

Near Optimal Decentralized \mathcal{H}_∞ Control: Bounded vs. Unbounded Controller Orders

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Abstract

It is shown that for a class of decentralized control problems there does not exist a sequence of controllers of bounded order which obtains near optimal control. Neither does there exist an infinite dimensional optimal controller. Using the insight of the line of proof of these results, a heuristic design algorithm is proposed for designing near optimal controllers of increasing orders.

1 Introduction

In a range of industrial environments, implementing a full multivariable controller which combines all measurements and all control signals, is not possible, practical, or desirable. For a distributed plant, installing a full multivariable controller could mean that a complex communication network had to be hardware. Moreover, in terms of reliability, a full multivariable controller could have the effect that a breakdown in a single unit, no matter how minor to the system, could have plant-wide consequences. Examples of application areas, where full multivariable controllers are unacceptable are: distributed power systems (where the controllers for each station should be independent), steel milling (where the controllers for each stand should not interfere), and large scale space systems (where the modules should be autonomous).

To formalize such requirements, known as decentralized control specifications, we consider a state space plant model of the form:

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u \\ z &= C_1x + D_{11}w + D_{12}u \\ y &= C_2x + D_{21}w + D_{22}u \end{aligned} \quad (1)$$

where u and y is partitioned as

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} \quad (2)$$

where each pair (u_i, y_i) are vectors of local actuator and sensor signals, respectively, corresponding to the i 'th subsystem of the plant.

Now, the problem is to design k controllers:

$$u_i = K_i y_i, \quad i = 1, \dots, k \quad (3)$$

such that the resulting transfer function from w to z meets the specifications. In this paper we shall assume that the specifications are posed in terms of an \mathcal{H}_∞ norm constraint of the transfer function from w to z . However, this choice is not crucial, and the argument found below would hold for many other types of performance specifications.

Rewriting (3) using (2), we get

$$u = Ky, \quad K = \begin{pmatrix} K_1 & 0 & \dots & 0 \\ 0 & K_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K_k \end{pmatrix}$$

This decentralized control problem is depicted in Figure 1. Note, that each controller $K_i(s)$ connects the measurement signal y_i and the control signal u_i , only.

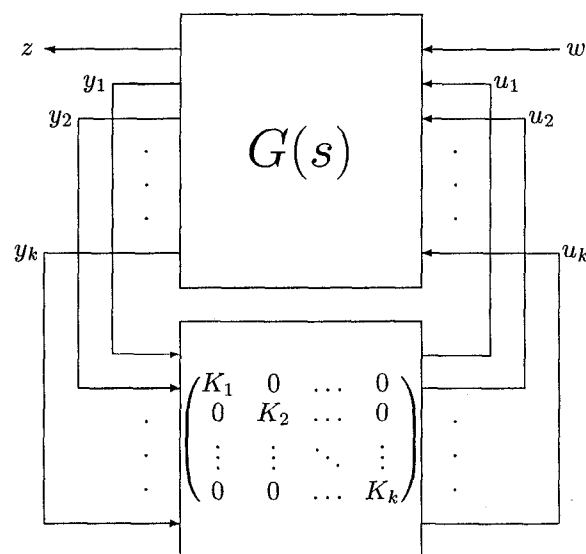


Figure 1: Decentralized control

The theory of decentralized control has been widely studied in the literature. The classical theory which especially addresses the issue of decentralized stabi-

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lization is surveyed in [Dav84]. Two excellent textbooks dealing with decentralized control are [Vid85] and [Ö94].

More recently, issues as \mathcal{H}_∞ decentralized control has been introduced [Paz93], and robust and reliable decentralized control has been studied, see e.g. [VMP92].

Most published results on decentralized control are based on sufficient conditions only. In contrast, [SM95] suggests an optimization based approach. The method in that paper uses a parameterization which enables an infinite dimensional optimization problem to be approximated by a finite dimensional one. In the example studies, controller orders grow rapidly, as the optimization reaches the optimum. The authors of [SM95] blame their method rather than the decentralized control problem itself. It is a common conjecture that decentralized control problems can be solved by fixed order controllers.

In this paper, we shall prove to the contrary that for decentralized control problems all controllers can have dynamic orders that tend to infinity as the optimum is approached.

2 Main Results

The main result of this paper is that near optimal decentralized \mathcal{H}_∞ control problems can require controllers of arbitrarily large orders as the optimum is approached. To state this in more precise terms, we introduce the following two sets of controllers:

$$\mathcal{K}_\gamma(G) = \left\{ K = \begin{pmatrix} K_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & K_k \end{pmatrix} : K \text{ is internally stabilizing, and } \|\mathcal{F}_l(G, K)\|_\infty < \gamma \right\}$$

$$\mathcal{K}_\gamma^N(G) = \{K = \text{diag}\{K_i\} \in \mathcal{K}_\gamma(G) : K_i \text{ is of dynamical order } \leq N\}$$

Theorem 1 *There exists a nonempty class of systems \mathcal{G}^∞ such that for each $G \in \mathcal{G}^\infty$ the inequality*

$$\inf\{\gamma : \mathcal{K}_\gamma(G) \neq \emptyset\} < \inf\{\gamma : \mathcal{K}_\gamma^N(G) \neq \emptyset\}$$

holds for any N .

The interpretation of Theorem 1 is that there exists systems for which no sequence of *fixed* order decentralized controllers approach the optimal value.

Remark 1 It is tempting, yet incorrect to conclude from Theorem 1 that this implies the existence of an optimal infinite dimensional decentralized controller.

We shall prove that in general there does not exist optimal decentralized controllers that give closed loop systems that are analytical in the open right half plane, which establishes the nonexistence of such controllers.

Before embarking the proof of Theorem 1, we shall need the following result from functional analysis

Corollary 2 *Let \mathcal{D} denote a closed subset of the complex plane. Consider $f(z) \in \mathcal{H}^p$, f analytic in \mathcal{D} , and assume $f(t) = 0$ on a set of positive measure on the boundary of \mathcal{D} . Then $f \equiv 0$.*

This observation is evident from the following result, which can be found in [Jen99]. (This paper illustrates how use of the word “new” in the title can be misleading.)

Theorem 3 *Let \mathcal{D} denote the unit disc or \mathbb{C}^+ . If $f(z) \in \mathcal{H}^p(\mathcal{D})$, $f \neq 0$ then*

$$\frac{1}{2\pi} \int_{\partial\mathcal{D}} \log |f(e^{i\theta})| d\theta > -\infty$$

This theorem states that only the zero function is identically zero on a (part of a) closed contour in the complex plane.

Finally, we shall use the following technical result.

Lemma 4 *Let*

$$G(s) = \frac{B(s)}{A(s)} = \frac{\beta_0 s^N + \beta_1 s^{N-1} + \dots + \beta_N}{s^N + \alpha_1 s^{N-1} + \dots + \alpha_N}$$

be an irreducible N 'th order proper rational function, and let $\{\omega_1, \dots, \omega_{2N+1}\}$ be a set of distinct real values for which $A(i\omega_i) \neq 0$, $i = 1, \dots, 2N+1$. Define the numbers

$$\gamma_i = \frac{B(i\omega_i)}{A(i\omega_i)} = \frac{\beta_0 (i\omega_i)^N + \beta_1 (i\omega_i)^{N-1} + \dots + \beta_N}{(i\omega_i)^N + \alpha_1 (i\omega_i)^{N-1} + \dots + \alpha_N} \quad (4)$$

Then there exists a neighborhood of $(\gamma_1, \dots, \gamma_{2N+1})$ such that the map $F : \mathbb{C}^{2N+1} \rightarrow \mathbb{C}^{2N+1}$ which maps the $2N+1$ complex numbers $(\gamma_1, \dots, \gamma_{2N+1})$ to the $2N+1$ (possibly complex) parameters $(\alpha_1, \dots, \alpha_N, \beta_0, \dots, \beta_N)$ of a rational function in the form (4), is a continuous bijection.

Proof. Let us first establish uniqueness of F at $(\gamma_1, \dots, \gamma_{2N+1})$. To that end assume that the parameters $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_N, \tilde{\beta}_0, \dots, \tilde{\beta}_N)$ satisfy (4), i.e.

$$\gamma_i = \frac{\tilde{B}(i\omega_i)}{\tilde{A}(i\omega_i)} = \frac{\tilde{\beta}_0 (i\omega_i)^N + \tilde{\beta}_1 (i\omega_i)^{N-1} + \dots + \tilde{\beta}_N}{(i\omega_i)^N + \tilde{\alpha}_1 (i\omega_i)^{N-1} + \dots + \tilde{\alpha}_N} \quad (5)$$

However, from (4) and (5) we infer

$$A(i\omega_i)\tilde{B}(i\omega_i) - \tilde{A}(i\omega_i)B(i\omega_i) = 0, \quad i = 1, \dots, 2N+1$$

The only polynomial of degree less than or equal to $2N$ having $2N+1$ zeros is the zero polynomial, hence:

$$A(s)\tilde{B}(s) - \tilde{A}(s)B(s) \equiv 0 \quad (6)$$

Since $A(s)$ and $B(s)$ were assumed to be coprime, the only solutions of order less than or equal to N to (6) are

$$\tilde{A} = k \cdot A, \quad \tilde{B} = k \cdot B$$

where k is a unit in the ring of polynomials, i.e. a constant. Finally, since the coefficients of highest order in A and \tilde{A} are fixed to 1, we conclude $k = 1$.

From this argument it follows that the map

$$F: \mathcal{C}^{2N+1} \rightarrow \mathcal{C}^{2N+1}, \\ F: (\gamma_1, \dots, \gamma_{2N+1}) \mapsto (\alpha_1, \dots, \alpha_N, \beta_0, \dots, \beta_N)$$

is well defined in any neighborhood of $(\gamma_1, \dots, \gamma_{2N+1})$ where the corresponding transfer function remains irreducible. Such a neighborhood exists due to the continuity of the roots of a polynomial as functions of the coefficients, and due to the fact that the coefficients are computable by solving linear equations that depend continuously on the γ_i 's. This also establishes continuity. Obviously, the inverse map is injective, due to the definition of the γ_i 's. \square

We are now able to prove our main result.

Proof of Theorem 1. To establish nonemptiness of \mathcal{G}^∞ we shall study the decentralized control problem in Fig. 2. The system is a series connection of two 'model matching problems', which can be thought of as a prototype of decentralized production line control. In this interpretation w is an impurity of the product eliminated in part by the controller Q_1 which is then transferred downstream, where the product is further refined by Q_2 before it is fully processed as z . The notation Q_i rather than K_i is introduced because we think of the Q_i 's as Youla parameters ([YJB71]) rather than controllers.

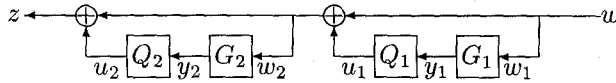


Figure 2: Decentralized control for series connected model matching problems

Specifically, we shall choose:

$$G_1(s) = \frac{s - z_1}{s + z_1}, \quad G_2(s) = \frac{s - z_2}{s + z_2}, \quad z_2 > z_1 > 0$$

Note, that internal stability is equivalent to stability of Q_1 and Q_2 since the G_i 's are stable (though non-minimum phase.)

For this system we shall prove that any sequence of fixed order controllers stays boundedly away from the optimal value of γ which for this example is 0 (see below).

To that end, let N be fixed and assume to the contrary that we have a sequence of controllers $Q^\gamma = \begin{pmatrix} Q_1^\gamma & 0 \\ 0 & Q_2^\gamma \end{pmatrix}$ with Q_i^γ being N 'th order transfer functions which satisfies $\|T_{zw}^\gamma(\cdot)\|_\infty < \gamma$ for all γ , where $T_{zw}^\gamma(\cdot)$ is the closed loop transfer function from w to z :

$$T_{zw}^\gamma(\cdot) = T_2^\gamma(\cdot)T_1^\gamma(\cdot) \\ = [I + Q_2^\gamma(\cdot)G_2(\cdot)][I + Q_1^\gamma(\cdot)G_1(\cdot)]$$

For any $\delta > 0$ we can perturb $G_1(\cdot)$ and $G_2(\cdot)$ by two irreducible N 'th order stable proper rational functions $\tilde{G}_1(\cdot)$ and $\tilde{G}_2(\cdot)$:

$$\tilde{G}_1 = G_1 + \hat{G}_1, \quad \tilde{G}_2 = G_2 + \hat{G}_2 \quad (7)$$

such that \tilde{G}_1 and \tilde{G}_2 are N 'th order stable non-strictly proper rational functions which have zeros in the right half plane and satisfy:

$$\|\tilde{T}_{zw}^\gamma(\cdot)\|_\infty = \|\tilde{T}_2^\gamma(\cdot)\tilde{T}_1^\gamma(\cdot)\|_\infty = \\ \|[I + Q_2^\gamma(\cdot)\tilde{G}_2(\cdot)][I + Q_1^\gamma(\cdot)\tilde{G}_1(\cdot)]\|_\infty < \gamma + \delta$$

Obviously, $\|\tilde{T}_{zw}^\gamma\|_\infty < \gamma + \delta$ implies that for each frequency ω either

$$\|\tilde{T}_1^\gamma(i\omega)\| < \sqrt{\gamma + \delta} \quad \text{or} \quad \|\tilde{T}_2^\gamma(i\omega)\| < \sqrt{\gamma + \delta}$$

Now, choose $4N + 2$ arbitrary, but different frequencies. Then for each γ either $\|\tilde{T}_1^\gamma(i\omega_i)\| < \sqrt{\gamma + \delta}$ or $\|\tilde{T}_2^\gamma(i\omega_i)\| < \sqrt{\gamma + \delta}$ will be satisfied for at least $2N + 1$ of the chosen frequencies. Since there is only finitely many ways to choose $2N + 1$ frequencies among $4N + 2$ frequencies, there exists a subsequence $\{Q_i^{\hat{\gamma}}\}$ of $\{Q^\gamma\}$ for which one of the $Q_i^{\hat{\gamma}}(\cdot)$'s, which can be taken to be $Q_1^{\hat{\gamma}}(\cdot)$ without loss of generality, satisfy

$$\|\tilde{T}_1^{\hat{\gamma}}(i\omega_i)\| < \sqrt{\hat{\gamma} + \delta}$$

for $2N + 1$ fixed frequencies, $\{\omega_1, \dots, \omega_{2N+1}\}$.

Hence, for these $2N + 1$ frequencies

$$\lim_{\hat{\gamma} \rightarrow 0} Q_1^{\hat{\gamma}}(i\omega_i) \in \mathbf{B}(-\tilde{G}_1^{-1}(i\omega_i), \delta) \quad (8)$$

where $\mathbf{B}(c, r)$ denotes the complex ball of radius r centered in c .

Let us consider a transfer function¹ representation of $Q_1^{\hat{\gamma}}$:

$$Q_1^{\hat{\gamma}}(s) = \frac{\beta_0 s^N + \beta_1 s^{N-1} + \dots + \beta_N}{s^N + \alpha_1 s^{N-1} + \dots + \alpha_N}$$

Now, since \tilde{G}_1^{-1} is irreducible, we can apply Lemma 4. Indeed, by selecting δ sufficiently small

¹The controller is allowed to be a *complex* transfer function in this argument. Thereby we prove a slightly stronger result.

$\mathbf{B}(-\tilde{G}_1^{-1}(i\omega_i), \delta)$ will be contained in some neighborhood of $-\tilde{G}_1^{-1}(i\omega_i)$ in which the operator F mentioned in Lemma 4 is continuous.

Finally, by the continuity of the roots of a polynomial, the denominator of $Q_1^{\hat{\gamma}}(s)$ will have roots in the open right half plane for $\hat{\gamma}$ and δ sufficiently small, since the denominator of $-\tilde{G}_1^{-1}(s)$ has. That is a contradiction, since $Q_1^{\hat{\gamma}}(s)$ was assumed to be stable.

Hence, no fixed order sequence of controllers achieve the infimal value of γ .

To establish the nonexistence of an infinite dimensional optimal decentralized controller as mentioned in Remark 1 we assume to the contrary the existence of an optimal analytical function $Q^* = \begin{pmatrix} Q_1^* & 0 \\ 0 & Q_2^* \end{pmatrix}$, i.e. a function, which is analytical in the open right half plane, and which makes the closed loop transfer function from w to z equal to 0:

$$T_{zw}(\cdot) = [I + Q_2^*(\cdot)G_2(\cdot)][I + Q_1^*(\cdot)G_1(\cdot)] = 0$$

From continuity of the transfer function $I + Q_2^*(\cdot)G_2(\cdot)$, the transfer function $I + Q_1^*(\cdot)G_1(\cdot)$ has to be identically equal to zero in a neighborhood of $s = z_2$. Applying Corollary 2 for a set \mathcal{D} contained in the (nonempty) intersection between this neighborhood and the open right half plane, it follows from Corollary 2 that $I + Q_1^*(\cdot)G_1(\cdot) \equiv 0$. This implies that $Q_1^*(\cdot) = -G_1(\cdot)^{-1}$ which is a contradiction, since $G_1(\cdot)^{-1}$ is not analytic in the right half plane.

On the other hand, there does exist a sequence of controllers of increasing orders that makes T_{zw} tend to zero in \mathcal{H}_∞ norm topology.

Such a sequence is relatively easy to design. The main idea is to design $I + Q_1^{\gamma}(\cdot)G_1(\cdot)$ to have low pass characteristics and $I + Q_2^{\gamma}(\cdot)G_2(\cdot)$ to have high pass characteristics. Then the overall \mathcal{H}_∞ norm is determined only at frequencies between z_1 and z_2 by the the roll off rates of these two transfer functions.

To achieve this, we introduce $P_{BW}^N(s, \omega_{BW})$ to denote the N 'th order Butterworth polynomial with characteristic frequency ω_{BW} . In terms of these polynomials, we can give explicit expressions for a possible controller sequence:

$$Q_1^N(s) = \frac{(s + z_1)(P_{BW}^N(z_1, z_1) - P_{BW}^N(s, z_1))}{(s - z_1)P_{BW}^N(s, z_1)}$$

$$Q_2^N(s) = \frac{(s + z_2) \left(s^N P_{BW}^N(z_2, \frac{1}{z_2}) - z_2^N P_{BW}^N(s, \frac{1}{z_2}) \right)}{(s - z_2)z_2^N P_{BW}^N(s, \frac{1}{z_2})}$$

Note, that Q_1^N and Q_2^N are stable, considered as rational functions, since the two unstable denominator factors are cancelled by the numerators.

Now, it can be verified using a symbolic manipulation package that the maximal value of $|T_{zw}(i\omega)|$ appears for $\omega = \sqrt{z_1 z_2}$, and that this maximal value

tends to zero as N tends to infinity. The resulting design for $N = 5$ can be seen in Fig. 3. The dotted lines are the magnitudes of the two transfer functions $I + Q_1^N(i\omega)G_1(i\omega)$ and $I + Q_2^N(i\omega)G_2(i\omega)$, and the solid line is magnitude of their product, $T_{zw}(i\omega)$. The vertical lines indicate the two non-minimum phase zeros z_1 and z_2 .

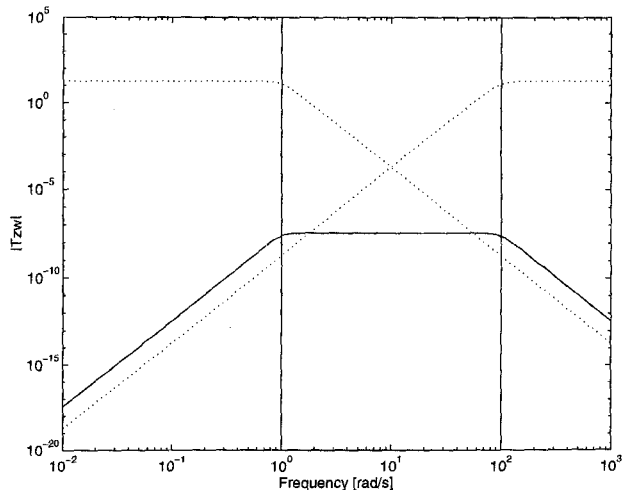


Figure 3: 5'th order Butterworth design

□

Remark 2 It is not easy to determine the exact contents of the class \mathcal{G}^∞ . Theorem 1 shows that the class is nonempty. Indeed, from the line of proof, it could be anticipated that a majority of non-minimum phase systems would be in the class. On the other hand, if G_1 or G_2 would be minimum phase in the configuration in Fig. 2, there would exist a fixed dimensional sequence of controllers, so the class does not comprise all decentralized control problems.

3 Near Optimal Design of Decentralized Controllers

In the literature, few algorithms can be found for near optimal decentralized control for arbitrary plants. The reason for this is likely to be found in the result above, which eliminates the possibility of Riccati-type necessary and sufficient conditions for near optimal problems.

One result that facilitates design for near optimal control can be found in [SM95]. This method, however, is based on a complex optimization procedure, and might be numerically infeasible for large scale systems.

Based on the line of proof above, however, a heuristic algorithm can be devised, which works for systems, where individual subsystems have only a limited number of non-minimum phase zeros, and where subsystems are only lightly coupled.

First, without loss of generality we will rewrite (1) in k subsystems of the form:

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + B_{1,i} w + \sum_{j \neq i} B_{2,ij} z_j + B_{3,i} u_i \\ z_i = C_{1,i} x_i + D_{11,i} w + \sum_{j \neq i} D_{12,ij} z_j + D_{13,i} u_i \\ y_i = C_{2,i} x_i + D_{21,i} w + \sum_{j \neq i} D_{22,ij} z_j + D_{23,i} u_i \end{cases}$$

The intuition of this form is that each controller “looks into” a subsystem with two kinds of disturbances: the original exogenous signals w and the artificial set of disturbances:

$$\tilde{w}_i = \begin{pmatrix} z_1 \\ \vdots \\ z_{i-1} \\ z_{i+1} \\ \vdots \\ z_k \end{pmatrix}$$

which determine how the subsystems influence one another.

Expanding the idea of the proof of Theorem 1 we obtain the following algorithm:

Algorithm 1

1. Determine the non-minimum phase zeros for each subsystem Σ_i with respect to each component of the input \tilde{w}_i
2. Sort the zeros of all subsystems by magnitude, and assign either a low pass (LP), a band pass (BP), or a high pass (HP) attribute to each input of each subsystem based on this. The assignment should consider the signals w_i also, i.e. by computing the zeros related to these inputs the local interpolation constraints should be taken into account
3. Design weightings for each subsystem such that the outputs for a subsystem with a HP attribute is input only to LP loops and vice versa
4. Compute a controller for each subsystem with these weightings using \mathcal{H}_∞ optimization
5. Iterate from Step 3 by increasing the roll-off rate of the weightings until the specifications are met

At each iteration of the algorithm, the controller order will increase due to increased order of the weightings.

Remark 3 Obviously, if any of the involved systems are minimum phase w.r.t. all input/output pairs, these subsystems can be made *uniformly* small (at the possible cost of robustness).

It is interesting to observe that e.g. for systems with three subsystems with each just one non-minimum phase zero it might be the case that (LP,LP,HP) and (LP,HP,HP) are both admissible sets of attributes, leading to the same optimum. In fact, for a series connection, if it is possible to design two loops to have disjoint LP and HP characteristics, the remaining loops are completely free. Needless to say, the corresponding controllers will then be rather different. This type of non-uniqueness does not exist always in a full multivariable near optimal design.

4 Conclusions

We have shown that for a class of systems, the controller order of a decentralized \mathcal{H}_∞ controller will not remain bounded as the \mathcal{H}_∞ optimization tends to the optimum.

In such cases, no sequence of controllers will converge, not even to an infinite dimensional controller. The ‘optimal’ controller will be non-causal.

We believe that the proof of the main result in this paper provides insight which can guide the design of decentralized controllers. In particular, a heuristic design algorithm has been devised, which works for systems which are not too strongly coupled, or have too many non-minimum phase zeros.

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