LOOP TRANSFER RECOVERY FOR SAMPLED-DATA SYSTEMS [1]

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Abstract. This paper considers the design of loop transfer recovery (LTR) controller for sampled-data systems. The LTR design problem is formulated by using the 2 x 2 setup formulation. Following the standard LTR theory, the difference between the target loop and the full-loop is defined as the recovery error, which is equal to the target loop multiplied by the recovery matrix. The minimization of the recovery error is derived by using $\mathcal{H}_2$ and $\mathcal{H}_\infty$ designs.

Keywords. Sampled-data systems, Lifting, Observer based controllers, LTR design, Discrete-time $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design, Riccati equations.

1. INTRODUCTION

The problem of Loop Transfer Recovery (LTR) was originally introduced in Doyle and Stein (1979), Doyle and Stein (1981), and since then many papers have been published in this area. The majority of these papers have been cited in the reference list of Saberi *et al.* (1993). All these papers deal either with continuous-time systems or with discrete-time systems. However, the LTR design of sampled-data systems has not been tackled in the literature except one paper Shi *et al.* (1994). The approach taken in Shi *et al.* (1994) is based on the result in Sun *et al.* (1991), which is distinctly different from the lifting approach proposed in Bamieh *et al.* (1991), Bamieh and Pearson (1992*b*). In the former case, the controller turns out to be linear time-varying and it generates continuously varying input signals rather than a piecewise constant input signal. On the other hand, in the latter case, the controller is assumed to be a finite dimensional shift invariant system which is interfaced with the continuous-time plant using a zero-order hold and an ideal sampler. After certain transformation steps, the controller is designed for the lifted system.

Our solution to LTR problem for sampled-data systems is based on the lifting approach of Bamieh and Pearson (1992*b*). The disadvantage of this approach is that it is slightly more difficult to formulate than the purely continuous-time or discrete-time cases. However, as it will be shown in this paper, the LTR design of sampled-data systems can directly be tied to the conventional discrete-time LTR design methods.

2. AN OVERVIEW OF LTR DESIGN

Let us consider the following system:

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{11} w + D_{12} u$$

$$y = C_2 x + D_{21} w$$

or in a short description form by

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right)$$

where $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}^r$, is the control input, $w \in \mathcal{R}^k$ is the external input or disturbance, $z \in \mathcal{R}^l$ is the controlled output and $y \in \mathcal{R}^m$ is the measurement output. It is assumed that (A, B_2) is stabilizable and (C_2, A) is detectable. Suppose the LTR design methodology is applied at the input loop breaking point. We first design a target feedback loop with the static state feedback gain F, for the system described by:

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$$G_{SF}(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right)$$

such that the design specifications are satisfied. It is assumed that the state feedback loop is asymptotically stable, i.e. all the eigenvalues of $A_F = A + B_2F$ lie in the left half plane. The target loop transfer function is then given by:

$$T_{zw,T}(s) = (C_1 + D_{12}F)(sI - A_F)^{-1}B_1 + D_{11}$$

which satisfies the closed-loop design specifications for the transfer function from w to z. Now, let the plant be controlled by a full-order observer based controller given by:

$$C(s) = -F(sI - A - B_2F - KC_2)^{-1}K$$

where K is the observer gain. Then, the resulting closed loop transfer function, in general, is not the same as the target loop transfer function $T_{zw,T}(s)$. In the LTR step the observer based controller is designed so as to recover either exactly (perfectly) or asymptotically (approximately) the target loop transfer function.

For a more careful analysis, we define the closed loop transfer recovery error as

$$E_{cl}(s) = T_{zw}(s) - T_{zw,T}(s)$$

where T_{zw} is the closed-loop transfer function from w to z when a full order observer is applied. The closed-loop recovery error is related to the so-called recovery matrix $M_I(s)$ given in Niemann *et al.* (1991) by the equation

$$E_{cl}(s) = T_{zu,T}(s)M_I(s) \ .$$

where $T_{zu,T}(s)$ is the closed-loop transfer function from u to z under the target design given by:

$$T_{zu,T}(s) = (C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_2 + D_{12}$$

and the recovery matrix M_I is given by

$$M_I(s) = F(sI - A - KC_2)^{-1}(B_1 + KD_{21})$$

We will say that exact loop transfer recovery at the input point (ELTRI) is achieved if the closed-loop system comprised of $C(s)$ and $G(s)$ is asymptotically stable and $E_{cl}(s) = 0$ or $M_I(s) = 0$ when $T_{zu,T}$ is left invertible, (they are equivalent in this case). For obtaining asymptotic LTR at the input point (ALTRI), Doyle and Stein (1981), Stein and Athans (1987), we parameterize a family of controllers with a positive scalar q, and say that ALTRI is achieved if the closed-loop system is asymptotically stable and $E_{cl}(s,q) \to 0$ pointwise in s as $q \to \infty$.

3. SAMPLED-DATA SYSTEMS

In the following, the sampled-data system setup is shortly introduced together with the lifting technique.

3.1 *System Setup*

Let us consider the sampled-data system described by:

$$G_{SD} = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right) \quad (1)$$

Note that D_{21} and D_{22} are not present, however, they can be incorporated by adding filters at the measurement outputs if necessary, see Bamieh and Pearson (1992*b*). The discrete time controller C is described by:

$$C(z) = \left(\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right)$$

Furthermore, $\mathcal{S}_\tau$ and $\mathcal{H}_\tau$ represent the sampler and the hold device, where τ is the sampling period.

The design of a sampled-data controller for the system given by (1) can be derived by including the sampler and the hold in the general system as shown in, see Yamamoto (1990), Bamieh *et al.* (1991), Bamieh and Pearson (1992*b*) for the description of the lifting technique.

Using the lifting technique, we have the following description of $\tilde{G}$:

$$\tilde{G} = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12}\hat{\mathcal{H}}_\tau \\ \hat{\mathcal{S}}_\tau\hat{G}_{21} & \hat{\mathcal{S}}_\tau\hat{G}_{22}\hat{\mathcal{H}}_\tau \end{bmatrix}$$

which has the following realization, Bamieh and Pearson (1992*b*):

$$\hat{G}_{SD}(z) = \left(\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \tilde{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & 0 & 0 \end{array}\right) \quad (2)$$

where the calculation of the operators in (2) can be found in Bamieh and Pearson (1992*b*).

Note that $\tilde{G}_{22}$ is the simply discretized transfer function of $G_{22}(s)$.

The lifted system (2) satisfies the following:

(1) $\mathcal{H}_\tau C\mathcal{S}_\tau$ internally stabilizes G if C internally stabilizes $\tilde{G}$
(2) $\|\mathcal{F}(G, \mathcal{H}_\tau C\mathcal{S}_\tau)\| = \|\mathcal{F}(\tilde{G}, C)\|$

3.2 $\mathcal{H}_2$ *and* $\mathcal{H}_\infty$ *Controller Design*

The lifted sampled-data system described by (2) can not be applied directly for an $\mathcal{H}_2$ or an $\mathcal{H}_\infty$ controller design, because the operators $\hat{B}_1, \hat{C}_1, \hat{D}_{11}$ and $\hat{D}_{12}$ are infinite-dimensional. However, by using operator theory, it is possible to derive an equivalent discrete-time finite dimensional system $\bar{G}$ such that the following two statements are equivalent for $\mathcal{H}_2$ optimization (see Bamieh and Pearson (1992*a*) for result and notation):

(1) $\mathcal{F}(\tilde{G}, C)$ is internally stable and $\mathcal{F}(\bar{G}, C)$ is internally stable
(2) $\|\mathcal{F}(\tilde{G}, C)\|^2_{\mathcal{H}^2_{HS}} = \frac{1}{\tau}(\|\hat{D}_{11}\|^2_{HS^2} + \|\mathcal{F}(\bar{G}, C)\|^2_{\mathcal{H}^2})$

In this case, it is required that $\hat{D}_{11} = 0$ for making the closed-loop strictly causal.

The equivalent finite dimensional discrete-time system $\bar{G}$ is given by:

$$\bar{G}(z) = \left(\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right)$$

$\bar{G}$ can be calculated by using the equations given in Bamieh and Pearson (1992*a*). Now, the $\mathcal{H}_2$ design follows standard discrete-time $\mathcal{H}_2$ controller design.

However, if we are interested to apply an $\mathcal{H}_\infty$design instead, we need to consider another finite dimensional discrete-time system $\check{G}$. This system can be derived by using the equations given in Bamieh and Pearson (1992*b*). The relation between the lifted system $\tilde{G}$ and the equivalent finite dimensional system $\check{G}$ is given by:

(1) $\mathcal{F}(\tilde{G}, C)$ is internally stable and $\|\mathcal{F}(\tilde{G}, C)\|_\infty < 1$
(2) $\mathcal{F}(\check{G}, C)$ is internally stable and $\|\mathcal{F}(\check{G}, C)\|_\infty < 1$

Now, the $\mathcal{H}_\infty$ design follows standard discrete-time $\mathcal{H}_\infty$ controller design, see e.g. Stoorvogel (1992).

4. LTR DESIGN FOR SAMPLED-DATA SYSTEMS

Based on section 3, the LTR design problem for sampled-data systems can be solved by using the equivalent discrete-time system. In the following, we shall describe the discrete-time LTR problem and suggest $\mathcal{H}_2$/LTR and $\mathcal{H}_\infty$/LTR design methods for discrete-time systems.

4.1 *The LTR Design Problem for Sampled-Data Systems*

To apply the LTR design methodology on sampled-data systems, we consider the following sampled-data system for state feedback design:

$$G_{SD,SF} = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right)$$

Let the target design be a state feedback controller given by:

$$u_k = F y_k = F x_k$$

with the resulting target closed-loop transfer operator given by:

$$\mathcal{G}_{zw,T} = \mathcal{F}(G_{SD,SF}, \mathcal{H}_\tau F \mathcal{S}_\tau).$$

It is assumed that the target closed loop is internally stable and it satisfies the design specifications.

As in the continuous-time case, the target controller can not be implemented, so we need to recover the target operator by using a dynamic controller $C(z)$. With the sampled-data system given by:

$$G_{SD} = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right)$$

the closed-loop operator with the controller $C(z)$ is then given by:

$$\mathcal{G}_{zw} = \mathcal{F}(G_{SD}, \mathcal{H}_\tau C \mathcal{S}_\tau).$$

Based on these two closed-loop operators, we can define the recovery operator by:

$$\mathcal{E}_{SD,I} = \mathcal{G}_{zw} - \mathcal{G}_{zw,T} \tag{3}$$

From the recovery error operator in (3), we can now define the LTR design problem for sampled-data systems.

Problem 1. Let the target loop operator, the full loop operator and the recovery operator be given by $\mathcal{G}_{zw,T}$, $\mathcal{G}_{zw}$ and $\mathcal{E}_{SD,I}$ respectively. The LTR design problem is then to design a dynamic controller $C(z)$ that internally stabilize the sampled-data system and make a suitable norm of the recovery operator small in some sense.

It is not possible to minimize a suitable norm of the recovery operator directly. Instead, by using lifting of the sampled-data system, the design problem can be transformed into an equivalent discrete-time design problem as described in the previous section. In order to apply the lifting technique to the recovery error operator given by (3), we need to make a joint state space description before the system is lifted. If the system is lifted directly, we will not get the right equivalent discrete-time system to work with. A state space description of the recovery error is given by:

$$\mathcal{G}_{E_I,SD} = \left(\begin{array}{c|cc} A_E & B_{E,1} & B_{E,2} \\ \hline C_{E,1} & 0 & D_{E,12} \\ C_{E,2} & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|ccc} A & 0 & B_1 & B_2 & 0 \\ 0 & A & B_1 & 0 & B_2 \\ \hline -C_1 & C_1 & 0 & -D_{12} & D_{12} \\ I & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \end{array}\right) \tag{4}$$

with the controller given by

$$u_k = C_E(z) y_k = diag(F, C(z)) y_k \tag{5}$$

The recovery error described by (4) is now given in the standard description, which makes it possible to find an equivalent finite dimensional discrete time system by using the lifting technique. Performing this task, the following equivalent discrete time system for the recovery error is obtained:

$$\bar{G}_{E_I} = \left(\begin{array}{c|cc} \bar{A}_E & \bar{B}_{E,1} & \bar{B}_{E,2} \\ \hline \bar{C}_{E,1} & 0 & \bar{D}_{E,12} \\ \bar{C}_{E,2} & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|ccc} \bar{A} & 0 & \bar{B}_1 & \bar{B}_2 & 0 \\ 0 & \bar{A} & \bar{B}_1 & 0 & \bar{B}_2 \\ \hline -\bar{C}_1 & \bar{C}_1 & 0 & -\bar{D}_{12} & \bar{D}_{12} \\ I & 0 & 0 & 0 & 0 \\ 0 & \bar{C}_2 & 0 & 0 & 0 \end{array}\right)$$

It is important to note that the equivalent discrete-time state space description for the recovery error has exactly the same structure as the sampled-data description. This structure allows to rewrite the recovery error as a target loop transfer function multiplied by a recovery matrix as described in section 2. Using the controller $C_E(z)$ given by (5), we can express the recovery error

$$\bar{E}_I(z) = \mathcal{F}(\bar{G}_E(z), C_E(z))$$

in the standard form as:

$$\bar{E}_I(z) = \mathcal{F}(\bar{G}(z), C(z)) - \mathcal{F}(\bar{G}_T(z), F)$$

where

$$\bar{G}(z) = \left(\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ \bar{C}_2 & 0 & 0 \end{array}\right), \bar{G}_T = \left(\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & 0 & \bar{D}_{12} \\ I & 0 & 0 \end{array}\right)$$

The lifting guarantees that the norm (in consideration) of the recovery error is preserved, i.e.

$$\|\mathcal{E}_{SD,I}\| = \|\bar{E}_I\|$$

First, consider a full-order prediction observer based controllers given by:

$$C(z) = -F(zI - \bar{A} - \bar{B}_2 F - K\bar{C}_2)^{-1} K$$

Based on the above description of the recovery error $\bar{E}_I$, and the result from section 2, we get directly

$$\bar{E}_I(z) = T_{zu,T}(z)\bar{M}_I(z) \ . \quad (6)$$

where $T_{zu,T}(z)$ is the closed-loop transfer function from u to z under the target design given by:

$$T_{zu,T}(z) = (\bar{C}_1 + \bar{D}_{12}F)(zI - \bar{A} - \bar{B}_2 F)^{-1}\bar{B}_2 + \bar{D}_{12}$$

and $\bar{M}_I$ is the recovery matrix given by

$$\bar{M}_I(z) = F(zI - \bar{A} - K\bar{C}_2)^{-1}\bar{B}_1 \quad (7)$$

It is important to note that the design of the target gain F is free. It can be derived by e.g. an optimization method.

4.2 *Recovery Conditions*

Based on the recovery error (6), it is possible to give conditions for obtaining exact recovery. As in the continuous-time case, exact recovery is obtained if $\bar{E}_I(z) = 0$. Thus, we have the following result, Niemann *et al.* (1991), Saberi *et al.* (1993)

Lemma 1. Let $T_{zw,T}(z)$ be an admissible closed-loop target transfer function and let $T_{zu,T}(z)$ be left invertible. Exact LTR, i.e. $\bar{E}_I(z) = 0$, can be obtained if and only if $\bar{M}_I(z) = 0$.

Proof of Lemma 1. It follows directly from (6). □

In the rest of this paper we will concentrate on the $\mathcal{H}_2$/LTR and $\mathcal{H}_\infty$/LTR design methods. It is possible to minimize the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$ norm of the recovery error directly or indirectly by minimization of the recovery matrix $\bar{M}_I(z)$, Niemann *et al.* (1991), Niemann *et al.* (1993), Stoustrup and Niemann (1993). This is equivalent to the standard LQG/LTR design, Niemann *et al.* (1991), Niemann *et al.* (1995). Here, we only consider the case of minimizing the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$ norm of the recovery matrix. The minimization of the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the recovery matrix is based on the following norm inequality:

$$\|\bar{E}_I\| = \|T_{zu,T}\bar{M}_I\| \le \|T_{zu,T}\| \times \|\bar{M}_I\|$$

As a direct concequence of the above norm inequality, the norm of the recovery matrix should satisfy:

$$\|\bar{M}_I\| \le \|\bar{E}_I\| / \|T_{zu,T}\|$$

when the norm of the recovery error is specified.

4.3 *$\mathcal{H}_2$/LTR Design*

Let the equivalent discrete-time state space description of the recovery matrix be given by (7). Then, the $\mathcal{H}_2$/LTR design problem is formulated as follows.

Problem 2. Let the recovery matrix, $\bar{M}_I(z)$, for the observer design be given by (7). Find an observer gain K such that $\bar{A} + K\bar{C}_2$ is stable and the $\mathcal{H}_2$ norm of $\bar{M}_I(z)$ is minimized.

To calculate the $\mathcal{H}_2$/LTR observer gain, we consider the recovery matrix with the following state space realization:

$$\bar{M}_I(z) = \left(\begin{array}{c|cc} \bar{A} & \bar{B}_1 & I \\ \hline F & 0 & 0 \\ \bar{C}_2 & 0 & 0 \end{array}\right) \quad (8)$$

This design of observer gain $u_k = K_2 y_k$ can be obtained by using the discrete-time $\mathcal{H}_2$ design method of Trentelman and Stoorvogel (1993). From Trentelman and Stoorvogel (1993) we have the following result.

Lemma 2. Consider the system given by (8). It is assumed that $(\bar{C}_2, \bar{A})$ is detectable. The there exist an observer $u_k = K_2 y_k$ which stabilize the system (8) and minimize the $\mathcal{H}_2$ norm of the closed loop transfer function $\bar{M}_I$ if and only if there exist a symmetric matrix positive semidefinte Q_2 such that

$$Q_2 = \bar{A}Q_2\bar{A}^T + \bar{B}_1\bar{B}_1^T - \bar{A}Q_2\bar{C}_2^T(\bar{C}_2Q_2\bar{C}_2^T)^{-1}\bar{C}_2Q_2\bar{A}^T$$

Moreover, the observer gain K_2 is given by:

$$K_2 = -\bar{A}Q_2\bar{C}_2^T(\bar{C}_2Q_2\bar{C}_2^T)^{-1}.$$

It is important to note that it is in general required that $D_{11} = 0$ for the original system due to the condition on strict causality. However, this is not a condition in connection with the $\mathcal{H}_2$/LTR design method, because the D_{11} term does not appear in the recovery error equation.

Lemma 2 gives necessary and sufficient conditions for the existence of an observer gain K_2 such that internal stability is obtained and the $\mathcal{H}_2$ norm of the recovery matrix is minimized.

4.4 $\mathcal{H}_\infty$/LTR Design

Now, let us use an $\mathcal{H}_\infty$ optimization method instead. In this case, it is assumed that the equivalent discrete-time system (8) preserves the $\mathcal{H}_\infty$ norm. Then, we have the following $\mathcal{H}_\infty$/LTR design problem.

Problem 3. Let $\gamma > 0$ be given. Design, if possible, an observer $u = K_\infty y_k$ which internally stabilize the system (8) and makes the $\mathcal{H}_\infty$ norm of the closed loop transfer function $\bar{M}_I$ smaller than γ.

This design can be performed by using the approach in Stoorvogel (1992), Stoorvogel *et al.* (1994). Thus, we have the following result from Stoorvogel (1992):

Lemma 3. Consider the system given by (8). Assume that $(\bar{A}, \bar{B}_1, \bar{C}_2, 0)$ is left invertible and has no invariant zeros on the unit circle. Then, there exist an observer $u = K_\infty y_k$ which stabilizes the system (8) and makes the $\mathcal{H}_\infty$ norm of the closed loop transfer function from w to z less than γ, if and only if there exist a symmetric matrix $Q \geq 0$ such that:

$$R = \gamma^2 I - FQF^T > 0$$

$$Q = \bar{A}Q\bar{A}^T + \bar{B}_1\bar{B}_1^T - \begin{pmatrix} \bar{C}_2Q\bar{A}^T \\ FQ\bar{A}^T \end{pmatrix}^T G(Q)^\dagger \begin{pmatrix} \bar{C}_2Q\bar{A}^T \\ FQ\bar{A}^T \end{pmatrix}$$

where

$$G(Q) = \begin{pmatrix} \bar{C}_2Q\bar{C}_2^T & \bar{C}_2QF^T \\ FQ\bar{C}_2^T & FQF^T - \gamma^2 I \end{pmatrix}$$

and the eigenvalues of $\bar{A}_{cl}$, where

$$\bar{A}_{cl} = \bar{A} - \begin{pmatrix} \bar{A}Q\bar{C}_2^T & \bar{A}QF^T \end{pmatrix} G^T(Q)^{-1} \begin{pmatrix} \bar{C}_2 \\ -F \end{pmatrix}$$

are inside the unit circle.

Moreover, an observer gain K_∞ is given by:

$$K_\infty = -(\bar{A}Q\bar{C}_2^T + \bar{A}QF^TR^{-1}FQ\bar{C}_2^T)H^{-1}$$

where $H = \bar{C}_2Q\bar{C}_2^T + \bar{C}_2QF^TR^{-1}FQ\bar{C}_2^T$.

Lemma 3 gives necessary and sufficient conditions for the existence of an observer gain K_∞ such that the stability and the $\mathcal{H}_\infty$ norm conditions are satisfied.

5. EXAMPLE

A LTR design example for sampled-data systems is given in this section. The $\mathcal{H}_\infty$/LTR design method is applied for both a traditional discrete-time LTR design and for a sampled-data LTR design. The sampled-data system is given by:

$$G_{SD} = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right) = \left(\begin{array}{cccc|cc} -1000 & 0 & 39.478 & 0 & 0 & 1 \\ 0 & -.62832 & -39.478 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 39.478 & -100 & 1 & 1 \\ \hline 1000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The sample period is 0.1 sec. The target design is given by $F = \begin{bmatrix} 0.00 & -3.5495 & -32.2333 & 0 \end{bmatrix}$. The target loop $T_{zw,T}$ has an $\mathcal{H}_\infty$ norm of 3.61.

When we apply the $\mathcal{H}_\infty$/LTR design method on the discretized system of the continuous-time system, we get the following controller:

$$K_D(z) = \left(\begin{array}{c|c} A_D & B_D \\ \hline C_D & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 0 & 0 & -.041112 & -.00041112 \\ 0 & .75596 & -3.5795 & -4.4684 & -.044684 \\ 0 & .09067 & .81293 & -1.0459 & -.010459 \\ 0 & -.0028312 & .01149 & -.39358 & -.0039362 \\ \hline 0 & 3.5495 & 32.233 & 0 & 0 \end{array}\right)$$

By using this controller to recover the target loop result in that the $\mathcal{H}_\infty$ norm of the sampled-data recovery error is 5.48 and that the final closed loop from w to z has a sampled-data $\mathcal{H}_\infty$ norm of 5.49. If we instead apply the $\mathcal{H}_\infty$/LTR design method on the an equivalent discrete-time system based on lifting, we get the following sampled-data LTR controller given by:

$$K_{SD}(z) = \left(\begin{array}{c|c} A_{SD} & B_{SD} \\ \hline C_{SD} & 0 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 0 & 0 & 0 & -.098694 & -.00098694 \\ 0 & .75596 & -3.5795 & 4.0535 & .040535 \\ 0 & .09067 & .81293 & -2.496 & -.02496 \\ 0 & -.0028312 & .01149 & -.99748 & -.0099753 \\ \hline 0 & 3.5495 & 32.233 & 0 & 0 \end{array}\right)$$

When we apply this sampled-data LTR controller, the $\mathcal{H}_\infty$ norm of the sampled-data recovery error is reduced to 2.18 and the sampled-data $\mathcal{H}_\infty$ norm of the closed loop from w to z is reduced to 4.51 compared with the discrete-time designed LTR controller. In this example it is possible to reduce the $\mathcal{H}_\infty$ norm of the closed loop operator by 18% by using the lifting technique.

6. CONCLUSION

The LTR design concept from continuous-time and discrete-time systems has been extended in a straightforward way to handle the sampled-data case. It turns out that the sampled-data LTR design problem can be transformed into an equivalent finite dimensional discrete-time LTR design problem by using lifting technique. The calculation of the equivalent discrete-time systems from the sampled-data systems depend on the applied optimization method as e.g. $\mathcal{H}_2$ or $\mathcal{H}_\infty$ optimization.

Using the equivalence between a sample-data system and a discrete-time system, the conditions for obtaining exact recovery are given with respect to the equivalent discrete-time system. Due to the fact that sampling normally introduce non-minimum phase zeros, exact recovery is in general impossible.

It can be shown that there does not exist a straightforward duality between the input loop breaking point and the output loop breaking point for $\mathcal{H}_\infty$/LTR design. It turns out that it is not possible to make a standard recovery design for the output loop breaking point by using a standard full order observer based controller as it is possible for the input loop breaking point. Instead, controllers of order $2n$ need to be applied for the output loop breaking point.

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