is in focus.

Figure 1 shows such a design following the above steps, where γ has been found by iteration. For comparison is shown a traditional LQG/LTR design. The LQG/LTR design has much poorer performance for low frequencies, which is natural since this requirement was not built into the design. The bandwidth of the suggested design is slightly poorer than that of the LQG/LTR. This is due to a near-optimal choice of γ. If γ was chosen even closer to the optimum, the bandwidth would deteriorate accordingly. And vice versa: if γ would be increased to the value of the LQG/LTR design, the bandwidths would be comparable.

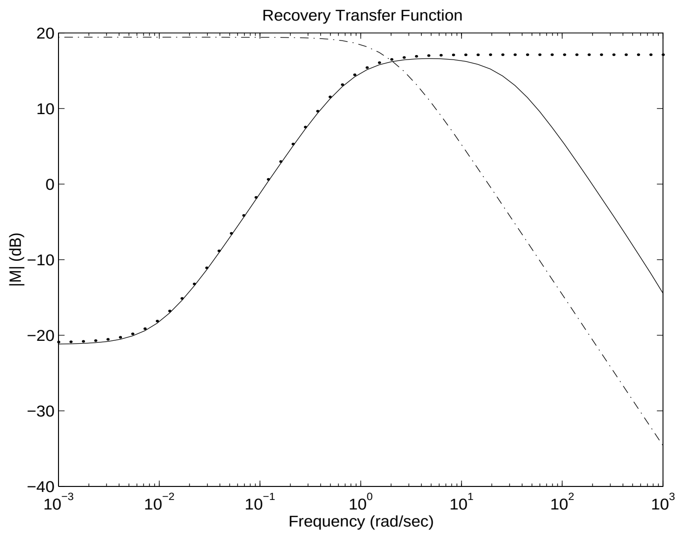

Fig. 1. Nonminimum Phase Example. Solid line: improved design. Dotted line: reciprocal of weighting. Dash-dotted line: LQG/LTR design

If desired, a bandwidth constraint could be built into the design directly by selecting a second order weight instead of a first order one.

6. CONCLUSION

A new method for LTR design of observer based controllers has been proposed in this paper. The method is based on multiplying the recovery matrix by a weight function for obtaining good recovery in specified frequency ranges. It turns out that the LTR design problem can be solved by using only one Quadratic Matrix Inequality or one Riccati equation. The controller can be given in explicit form. In this paper, the LTR design problem has been considered at the input loop breaking point, but the dual result for the output loop breaking point can also be derived without any further conditions.

It is important to point out that there is no requirement on the system to be minimum phase for achieving good recovery in a specified frequency range, except that the frequency range should not include non minimum phase zeros. In contrast, in standard LTR design methods, it is in general possible to obtain good recovery only if the non minimum phase zeros are outside the bandwidth for the target loop.

7. REFERENCES

Athans, M. (1986). A tutorial on the LQG/LTR method. In: *ACC*. Seattle, WA. pp. 1289–1296.

Doyle, J.C. and G. Stein (1979). Robustness with observers. *IEEE Transactions on Automatic Control* **24**, 607–611.

Doyle, J.C. and G. Stein (1981). Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Transactions on Automatic Control* **26(1)**, 4–16.

Luenberger, D.G. (1971). An introduction to observers. *IEEE Transactions on Automatic Control* **16(6)**, 596–602.

Moore, J.B. and T.T. Tay (1989). Loop recovery via $\mathcal{H}_\infty/\mathcal{H}_2$ sensitivity recovery. *International Journal of Control* **49(4)**, 1249–1271.

Niemann, H.H., J. Stoustrup, B. Shafai and S. Beale (1995). LTR of propertional integral observer architectures. *International Journal of Robust and Nonlinear Control*.

Niemann, H.H., P. Søgaard-Andersen and J. Stoustrup (1991). Loop transfer recovery for general observer architectures. *International Journal of Control* **53**, 1177–1203.

Niemann, H.H., P. Søgaard-Andersen and J. Stoustrup (1993). $\mathcal{H}_\infty$ optimization of the recovery matrix. *Control - Theory and Advanced Technology, C-TAT* **9(2)**, 547–564.

Saberi, B. M. Chen and P. Sannuti (1993). *Loop Transfer Recovery: Analysis and Design*. Springer-Verlag.

Shafai, B., S. Beale, H.H. Niemann and J. Stoustrup (1994). LTR design of discrete-time proportional-integral observers. Submitted for publictation.

Stein, G. and M. Athans (1987). The LQG/LTR procedure for multivariable feedback control design. *IEEE Transactions on Automatic Control* **32**, 105–114.

Stoorvogel, Anton (1992). *The $\mathcal{H}_\infty$ Control Problem: A State Space Approach*. Prentice Hall.

Stoustrup, J. and H.H. Niemann (1993). State space solution to the $\mathcal{H}_\infty$/LTR design problem. *International Journal of Robust and Nonlinear Control* **3(1)**, 1–46.