ROBUST $\mathcal{H}_2$ PERFORMANCE ANALYSIS FOR SYSTEMS WITH NONLINEAR PARAMETRIC UNCERTAINTIES

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Abstract

In this paper algorithms for calculating the maximal perturbation bounds under $\mathcal{H}_2$ performance constraints for systems with parametric uncertainties are presented. A family of systems is considered, described by state space models which depend nonlinearly on real uncertain parameters. The stability and performance robustness analysis are based on the same matrix algebra results, and the corresponding algorithms therefore are very similar in style. An example illustrates the algorithms and calculations.

1 Introduction

Robust performance analysis for uncertain control systems, which is now receiving a great deal of attention (see [4, 9] and references therein), is a relatively new area in comparison with robust stability analysis. For linear time-invariant systems, the $\mathcal{H}_2$ performance metric arises naturally in a number of different physically meaningful situations, see [4, 6, 3]. The $\mathcal{H}_2$ performance of a linear time-invariant system is measured via the $\mathcal{H}_2$ norm of its transfer matrix. As long as this $\mathcal{H}_2$ norm is less than a given upper bound, the design can stop, and there is no need to seek the minimal norm due to robustness considerations. Suppose the $\mathcal{H}_2$ norm of a nominal (stable) system is less than a given upper bound. Then the question is whether this is still less than this upper bound after suffering the parameter perturbation? or alternatively, how to find the "maximal domain" for perturbation parameters under stability and $\mathcal{H}_2$ norm constraints? This paper will consider the latter problem, and calculate the maximal (nonlinear) perturbation interval or radius in perturbation parameter space. The results obtained are not only sufficient, but are also necessary. The paper is different from most of published papers which deal with a fixed parameter domain and affinely linear perturbations. One of our motivations comes from [4] which computed the supremum of the $\mathcal{H}_2$ norm in the case of an affinely linear perturbation with perturbation parameter $q \in [0, 1]$.

The notation used throughout the paper is as follows. Denote the real number set by $\mathbf{R}$, and the complex plane (the complex open left half plane) by $\mathbf{C}$ ($\mathbf{C}^-$). Let cs: $\mathbf{R}^{m\times n} \to \mathbf{R}^{mn}$ be the column stacking operator on a matrix, $\otimes$: $\mathbf{R}^{n\times n} \times \mathbf{R}^{m\times m} \to \mathbf{R}^{mn\times mn}$ the standard matrix Kronecker product, and $\oplus$: $\mathbf{R}^{n\times n} \times \mathbf{R}^{m\times m} \to \mathbf{R}^{mn\times mn}$ the standard matrix Kronecker sum defined in [2]. Finally, let $\lambda_k(\cdot)$ be the kth eigenvalue of a square matrix.

2 Problem formulation

Consider a linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= A(q)x(t) + B(q)w(t), \\ z(t) &= C(q)x(t), \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^m$, and $z \in \mathbf{R}^p$ are state, disturbance, and performance vectors, respectively; $A(q)$, $B(q)$, and $C(q)$ are (of compatible dimension) continuous matrix functions of the perturbation parameter vector $q = [q_1, q_2, \cdots, q_l]^T \in \mathbf{R}^l$. The transfer function matrix from w to z can be expressed as $T(s,q) = C(q)(sI - A(q))^{-1}B(q)$. A square constant matrix is called stable if all of its eigenvalues lie in $\mathbf{C}^-$. The corresponding transfer function $T(s,q)$ is said to be stable for a given q if $A(q)$ is stable, its $\mathcal{H}_2$ norm is defined by:

$$\|T(s,q)\|_2 \doteq \left\{ \frac{1}{2\pi}\int_{-\infty}^{+\infty} Trace[T(j\omega,q)T^*(j\omega,q)]d\omega \right\}^{1/2} \tag{2}$$

where $T^*(s,q) \doteq T'(-s,q)$ and $(\cdot)'$ denotes transpose.

Assume that the following two assumptions are satisfied for the nominal system $(A(0), B(0), C(0))$

AS1. $A(0)$ is stable,

AS2. $\|T(s,0)\|_2^2 < \gamma$,

where γ is a known positive constant which reflects the tolerance of the system as measured by the $\mathcal{H}_2$ performance (for instance, an acceptable output variance of (1) to a white noise signal w). The goal is to find "the maximal domain" in $\mathbf{R}^l$ so that $\|T(s,q)\|_2^2 < \gamma$ for every q in the domain. A prerequisite for doing this is that $A(q)$ must be stable for all q in this domain. This means that the robust stability analysis must be completed first (see relevant results in [1, 5, 7, 8]). The relevant problems will be considered for two cases, i. e., $l = 1$ and $l = 2$, respectively.

2.1 Single parameter case

Define

$$\begin{aligned} r_s^- &\doteq \inf\{r < 0 : A(q) \text{ is stable } \forall q \in (r, 0)\}, \\ r_s^+ &\doteq \sup\{r > 0 : A(q) \text{ is stable } \forall q \in (0, r)\}, \\ r_2^- &\doteq \inf\{r < 0 : A(q) \text{ is stable and } \|T(s,q)\|_2^2 < \gamma \\ &\qquad\qquad \forall q \in (r, 0)\}, \\ r_2^+ &\doteq \sup\{r > 0 : A(q) \text{ is stable and } \|T(s,q)\|_2^2 < \gamma \\ &\qquad\qquad \forall q \in (0, r)\}. \end{aligned}$$

Then (r_s^-, r_s^+) is the maximal perturbation interval of q while keeping the stability of $A(q)$; and (r_2^-, r_2^+) is the maximal perturbation interval of q while keeping $\|T(s,q)\|_2^2 < \gamma$.

Problem 1 *Suppose that system (1) satisfies AS1, AS2, and the system matrices may be parameterised as:*

$$\begin{aligned} A(q) &\doteq A_0 + qA_1 + \cdots + q^{m_1}A_{m_1}, \\ B(q) &\doteq B_0 + qB_1 + \cdots + q^{m_2}B_{m_2}, \\ C(q) &\doteq C_0 + qC_1 + \cdots + q^{m_3}C_{m_3}, \end{aligned}$$

where all of A_k, B_k, and C_k are given constant matrices.

(a). Find r_s^- and r_s^+.

(b). Find r_2^- and r_2^+.

Remark 1 Obviously, $(r_2^-, r_2^+) \subset (r_s^-, r_s^+)$.

2.2 Two parameter cases

Denote by $U(r)$ and $\partial U(r)$ the disk $\{q = [q_1, q_2]' : \sqrt{q_1^2 + q_2^2} < r,\} \subset \mathbf{R}^2$ and its boundary circle, respectively. Define

$$\begin{aligned} r_s &\doteq \sup\{r : A(q) \text{ is stable } \forall q \in U(r)\}, \\ r_2 &\doteq \sup\{r : A(q) \text{ is stable and } \|T(s,q)\|_2^2 < \gamma \\ &\qquad\qquad \forall q \in U(r)\}. \end{aligned}$$

Then $U(r_s)$ is the maximal perturbation disk for q while keeping the stability of $A(q)$; and $U(r_2)$ is the maximal perturbation disk for q while keeping $\|T(s,q)\|_2^2 < \gamma$.

Problem 2 *Suppose that system (1) satisfies AS1, AS2, and*

$$\begin{aligned} A(q) &\doteq A_0 + q_1A_{10} + q_2A_{01} + q_1^2A_{20} + q_1q_2A_{11} \\ &\quad + q_2^2A_{02} + \cdots + \textstyle\sum_{i+j=m_1} q_1^i q_2^j A_{i,j}, \\ B(q) &\doteq B_0 + q_1B_{10} + q_2B_{01} + q_1^2B_{20} + q_1q_2B_{11} \\ &\quad + q_2^2B_{02} + \cdots + \textstyle\sum_{i+j=m_2} q_1^i q_2^j B_{i,j}, \\ C(q) &\doteq C_0 + q_1C_{10} + q_2C_{01} + q_1^2C_{20} + q_1q_2C_{11} \\ &\quad + q_2^2C_{02} + \cdots + \textstyle\sum_{i+j=m_3} q_1^i q_2^j C_{i,j}, \end{aligned}$$

where A_0, B_0, C_0 and all of $A_{i,j}$, $B_{i,j}$, and $C_{i,j}$ are given constant matrices.

(a). Find r_s.

(b). Find r_2.

Remark 2 Obviously, $0 < r_2 \le r_s$.

3 Preliminaries

Let $M(r) = M_0 + rM_1 + \cdots + r^m M_m$ where all of the M_k are nxn constant matrices, and $|M_0| \ne 0$ ($|\cdot|$ denotes the determinant). Let,

$$\begin{aligned} r^- &\doteq \sup\{r < 0 : |M(r)| = 0\}, & (3)\\ r^+ &\doteq \inf\{r > 0 : |M(r)| = 0\}, & (4)\end{aligned}$$

be the maximal perturbation bounds for nonsingularity of matrices. By simple operations on the matrix and its determinant (see [8]), it can be shown that,

$$\begin{aligned} r^- &= \frac{1}{\lambda^-_{min}(\mathbf{M})}, & (5)\\ r^+ &= \frac{1}{\lambda^+_{max}(\mathbf{M})}, & (6)\end{aligned}$$

where $\mathbf{M}$ is an $mn \times mn$ matrix given by

$$\mathbf{M} \doteq -\begin{pmatrix} \mathbf{O} & -\mathbf{I} & \mathbf{O} & \cdots \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \cdots \mathbf{O} \\ \vdots & \vdots & \vdots & \ddots \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots -\mathbf{I} \\ M_0^{-1}M_m & M_0^{-1}M_{m-1} & M_0^{-1}M_{m-2} & \cdots \ M_0^{-1}M_1 \end{pmatrix} \tag{7}$$

and $\lambda^-_{min}(\cdot)$ stands for the minimal value of the negative real eigenvalues (let $\lambda^-_{min}(\cdot) = 0^-$ if there exist no negative real eigenvalues), and $\lambda^+_{max}(\cdot)$ the maximal value of the positive real eigenvalues (let $\lambda^+_{max}(\cdot) = 0^+$ if no positive real eigenvalues), respectively.

Formulas (5) and (6) suggests the following algorithm:

Algorithm 1 (the max. perturbation bounds for nonsingularity of matrices)

Step 1. Input M_k, $k = 0, 1, \cdots, m$ where $|M_0| \ne 0$;

Step 2. Define $\mathbf{M}$ as in (7);

Step 3. Calculate all the eigenvalues of $\mathbf{M}$;

Step 4. Find r^- and r^+ based on (5) and (6), then output.

The following lemma helps us to transform Problems 1a and 2a into that of the maximal perturbation bounds for the nonsingularity of matrices.

Lemma 3 *Suppose that*

(i). Q is a single connected domain in $\mathbf{R}^l$, and $0 \in Q$,

(ii). $A(0)$ is stable.

Then $A(q)$ are stable for all $q \in Q$ if and only if $|A(q) \oplus A(q)| \ne 0$ for all $q \in Q$.

Proof: Recall the continuity of $A(q)$, $B(q)$, $C(q)$ to q, and

$$\begin{aligned}\lambda_k(A(q) \oplus A(q)) &= \lambda_i(A(q)) + \lambda_j(A(q)) \\ & \qquad k = 1, \cdots, n^2;\ i, j = 1, \cdots, n.\end{aligned}$$

from this observation the lemma becomes obvious. □

By using Lemma 3 it may be shown that

$$\begin{aligned} r_s^- &= \sup\{q < 0 : |A(q) \oplus A(q)| = 0\}, & (8)\\ r_s^+ &= \inf\{q > 0 : |A(q) \oplus A(q)| = 0\}, & (9)\\ r_s &= \inf\{r : |A(q) \oplus A(q)| = 0 \\ & \qquad \text{for some } q \in \partial U(r)\}. & (10)\end{aligned}$$

Instead of (2) in the frequency domain, we use the state space approach to compute :

$$\|T(s,q)\|_2^2 = Trace\{C'(q)C(q)Q(q)\}$$

where $Q(q) = Q(q)'$ satisfies

$$A(q)Q(q) + Q(q)A(q)' + B(q)B(q)' = 0$$

It is easy to show the following compact formula (or see [4])

$$\|T(s,q)\|_2^2 = -cs[C'(q)C(q)]' \cdot [A(q) \oplus A(q)]^{-1} \cdot cs[B(q)B'(q)] \tag{11}$$

Going one step from (11), the following result is obtained which helps transform Problem 1b and 2b into that of the maximal perturbation bounds for nonsingularity of matrices.

Lemma 4 *Suppose that*
(i) Q is a single connected domain in $\mathbf{R}^l$, and $0 \in Q$,
(ii) $A(q)$ are stable $\forall q \in Q$,
(iii) $\|T(s,0)\|_2^2 < \gamma$.

Then $\|T(s,q)\|_2^2 < \gamma$ *for all* $q \in Q$ *if and only if* $|\mathbf{M}_\gamma(q)| \neq 0$ *for all* $q \in Q$*, where*

$$\mathbf{M}_\gamma(q) \doteq A(q) \oplus A(q) + \frac{1}{\gamma} cs[B(q)B'(q)] \cdot cs[(C'(q)C(q)]' \tag{12}$$

Proof: $\|T(s,q)\|_2^2 < \gamma$ for all $q \in Q$
$\Leftrightarrow \gamma + cs[C'(q)C(q)]' \cdot [A(q) \oplus A(q)]^{-1} \cdot cs[B(q)B'(q)] > 0$ for all $q \in Q$. (from (11))
$\Leftrightarrow |\gamma I + [A(q) \oplus A(q)]^{-1} \cdot cs[B(q)B'(q)] \cdot cs[C'(q)C(q)]'| > 0$ for all $q \in Q$.

(use equality $|\gamma I + XY| = |\gamma I + YX|$)

$\Leftrightarrow |\gamma[A(q) \oplus A(q)]^{-1}| \cdot |\mathbf{M}_\gamma(q)| > 0$ for all $q \in Q$ (from (12))
$\Leftrightarrow |\mathbf{M}_\gamma(q)| \neq 0$ for all $q \in Q$ (due to the continuity of $A(q), B(q), C(q)$ to q, and Lemma 3)

The rest of the proof is trival and omitted. □

By using Lemma 4 we obtain the following formulas being suited for calculating.

$$r_2^- = \sup\{q \in (r_s^-, 0) : |\mathbf{M}_\gamma(q)| = 0\}, \tag{13}$$
$$r_2^+ = \inf\{q \in (0, r_s^+) : |\mathbf{M}_\gamma(q)| = 0\}, \tag{14}$$
$$r_2 = \inf\{r : r \leq r_s \text{ and } |\mathbf{M}_\gamma(q)| = 0 \text{ for some } q \in \partial U(r)\}. \tag{15}$$

In Section 2 we presented two types of problems. One is the maximal perturbation bounds for system stability; the other is the maximal perturbation bounds for system performance. Lemma 3 and 4 help us to transform these two into the maximal perturbation bounds for nonsingularity of matrices. This means that the resulting algorithms will be similar in spirit.

4 The Main results

This section will describe the main formulas and algorithms.

4.1 Single parameter case

By using matrix multiplication and the expressions of $A(q)$, $B(q)$, $C(q)$ in problem 1, then it can be seen that

$$A(q) \oplus A(q) = \mathbf{A}_0 + q\mathbf{A}_1 + \cdots + q^{m_1}\mathbf{A}_{m_1} \tag{16}$$
$$cs[B(q)B'(q)] = \mathbf{b}_0 + q\mathbf{b}_1 + \cdots + q^{2m_2}\mathbf{b}_{2m_2} \tag{17}$$
$$cs[C'(q)C(q)] = \mathbf{c}_0 + q\mathbf{c}_1 + \cdots + q^{2m_3}\mathbf{c}_{2m_3} \tag{18}$$

where

$$\mathbf{A}_k = A_k \oplus A_k,\ k = 0, 1, \cdots, m_1$$
$$\mathbf{b}_0 = cs[B_0B_0'],\ \mathbf{b}_{2m_2} = cs[B_{m_2}B_{m_2}'],$$
$$\mathbf{c}_0 = cs[C_0'C_0],\ \mathbf{c}_{2m_3} = cs[C_{m_3}'C_{m_3}],$$

(the expressions for $\mathbf{b}_k$ and $\mathbf{c}_k$ are omitted due to space limitations). Substituting the above expressions for $A(q)$, $B(q)$, $C(q)$ in (12), then it can be written as :

$$\mathbf{M}_\gamma(q) = M_{0\gamma} + qM_{1\gamma} + \cdots + q^m M_{m\gamma} \tag{19}$$

where $m = max\{m_1, 2(m_2 + m_3)\}$, and

$$M_{0\gamma} = (A_0 \oplus A_0) + \frac{1}{\gamma} cs[B_0B_0'] \cdot cs[C_0'C_0]', \tag{20}$$

and all of other $M_{k\gamma}$ (the detailed expressions are omitted) depend on $\mathbf{A}_i$, $\mathbf{b}_j$, and $\mathbf{c}_k$ in a similar fashion.

By recalling Algorithm 1, and using (9), (10) and (17), then the following is obtained:

Algorithm 2 (the max. perturbation bounds for Problem 1a)

Step 1. Input A_k, $k = 0, 1, \cdots, m$ where A_0 must be stable;

Step 2. Calculate $\mathbf{A}_k$, $k = 0, 1, \cdots, m_1$;

Step 3. Let $M_k = \mathbf{A}_k$, recall Algorithm 1, then compute r^- and r^+;

Step 4. Let $r_s^- = r^-$ and $r_s^+ = r^+$, and output.

From AS2, Lemma 4, and (20), it can be shown that $|M_{0\gamma}| \neq 0$. By recalling Algorithm 1, and using (14), (15) and (19), the following algorithm is obtained:

Algorithm 3 (the max. perturbation bounds for Problem 1b)

Step 1. Input A_i, B_j, and C_k where we must have AS1 and AS2;

Step 2. Calculate $\mathbf{A}_i$, $\mathbf{b}_j$ and $\mathbf{C}_k$, and also m;

Step 3. Calculate $M_{k\gamma}$;

Step 4. Let $M_k = M_{k\gamma}$, and recall Algorithm 1 to get r^- and r^+;

Step 5. Output $r_2^- = \max\{r_s^-, r^-\}$, and $r_2^+ = \min\{r_s^+, r^+\}$.

Remark 5 Algorithms 2 and 3 do not need any iteration.

The maximal perturbation bounds for Problem 1a in the simplest case (affine perturbations with a single parameter) were given in [5].

4.2 Two-parameter case

In order to solve Problem 2, introduce polar coordinates, namely, $q_1 = r\cos\theta$, $q_2 = r\cos\theta$, thus

$$\begin{aligned} A(q) = A(r,\theta) &= A_0 + rA_1(\theta) + \cdots + r^{m_1}A_{m_1}(\theta),\\ B(q) = B(r,\theta) &= B_0 + rB_1(\theta) + \cdots + r^{m_2}B_{m_2}(\theta),\\ C(q) = C(r,\theta) &= C_0 + rC_1(\theta) + \cdots + r^{m_3}C_{m_3}(\theta), \end{aligned}$$

where

$$\begin{aligned} A_k(\theta) &\doteq \textstyle\sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j A_{ij}, k = 1,\cdots,m_1\\ B_k(\theta) &\doteq \textstyle\sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j B_{ij}, k = 1,\cdots,2m_2\\ C_k(\theta) &\doteq \textstyle\sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j C_{ij}, k = 1,\cdots,2m_3 \end{aligned}$$

Obviously, for a fixed θ, Problem 2 is fully transformed into Problem 1. A grid for the interval $[0, 2\pi)$ is needed and finally

$$\begin{aligned} r_s &= \inf\{r_s^+(\theta), \theta \in [0,2\pi)\},\\ r_2 &= \inf\{r_2^+(\theta), \theta \in [0,2\pi)\}. \end{aligned}$$

The algorithms corresponding to Problem 2a and 2b are listed briefly below:

Algorithm 4 (the maximal stab. radius for Problem 2a)

Step 1. Select a large natural number p, and let $\theta_j = 2j\pi/p$, $j = 0, 1, \cdots, p$;

Step 2. Let $A_k = A_k(\theta_j)$, repeatedly recall Algorithm 2 to get r_{sj}^+, $j = 0, 1, \cdots, p$;

Step 3. Find $r_s = \min\{r_{sj}^+, j = 0, 1, \cdots, p\}$, then output it.

Algorithm 5 (the maximal stab. radius for Problem 2b)

Step 1. Select a large natural number p, and let $\theta_j = 2j\pi/p$, $j = 0, 1, \cdots, p$;

Step 2. Let $A_k = A_k(\theta_j)$, $B_k = C_k(\theta_j)$, and $A_k = C_k(\theta_j)$, repeatedly recall Algorithm 3 to get r_{2j}^+, $j = 0, 1, \cdots, p$;

Step 3. Find $r_2 = \min\{r_{2j}^+, j = 0, 1, \cdots, p\}$, then output it.

Remark 6 Solving Problem 2 needs iteration in one dimension.

5 Example

An example with a single perturbation parameter is cited below. Let

$$A(q) = \begin{bmatrix} -2 & 1 \\ 0 & -1.5 \end{bmatrix} + q\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + q^2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + q^3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B(q) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},\ C(q) = [1\ 1]$$

It is easy to show that $A_0 = \begin{bmatrix} -2 & 1 \\ 0 & -1.5 \end{bmatrix}$ is stable, and $T(s,0) = [\frac{1}{s+2}\ \ \frac{s+3}{(s+2)(s+1.5)}]$, and $\|T(s,0)\|_2^2 \approx 0.8214 < 1 = \gamma$. In this example it may be shown that

$$A(q) \oplus A(q) = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 0 & -3.5 & 0 & 1 \\ 0 & 0 & -3.5 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} + q\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + q^2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} + q^3\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$cs[B(q)B'(q)] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + q\begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \end{bmatrix} + q^2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix},$$

and $cs[C'(q)C(q)]' = [1\ 1\ 1\ 1]$, furthermore,

$$\mathbf{M}_\gamma(q) = \begin{bmatrix} -3 & 2 & 2 & 1 \\ 0 & -3.5 & 0 & 1 \\ 0 & 0 & -3.5 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix} + q\begin{bmatrix} 3 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 4 & 4 & 4 & 4 \end{bmatrix} + q^2\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 5 & 6 & 6 & 5 \end{bmatrix} + q^3\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

Finally $(r_s^-, r_s^+) = (-1.6711, 0.7683)$ can be calculated, which shows that the family $A(q)$ is stable for all $q \in (-1.6711, 0.7683)$, and $(r_2^-, r_2^+) = (-1.6711, 0.0433)$, meaning that $\|T(s,q)\|_2^2 < 1$ for all $q \in (-1.6711, 0.0433)$. These two intervals are furthermore the largest intervals with these properties.

6 Conclusions

In this paper, methods for calculating the maximal parameter perturbation bounds under $\mathcal{H}_2$ performance constraints for a family of systems described by state space models, with nonlinear dependence on real uncertain parameters, have been presented, as well as methods for computing similar bounds for the corresponding stability bounds. The results are not conservative as the information of the system structure is used completely.

The domains for robust performance are, obviously, subsets (usually strict) of the robust stability domains, but the algorithms for computation of the robust performance radii and for stability radii are similar in nature, since they are based on the same matrix algebra results.

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