SEMI-GLOBAL $\mathcal{H}_\infty$ STATE FEEDBACK CONTROL

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Abstract

Semi-global set-stabilizing $\mathcal{H}_\infty$ control is local $\mathcal{H}_\infty$ control within some given compact set Ω such that all state trajectories are bounded inside Ω, and are approaching an open loop invariant set $\mathcal{S} \subset \Omega$ as $t \to \infty$. Sufficient conditions for the existence of a continuous state feedback law are given, based on a new theorem.

1 Introduction

The standard formulation of local state feedback $\mathcal{H}_\infty$ control is mainly based on the theory of dissipative systems first introduced by Willems [Wil72]. In this paper we will approach the problem by the theory of differential games as outlined in the papers by Isidori [Isi92], and Isidori and Astolfi [IA92b, IA92a], but we allow for non-zero initial conditions following van der Schaft [vdS92b]. Recently, the local nonlinear state feedback $\mathcal{H}_\infty$ control problem has been solved for general nonlinear plants by Isidori and Kang [IK95], and Ball, Helton, and Walker [BHW93]. The standard nonlinear $\mathcal{H}_\infty$ control theory is briefly summarized in Section 3.1.

From an applicational point of view the theory of local $\mathcal{H}_\infty$ control has a severe drawback: It does not give a bound on the state trajectories, but merely states that it is valid for bounded trajectories. In fact, a linear controller based on the linearization in an equilibrium point might even do better in practice than a local nonlinear $\mathcal{H}_\infty$ controller. Moreover, it can be argued that the real motivation for nonlinear control theory is applications where the plant is operating in a significant range of operating points. Otherwise, linear control theory will work in most cases.

On the other hand, to compute a global nonlinear $\mathcal{H}_\infty$ control is not realistic in most practical cases since it basically requires finding an analytical expression for a global solution to a Hamilton-Jacobi equation or inequality.

This is the main motivation for the present paper which presents a method to design $\mathcal{H}_\infty$ controllers constraining state trajectories to a region of the state space rather than operating with local results without knowledge of boundedness of the state. The regions are specified in terms of invariant sets, and the results are generalizations of local $\mathcal{H}_\infty$ results. Moreover, the computational methods that apply to local $\mathcal{H}_\infty$ control extend directly to the obtained semi-global $\mathcal{H}_\infty$ results. This constitutes a much more practical theory for nonlinear control systems where also oscillating and other non-stationary modes of operation can be dealt with.

It is described in Section 3.2 how semi-global stability has been obtained for autonomous systems. The main idea of this paper is based on the proof of La Salle's invariance principle [SL61], here restated in Theorem 3.2.

The new contribution to the theory of semi-global stability and set stability by $\mathcal{H}_\infty$ control is found in Section 4. In order to prove the boundedness of state trajectories we have to restrict to a certain class of disturbances denoted $\mathcal{W}^\epsilon$. Given some solution V to the standard $\mathcal{H}_\infty$ Hamilton-Jacobi inequality, a new lemma shows how to compute the region of boundedness Ω, and the region of allowed initial conditions Ω^ϵ. A new theorem, based on La Salle's invariance principle, is the cornerstone of semi-global stability and set stability by $\mathcal{H}_\infty$ control provided that a certain detectability property is satisfied.

2 Problem formulation

Let $\mathbb{R}^+$ denote the real positive closed time-axis $[0, \infty[$. We consider the smooth, continuous time system

$$\dot{x} = X(x, u, w) \quad , \quad z = Z(x, u) \tag{1}$$

where $x(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is called the state, $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^m$ the input, $w(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^l$ the exogenous input, also called disturbance, and $z(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$ the performance, or to-be-controlled signal.

The open loop system

$$\dot{x} = X^{\mathrm{open}}(x) \equiv X(x, 0, w) \tag{2}$$

with constant disturbance $w(\cdot) = 0$ is autonomous, and its dynamic is therefore naturally assumed to have at least one connected, non-empty invariant set such as a closed periodic orbit or an equilibrium point.

The static state feedback used here is some vector valued function $a : \mathbb{R}^n \mapsto \mathbb{R}^m$

$$u = a(x) \quad , \tag{3}$$

thus the closed loop system is given by the equations

$$\begin{aligned} \dot{x} &= X^a(x, w) \equiv X(x, a(x), w) \\ z &= Z^a(x) \equiv Z(x, a(x)) \quad . \end{aligned} \tag{4}$$

Whenever convenient, we use the notation $x(\cdot)$ for the unique signal $x(\cdot, t_0, x_0, u(\cdot), w(\cdot))$ generated by the inputs $u(\cdot), w(\cdot)$, where the initial condition at time t_0 is x_0. It is assumed that all signals are L_2^{loc}, and that the state exist uniquely for all inputs, and is a C^1 signal except on a set of measure zero.

Define the L_2 norm for any locally square integrable signal $y(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$ for all $T \in \mathbb{R}^+$ by

$$\|y\|_T^2 \equiv \int_0^T |y(t)|^2 \, dt \quad , \tag{5}$$

where $|\cdot|$ is the usual Euclidean vector norm. By definition, the open or closed loop system (2) or (4) has local L_2 gain less than or equal to $\gamma \geq 0$ if there exists a neighbourhood $\Omega \subset \mathbb{R}^n$ around the origin, and a nonnegative and bounded function $V_{\mathrm{a}} : \mathbb{R}^n \mapsto [0, \infty[$, called available storage, depending only on the initial condition x_0, such that

$$\|z\|_T^2 \leq \gamma^2 \|w\|_T^2 + V_{\mathrm{a}}(x_0) \tag{6}$$

for all $T \in \mathbb{R}^+$, all initial conditions $x_0 \in \Omega$, and all $w(\cdot), z(\cdot) \in L_2^{\mathrm{loc}}$ such that the state trajectories never leave Ω [vdS92b, IA92a].

To allow for oscillatory or other non-stationary modes of operation we adopt the notion of set stability introduced in [Lin92], more precisely, we are interested in asymptotically stabilization of some open loop invariant set $\mathcal{S}$ such that the motions on $\mathcal{S}$ are unaltered by feedback.

2.1 Problem Formulation *Given a plant (1) whose open loop dynamics (2) subject to $w(\cdot) = 0$ has a nonempty invariant set $\mathcal{M}$ (e.g. a collection of closed orbits and equilibria), pick a to be stabilized union $\mathcal{S}$ of some components of $\mathcal{M}$, and a $\gamma > 0$. Find, if possible, a nonempty compact set Ω containing $\mathcal{S}$, and a state feedback law (3) such that the closed loop L_2 gain (6) is less than or equal to γ, and such that the closed loop system (4) subject to $w(\cdot) = 0$ asymptotically stabilizes the open loop invariant set $\mathcal{S}$.*

Find also a class of disturbances $\mathcal{W}^\epsilon$ such that the state trajectories never leave Ω if started inside some $\Omega^\epsilon \subset \Omega$, and such that all trajectories generated by $w(\cdot) \in \mathcal{W}^\epsilon$ are approaching the closed loop positive invariant set $\mathcal{S}$.

In other words: we want to solve a local $\mathcal{H}_\infty$ control problem in such a way that all trajectories are bounded inside some compact Ω, and that Ω is a basin of attraction for the to-be-stabilized, hence closed loop positive invariant set $\mathcal{S}$.

3 Background

3.1 Local $\mathcal{H}_\infty$ state feedback

The aim of standard local nonlinear $\mathcal{H}_\infty$ control is to design a controller (3), and to find a sufficient small $\gamma \geq 0$ such that the L_2 gain (6) is satisfied locally on a neighbourhood $\Omega \subset \mathbb{R}^n$ around the origin. In this subsection the equilibrium condition $X(0,0,0) = 0$ is assumed to hold.

It is known [vdS92b, vdS92a] that the local L_2 gain condition is implied by (equivalence is given subject a reachability condition [Wil72]) the existence of a non-negative, bounded storage function $V : \Omega \mapsto [0, \infty[$ satisfying the dissipation inequality

$$\begin{aligned} V(x_T) - V(x_0) &\leq \int_0^T (\gamma^2 |w(t)|^2 - |z(t)|^2) \, dt \\ &= \gamma^2 \|w\|_T^2 - \|z\|_T^2 \quad , \\ V(0) &= 0 \quad , \end{aligned} \tag{7}$$

where $x_T = x(T)$. Whenever convenient we denote in the following the value of V along a given path $x(\cdot)$ by the abuse of notation $V(\cdot) = V(x(\cdot, t_0, x_0, u(\cdot), w(\cdot)))$.

In case that V is continuously differentiable almost everywhere, it satisfies the closed loop differential inequality

$$\begin{aligned} \mathsf{H}_\gamma(u, w) &\equiv \tfrac{d}{dt} V - (\gamma^2 |w|^2 - |z|^2) \\ &= \tfrac{\partial V}{\partial x} X(x, u, w) \\ &\quad - \gamma^2 |w|^2 + |Z(x, u)|^2 \leq 0 \end{aligned} \tag{8}$$

for all $t \in \mathbb{R}^+$, where the Hamiltonian function H_γ is defined by equation (8). Assuming that $Z(x, u)$ is such that $\frac{\partial Z}{\partial u}(0,0)$ has rank m, it is known [IK95] that H_γ has a unique saddle point $(w_{\max}, u_{\min})$ for all x and all $\frac{\partial V}{\partial x}$ near zero, and the extremal functions $u_{\min}(x, \frac{\partial V}{\partial x})$, $w_{\max}(x, \frac{\partial V}{\partial x})$ are characterized by the equations

$$\begin{aligned} \frac{\partial \mathsf{H}_\gamma}{\partial u}(u_{\min}, w_{\max}) &= 0 \\ \frac{\partial \mathsf{H}_\gamma}{\partial w}(u_{\min}, w_{\max}) &= 0 \end{aligned} \tag{9}$$

$$\begin{aligned} u_{\min}(0,0) &= 0 \\ w_{\max}(0,0) &= 0 \quad . \end{aligned} \tag{10}$$

Clearly, $a(x) = u_{\min}(x) \equiv u_{\min}(x, \frac{\partial V}{\partial x}(x))$ is the best possible state feedback law, and $w_{\max}(x) \equiv w_{\max}(x, \frac{\partial V}{\partial x}(x))$ is the worst possible disturbance. Note that $u_{\min}$ and $w_{\max}$ vanish at the origin, hence the autonomous closed loop systems

$$\begin{aligned} \dot{x} &= X^{\min}(x) \equiv X(x, u_{\min}(x), 0) \quad \text{and} \\ \dot{x} &= X^{\substack{\min \\ \max}}(x) \equiv X(x, u_{\min}(x), w_{\max}(x)) \end{aligned} \tag{11}$$

do preserve the equilibrium point 0.

Thus, we seek a sufficient small $\gamma \geq 0$, and a C^1 storage function V defined on a sufficiently large neighbourhood Ω around the origin satisfying the Hamilton-Jacobi inequality [IK95]

$$\begin{aligned} & \mathsf{H}_{\gamma}^{\substack{\min \\ \max}}(x, \tfrac{\partial V}{\partial x}) \\ & \quad = \tfrac{\partial V}{\partial x} X(x, u_{\min}(x, \tfrac{\partial V}{\partial x}), w_{\max}(x, \tfrac{\partial V}{\partial x})) \\ & \qquad - \gamma^2 \left| w_{\max}(x, \tfrac{\partial V}{\partial x}) \right|^2 + \left| Z(x, u_{\min}(x, \tfrac{\partial V}{\partial x}) \right|^2 \\ & \quad \leq 0 \quad \text{for all } x \in \Omega \quad . \end{aligned} \tag{12}$$

In case that the locally linearized problem is solvable, it can easily be seen that any $\gamma > \gamma_*$ can be used, where γ_* is some sub-optimal gain of the linearized $\mathcal{H}_\infty$ control problem. See van der Schaft [vdS91, vdS92a] for further information.

The existence of a C^1 storage function satisfying (12) locally guarantees that the closed loop system is dissipative in the sense of (7), and the input-output map of the closed loop system has L_2 gain less than or equal to γ as defined in equation (6) *if and only if every closed loop state trajectory is bounded inside* Ω. Unfortunately, local theory does not give any a priori estimates on the boundedness of the state.

3.2 Set stability

The basic idea of this paper is that the storage function V satisfying (12) shall serve as a Lyapunov function to determine the stability properties of the closed loop trajectories $x(\cdot)$ not only locally, but semi-globally.

For this purpose it is beneficial to recall boundedness and invariance properties of smooth autonomous systems of the form

$$\dot{x} = X(x) \quad . \tag{13}$$

We assume that the integral curves of (13) are uniquely given on some suitable set, and we denote them $x(\cdot) = x(\cdot, t_0, x_0)$.

3.1 Definition *A set $\mathcal{M} \subset \mathbb{R}^m$ is called* ***invariant*** *if all trajectories starting in $\mathcal{M}$ are defined in the future and in the past, and evolve entirely inside $\mathcal{M}$.*

The set is called ***positive invariant*** *if all trajectories starting in $\mathcal{M}$ are defined in the future and never leave $\mathcal{M}$ as time increases.*

Note that invariance is a stronger property of a set than positive invariance.

It is our purpose to use a formal solution to the Hamilton-Jacobi inequality as a Lyapunov function in order to establish semi-global stability properties of the $\mathcal{H}_\infty$ state feedback problem. Our theorem in the next section will be based on a result published in the early sixties by La Salle and Lefschetz [SL61].

3.2 Theorem (La Salle and Lefschetz)
Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^1 function and let Ω denote a connected component of the pre-image $V^{-1}(]-\infty, c])$, $c \in \mathbb{R}$. Assume that Ω is bounded, and that

$$\tfrac{\partial}{\partial t} V \leq 0 \tag{14}$$

within Ω along any trajectory of the autonomous system (13). Let $\mathcal{R} \subset \Omega$ be the largest set where $\frac{\partial}{\partial t} V = 0$, and let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{R}$.

Then Ω is positive invariant and every solution in Ω tends to $\mathcal{M}$ as $t \to \infty$.

In other words: Ω is a basin of attraction for the stable invariant set $\mathcal{M}$. This is in fact a semi-global stability property of the type we want to establish for the $\mathcal{H}_\infty$ state feedback problem. Note that the original proof of Theorem 3.2 shows that any such C^1 function V satisfying $\frac{\partial}{\partial t} V \leq 0$ is not assumed to be positive definite. Every component of $\mathcal{M}$ is merely a local minimum of the function $V(x)$.

4 Set stability in $\mathcal{H}_\infty$ control

This section contains the new contribution to the theory of regional (semi-global) stability and set stability by $\mathcal{H}_\infty$ control. We want to modify Theorem 3.2 such that the property of set stability can be used in $\mathcal{H}_\infty$ control. We have to use condition (12) instead of (14), thereby ensuring the L_2 gain condition (6) to hold.

In order to prove the boundedness of state trajectories we have to restrict ourselves to the class of disturbances

$$\mathcal{W}^\epsilon \equiv \left\{ w(\cdot) \in L_2(\mathbb{R}^+) \;\middle|\; \|w\|_2^2 \leq \epsilon \right\} \quad . \tag{15}$$

Given some solution V to the standard $\mathcal{H}_\infty$ Hamilton-Jacobi inequality, the following new lemma will help us to construct some appropriate region of boundedness, denoted Ω, and the region of allowed initial conditions Ω^ϵ (see figure 1).

Given a formal C^1 solution V of the Hamilton-Jacobi inequality (12), pick some $c \in \mathbb{R}$ such that some connected component of the pre-image $V^{-1}(]-\infty, c])$, denoted Ω, is bounded. Since V is continuous it follows that Ω is closed, hence compact. Moreover (12) implies that V satisfies the dissipation inequality (7), therefore any trajectory $x(\cdot)$ with initial condition $x_0 \in \Omega$ subject to $w(\cdot) = 0$ fulfills

$$V(x_T) \leq V(x_0) - \|z\|_T^2 \leq c$$

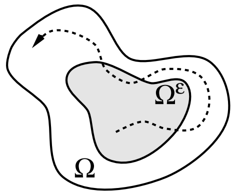

Figure 1: Boundedness of trajectories

for all $T \in \mathbb{R}^+$ (Note that then $\left\|z\right\|_T^2 \leq V(0) \leq c$ is always true). Therefore $x(T) \in \Omega$ for all $T \in \mathbb{R}^+$, and the trajectory can not leave Ω.

Now consider components of the sets $V^{-1}(]-\infty, c-\epsilon])$ with $\epsilon > 0$ which are subsets of Ω. These are clearly closed subsets of Ω, hence compact. Let $x(\cdot)$ be any closed loop trajectory with initial condition $x_0 \in \Omega^\epsilon \subset V^{-1}(]-\infty, c-\epsilon])$, and assume $w(\cdot) \in \mathcal{W}^\epsilon$. Then we have

$$V(x_T) \leq V(x_0) - \left\|z\right\|_T^2 + \gamma^2 \left\|w\right\|_T^2 \leq c - \epsilon + \epsilon$$

for all $T \in \mathbb{R}^+$ (Note that in this case $\left\|z\right\|_T^2 \leq V(0) + \epsilon \leq c$ is always true). We conclude that $x(\cdot)$ is bounded inside Ω. Formally we can restate our observations in the following lemma:

4.1 Lemma *Given a formal C^1 solution V of the Hamilton-Jacobi inequality (12), pick some $c \in \mathbb{R}$ such that some component of $V^{-1}(]-\infty, c])$, denoted Ω, is connected and bounded.*

Then Ω is compact and closed loop positive invariant by use of the state feedback law $a(x) = u_{min}(x)$ subject to the condition $w(\cdot) = 0$.

Pick some $\epsilon > 0$, then the appropriate subset $\Omega^\epsilon \subset \Omega$ of $V^{-1}(]-\infty, c-\epsilon])$ is such that any closed loop trajectory $x(\cdot)$ with initial condition $x_0 \in \Omega^\epsilon$ is bounded inside Ω if driven by the state feedback law $a(x) = u_{min}(x)$, and by any disturbance $w(\cdot) \in \mathcal{W}^\epsilon$.

Note that the formal solution V may be such that the pre-image $V^{-1}(]-\infty, c])$ never has a bounded component, in which case the approach proposed here is not applicable. Moreover, picking $\epsilon \geq 0$ too large may result in $\Omega^\epsilon = \emptyset$.

Having taken care of the boundedness of state trajectories, we proceed the discussion leading to the new theorem, which will be the cornerstone of semi-global set stability by $\mathcal{H}_\infty$ control.

Assume that the autonomous open loop system (2) subject to $w(\cdot) = 0$ has an invariant set $\mathcal{M} \subset \Omega$ consisting of a collection of disjoint components (for example periodic orbits and equilibrium points). If we wish to stabilize the motions on an invariant set $\mathcal{S} \subset \mathcal{M}$ consisting of some components of $\mathcal{M}$ without change of the motions on $\mathcal{S}$ (see figure 2), we have to use a feedback law $a(x)$ such that

$$a(x)|_{x \in \mathcal{S}} = 0 \quad .$$

In case that we want to destroy the open loop motions on $\mathcal{M}/\mathcal{S}$, we must have in addition that

$$a(x)|_{x \in \mathcal{M}/\mathcal{S}} \neq 0 \quad .$$

Moreover, in order to be able to fulfill the L_2 gain (6) for all desired motion of the open loop system (2), the performance measure $Z(x, u)$ must satisfy

$$Z(x, 0)|_{x \in \mathcal{S}} = 0 \quad .$$

Observability of the state trajectory on $\mathcal{S}$, that is $Z(x, 0)|_{x \in \Omega/\mathcal{S}} \neq 0$, may be too severe an assumption. Instead we will impose a weaker detectability assumption on the system:

4.2 Definition *Given some invariant set $\mathcal{S}$ of the open loop system (2) subject to $w(\cdot) = 0$, the control system (1) is called **$\mathcal{S}$-detectable** if all bounded trajectories $x(\cdot) = x(\cdot, t_0, x_0, u(\cdot), 0)$ (subject to $w(\cdot) = 0$) generating the zero-output $z(\cdot) = 0$ are approaching $\mathcal{S}$ as $t \to \infty$.*

*In case that $\mathcal{S}$ is the origin, we say the control system is **zero-detectable**.*

Assuming furthermore that $\frac{\partial Z}{\partial u}(x, 0)$ has rank m for all $x \in \Omega$, a similar argumentation as in the paper [IK95] shows that H_γ defined in (8) has a unique saddle point $(w_{\max}, u_{\min})$ for all x in Ω and all $\frac{\partial V}{\partial x}$ near zero, and the extremal functions $u_{\min}(x, \frac{\partial V}{\partial x})$, $w_{\max}(x, \frac{\partial V}{\partial x})$ are characterized by the equations (9) and

$$\begin{aligned} u_{\min}(x, 0)|_{x \in \mathcal{S}} &= 0 \\ w_{\max}(x, 0)|_{x \in \mathcal{S}} &= 0 \quad . \end{aligned} \tag{16}$$

Hence following the principal idea of the paper [IK95] as outlined in Section 3.1, we conclude that any C^1 function V satisfying the Hamilton-Jacobi inequality (12) will also satisfy the dissipation inequality (7), and therefore the L_2 gain (6) in case that the state is bounded inside Ω. We take advantage of Lemma 4.1 to state the following theorem, and to follow the main idea of Theorem 3.2 to prove it.

4.3 Theorem *Assume that some C^1 solution $V : \Omega \mapsto \mathbb{R}$ of the Hamilton-Jacobi inequality (12) is defined on a bounded and connected component Ω of $V^{-1}(]-\infty, c])$, $c \in \mathbb{R}$. Assume furthermore that $\frac{\partial Z}{\partial u}(x, 0)$ has rank m for all $x \in \Omega$.*

Then all closed loop trajectories $x(\cdot)$ subject to $a(x) = u_{min}(x)$ with initial condition $x_0 \in \Omega^\epsilon$ do not leave Ω if driven by some $w(\cdot) \in \mathcal{W}^\epsilon$, and consequently the system has L_2 gain less than or equal to γ.

Moreover, all such $x(\cdot)$ generated by $w(\cdot) \in \mathcal{W}^\epsilon$ which are identically zero for all times $t > t^*$, $t^* \in \mathbb{R}$, approach the biggest closed loop invariant set $\mathcal{A}$ contained in the null set

$$\mathcal{N} \equiv \left\{ x \in \Omega \;\middle|\; \mathsf{H}_\gamma^{\substack{min \\ max}}(x, \tfrac{\partial V}{\partial x}) = 0 \right\} \quad .$$

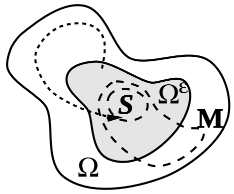

Figure 2: Set stability

Assume furthermore that the control system (1) is $\mathcal{S}$-detectable, where $\mathcal{S}$ is a collection of components of the maximal open loop autonomous invariant set $\mathcal{M} \subset \Omega$, then $x(\cdot)$ approaches $\mathcal{S}$ as $t \to \infty$.

Proof. By Lemma 4.1 all state trajectories $x(\cdot)$ are bounded inside Ω. Therefore, as outlined in the discussion before Theorem 4.3, the dissipation inequality (7) and the L_2 gain (6) are satisfied for all trajectories.

We show now that all such $x(\cdot)$ generated by $w(\cdot) \in \mathcal{W}^\epsilon$ which are identically zero for all times $t > t^*$, $t^* \in \mathbb{R}$, approach the biggest closed loop invariant set $\mathcal{A}$ contained in the null set $\mathcal{N}$. By boundedness of state trajectories and time invariance of the system, we can assume without loss of generality that $w(\cdot) = 0$ for all $t \in \mathbb{R}^+$. Then the saddle point property defined by (9) and (16) implies that the C^1 solution V serves as a Lyapunov function for the closed loop dynamics. More explicitly, we have

$$\mathsf{H}_\gamma(u_{\min}, w) \le \mathsf{H}_\gamma(u_{\min}, w_{\min}) = \mathsf{H}_\gamma^{\stackrel{\min}{\max}} \le 0 \tag{17}$$

for all $w(\cdot) \in \mathcal{W}^\epsilon$. Choosing $w(\cdot) = 0$ then gives with (8)

$$\tfrac{d}{dt}V - \gamma^2 \left|0\right|^2 + \left|z\right|^2 \le 0 \tag{18}$$

for all such trajectories. Hence we have $\frac{d}{dt}V < 0$ for all motions evolving on $\Omega/\mathcal{N}$. Trajectories on $\mathcal{N}$ are satisfying $\frac{d}{dt}V = 0$ if and only if $\left|z\right|^2 = \left|Z(x, u_{\min}(x)\right|^2 = 0$, and $\frac{d}{dt}V < 0$ else.

Now, observe that $V(x)$ by assumption is continuous and defined on the bounded set Ω, hence $V(x)$ is bounded from below. Given some particular state trajectory $x(\cdot)$, the storage function $V(\cdot)$ is decreasing and bounded from below, hence approaches some minimal value, say $c_\Gamma \in \mathbb{R}$, as $t \to \infty$. By continuity we conclude that $V(x) = c_\Gamma$ on the positive limit set Γ^+, and consequently $\frac{d}{dt}V = 0$ on Γ^+. Rearranging the inequalities (17) and (18) then shows that

$$0 \le \left|z\right|^2 \le \mathsf{H}_\gamma^{\stackrel{\min}{\max}} \le 0 \quad , \tag{19}$$

therefore we must conclude that Γ^+ is a (non-empty by boundedness of $x(\cdot)$) subset of the null set $\mathcal{N}$. But Γ^+ is an invariant set, hence contained in the maximal closed loop invariant set $\mathcal{A}$, and consequently any trajectory $x(\cdot)$ satisfying the conditions of the theorem are approaching $\mathcal{A}$ as $t \to \infty$.

We show finally that $\mathcal{S}$-detectability implies that $x(\cdot)$ approaches $\mathcal{S}$ as $t \to \infty$. Clearly any trajectory evolving entirely on Γ^+ satisfies by inequality (19) that $z(\cdot) = 0$, hence by $\mathcal{S}$-detectability $\mathcal{S}$ is approached. Finally, any trajectory with the same limit set Γ^+ is by continuity of the closed loop dynamics forced to approach $\mathcal{S}$ as $t \to \infty$.

Note, that in this case $\Gamma^+ \subset \mathcal{S}$, and that condition (16) shows that closing the loop with the feedback $a(x) = u_{\min}$ does not change the dynamics on the open loop invariant set $\mathcal{S}$.

Note too, that in the case that $\mathcal{S}$ is not connected (it may consist of several open loop positive limit sets for example), the proof indicates that each component of $\mathcal{S}$ is a local minimum of the function $V(x)$, but the constant value $V(x) = c_\Gamma$ will in general be different from component to component. In case that $\mathcal{S} = \{0\}$ we can always assume without loss of generality that $V(x)$ is positive definite. □

Following the proof of Theorem 4.3, it is clear that every connected component of $\mathcal{S}$ is a local minimum of any solution V of the Hamilton-Jacobi inequality (12), and that $\frac{\partial V}{\partial x}\dot{x} = 0$ along any trajectory evolving inside $\mathcal{S}$.

In case that $\mathcal{S} = \{0\}$, local solutions can be obtained by use of an approximation scheme originally developed by Lukes [Luk69] for quadratic cost functions. It has been used to compute solutions of the Hamilton-Jacobi inequalities associated with the local nonlinear state feedback $\mathcal{H}_\infty$ control problem [IK95]. An implementation in the symbolic language `MAPLE` is available for affine control systems [MP95], see [CMPP96] for a calculated example.

4.1 Extending the class of disturbances

From an engineering point of view, the theory so far developed is not yet entirely adequate for practical control purposes: in real systems the disturbance $w(\cdot)$ is often time persistent, and has therefore no finite L_2 norm. In linear $\mathcal{H}_\infty$ theory standard transformation results automatically translate the L_2 induced norm results into power seminorm induced or spectral seminorm induced equivalent results. This kind of equivalence does of course not hold for nonlinear systems.

In general the class of allowed disturbances $\mathcal{W}^\epsilon$ is not conservatively chosen as one might think. However, assuming that $\left|Z(x, u_{\min}(x)\right|$ is a function of class $\mathcal{K}_\infty$, and using the principal ideas of the input-to-state stability property as outlined in [Lin92, Son95] together with the improvements on $\mathcal{H}_\infty$ control mentioned here, it is possible to allow for input and disturbance signals which are time persistent, but bounded in $\mathcal{L}_\infty$ norm (essentially peak bounded). The price to pay is that asymptotic stability of the invariant set $\mathcal{S}$ only is obtained for $w(\cdot) = 0$, but $\mathcal{L}_\infty$ boundedness of $w(\cdot)$ implies then that the state trajectories are bounded in a neighbourhood of $\mathcal{S}$ and $x(\cdot) \to \mathcal{S}$

for $w(\cdot) \to 0$. The proof of a similar theorem involves decay estimates, and will be published later on.

5 Conclusion

In this paper it is shown that state feedback problems involving the stabilization of open loop invariant sets can successfully be recast as generalized formulations of nonlinear local state feedback $\mathcal{H}_\infty$ control problems. Given a formal solution V to a certain Hamilton-Jacobi inequality, the generalized problem is solved regionally (semi-globally) provided V is such that some connected component of the pre-image $V^{-1}(]-\infty, c)$ for some $c \in \mathbb{R}$ is bounded and includes the to-be-stabilized invariant set. The plant is assumed to have a certain detectability property (which is just the generalization of the standard zero-detectability assumption) to prove asymptotic stability of the obtained control law with respect to the invariant set of concern. Sets of allowable initial conditions and disturbance classes are specified.

Hence, the presented results constitute a natural extension of local $\mathcal{H}_\infty$ control theory which possess most of the advantages of global nonlinear control. In particular, performance is guaranteed in a range of operational conditions, in contrast to local $\mathcal{H}_\infty$ control. Non-stationary modes of operation such as stability of periodic orbits are included in this new theory. Numerical methods which apply to local $\mathcal{H}_\infty$ theory can without problem be applied in a semi-global context.

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