Computation of the maximal robust $\mathcal{H}_2$ performance radius for uncertain discrete time systems with nonlinear parametric uncertainties

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In this paper we address the problems of robust stability and robust $\mathcal{H}_2$ performance for uncertain discrete time systems with nonlinear parametric uncertainties. We consider two families of systems with parametric uncertainties described by state-space models which offer a fairly general representation of most uncertain systems with one or two parameters (the approach can be extended to more parameters). For these two families we obtain explicit expressions for the Schur stability radius and for the $\mathcal{H}_2$ robust performance radius in the case of uncertainties with a single parameter. Moreover, we provide a line search algorithm for these two problems in the case of two parameters. Both for the robust stability and the robust performance problem, explicit necessary and sufficient conditions are derived.

1. Introduction

In the dawn of robust control theory, most attention was paid to systems with unstructured uncertainty descriptions. It was soon realized, however, that in many applications the real uncertainties are better captured by structured uncertainty descriptions. This is definitely the case when the model applied is based on physical insight of the plant, such that the uncertainties are basically just an imperfect determination of physically meaningful parameters. But even in the case where the nominal model and the uncertainty are obtained entirely by identification methods, this still results in parametric uncertainty descriptions. The reason for this is that statistical methods will always have different preferences for different directions in the s-plane, thus providing phase information. Uncertain phase information is only representable by structured uncertainty models.

Moreover, robust control theory has had far more emphasis on the nominal performance/robust stability paradigm, rather than the robust performance paradigm, which of course is the problem of ultimate importance. This is not because the significance of robust performance problems have been overlooked, but simply because the research has had little success in this field so far. One reason is that some of these problems are NP-hard.

Many papers have been devoted to the topic of robust stability bounds under structured perturbations. Let us mention a few which also have comprehensive lists of references: Ackermann and Barmish (1988), Barmish (1994), Zhou *et al.* (1992), Hinrichsen and Pritchard (1986a, 1986b), Doyle *et al.* (1991).

For the $\mathscr{H}_\infty$ norm, robust performance bounds can be obtained by μ optimization, see Packard and Doyle (1993) for a survey or Young *et al.* (1991) for an exposition in the line of this paper. A convex optimization approach for robust $\mathscr{H}_\infty$ analysis and synthesis for systems with parametric uncertainties is given by Zhou *et al.* (1995).

For linear time-invariant systems, the $\mathscr{H}_2$ performance metric arises naturally in a number of different physically meaningful situations, see Doyle *et al.* (1991), and Chen and Francis (1995). The $\mathscr{H}_2$ performance of a linear time-invariant system is measured via the $\mathscr{H}_2$ norm of its transfer matrix. As long as this $\mathscr{H}_2$ norm is less than a given upper bound, we can stop, and need not seek the minimal one due to robustness consideration. Even if the $\mathscr{H}_2$ norm of a nominal (stable) system is less than a given upper bound, it might not be less than this bound after suffering parameter perturbation.

This paper will consider the problem of finding the 'maximal domain' for perturbation parameters under stability and $\mathscr{H}_2$ norm constraints, and calculate the maximal (nonlinear) perturbation interval or radius in perturbation parameter space. The obtained results are not only sufficient, but also necessary. The paper is different from most of the published papers which deal with a fixed parameter domain and affine perturbations. Although the extension from affine to polynomial perturbations is not surprising for experts, the authors find that its importance is still sufficiently significant to justify independent treatment. For recent advances on robust $\mathscr{H}_2$ performance analysis for uncertain control systems, see the papers of Friedman *et al.* (1995), Mustafa (1995) and references therein. In this paper we shall find the maximal allowable perturbation, given a bound on the $\mathscr{H}_2$ norm. In some papers, such as for example Stoorvogel (1993), the inverse problem has been studied, i.e. to bound the maximal $\mathscr{H}_2$ performance given a bounded perturbation.

This paper deals with discrete time uncertain systems. The corresponding problem in continuous time has been addressed by Zhao *et al.* (1996). The stability results are based on the paper of Zhao (1994).

Before we begin, we need to introduce some notation used throughout this paper. Denote the real number set by $\mathbb{R}$. Let $cs\colon \mathbb{R}^{m \times n} \to \mathbb{R}^{mn}$ be the column stacking operator on a matrix, $\otimes\colon \mathbb{R}^{n \times n} \times \mathbb{R}^{m \times m} \to \mathbb{R}^{mn \times mn}$ the standard matrix Kronecker product (see Brewer 1978), and let $\lambda_k(\cdot)$ be the kth eigenvalue of a square matrix.

2. Problem formulation

Consider a linear time-invariant discrete-time system described by

$$G(z, q) = \left[\begin{array}{c|c} A(q) & B(q) \\ \hline C(q) & O \end{array} \right] \tag{1}$$

where $A(q)$, $B(q)$ and $C(q)$ with dimensions $n \times n$, $n \times m$, $p \times m$, respectively, are continuous matrix functions of a perturbation parameter vector $q = [q_1, q_2, \ldots, q_l]^T \in \mathbb{R}^l$. A square constant matrix is called (Schur) stable if all of its eigenvalues lie in $\{z\colon |z| < 1\}$. We say $G(z, q)$ is (Schur) stable for a given q if $A(q)$ is stable, and the $\mathscr{H}_2$ norm is defined by

$$\|G(z, q)\|_2 \doteq \left\{ \frac{1}{2\pi \mathrm{j}} \oint_{|z| = 1} \mathrm{Trace}\left[G^*(z, q) G(z, q) \right] \frac{\mathrm{d}z}{z} \right\}^{1/2} \tag{2}$$

where $G^*(z, q) \doteq G(z^{-1}, q)$ and $(\cdot)'$ denotes transpose.

Suppose for $q = 0$, the nominal system of (1) satisfies

$$\text{AS1. } A(0) \text{ is stable,}$$

$$\text{AS2. } \|G(z,0)\|_2^2 < \gamma,$$

where γ is a known positive constant which reflects the tolerance of the system $\mathcal{H}_2$ performance (for instance, an acceptable output variance of (1) to a white noise signal). Our goal is to find 'the maximal domain' in $\mathbb{R}^l$ so that $\|G(z,q)\|_2^2 < \gamma$ for every q in it. A prerequisite for this is that $A(q)$ is stable for every q in this domain. This problem will be solved in the two cases $l = 1$ and $l = 2$. The method could, in principle, be extended for $l > 2$ but the computational costs would be quite considerable.

2.1. *Single parameter case*

Define

$$r_s^- \doteq \inf \{r < 0 : A(q) \text{ is stable } \forall q \in (r,0)\}$$

$$r_s^+ \doteq \sup \{r > 0 : A(q) \text{ is stable } \forall q \in (0,r)\}$$

$$r_2^- \doteq \inf \{r < 0 : A(q) \text{ is stable and } \|G(z,q)\|_2^2 < \gamma \ \forall q \in (r,0)\}$$

$$r_2^+ \doteq \sup \{r > 0 : A(q) \text{ is stable and } \|G(z,q)\|_2^2 < \gamma \ \forall q \in (0,r)\}$$

Then (r_s^-, r_s^+) is the maximal perturbation interval of q while keeping the stability of $A(q)$; and (r_2^-, r_2^+) the maximal perturbation interval of q while keeping $\|G(z,q)\|_2^2 < \gamma$.

Problem 1: *Suppose that system* (1) *satisfies AS1, AS2, and*

$$\text{AS3. } \begin{cases} A(q) \doteq A_0 + qA_1 + \cdots + q^{m_1} A_{m_1} \\ B(q) \doteq B_0 + qB_1 + \cdots + q^{m_2} B_{m_2} \\ C(q) \doteq C_0 + qC_1 + \cdots + q^{m_3} C_{m_3} \end{cases}$$

where all of A_k, B_k and C_k are given constant matrices.

(a) *Find r_s^- and r_s^+.*
(b) *Find r_2^- and r_2^+.*

Remark 1: Obviously, $(r_2^-, r_2^+) \subset (r_s^-, r_s^+)$.   □

2.2. *Two-parameter case*

Denote by $U(r)$ and $\partial U(r)$ the circular disc $\{q = [q_1, q_2] : \sqrt{q_1^2 + q_2^2} < r\} \subset \mathbb{R}^2$ and its boundary circle, respectively. Define

$$r_s \doteq \sup \{r : A(q) \text{ is stable } \forall q \in U(r)\}$$

$$r_2 \doteq \sup \{r : A(q) \text{ is stable and } \|G(z,q)\|_2^2 < \gamma \ \forall q \in U(r)\}$$

Then $U(r_s)$ is the maximal perturbation circular disc for q while keeping the stability of $A(q)$; and $U(r_2)$ is the maximal perturbation circular disc for q while keeping $\|G(z,q)\|_2^2 < \gamma$.

Problem 2:   *Suppose that system* (1) *satisfies AS1, AS2 and*

AS4.
$$
\begin{cases}
A(q) \doteq A_{00} + q_1 A_{10} + q_2 A_{01} + q_1^2 A_{20} + q_1 q_2 A_{11} + q_2^2 A_{02} + \cdots \\
\qquad + \sum_{i+j=n_1} q_1^i q_2^j A_{i,j} \\[2mm]
B(q) \doteq B_{00} + q_1 B_{10} + q_2 B_{01} + q_1^2 B_{20} + q_1 q_2 B_{11} + q_2^2 B_{02} + \cdots \\
\qquad + \sum_{i+j=n_2} q_1^i q_2^j B_{i,j} \\[2mm]
C(q) \doteq C_{00} + q_1 C_{10} + q_2 C_{01} + q_1^2 C_{20} + q_1 q_2 C_{11} + q_2^2 C_{02} + \cdots \\
\qquad + \sum_{i+j=n_3} q_1^i q_2^j C_{i,j}
\end{cases}
$$

where $A_{i,j}$, $B_{i,j}$, and $C_{i,j}$ are given constant matrices for all i, j.

   (*a*) *Find r_s.*

   (*b*) *Find r_2.*

Remark 2:   Obviously, $0 < r_2 \le r_s$.   □

Remark 3:   The polynomial perturbation sets described in Problems 1 and 2 are very general in the sense that any nonlinear perturbation set which depends continuously on the parameters, defined on a compact set in parameter space, can be approximated arbitrarily well by these types of uncertainties. The cost of a good approximation is that the computational requirements will be extensive, since the computational time involved with the solutions presented below, grows rapidly with increasing polynomial order.   □

     The polynomial perturbation sets described here can be seen as generalizations of the affine sets discussed by Barmish (1994).

3.   Preliminaries

     By doing simple operations on a matrix and its determinant (see Zhao 1994), we can get the maximal perturbation bounds for the non-singularity of matrices.

Lemma 4:   *Let $M(r) = M_0 + rM_1 + \cdots + r^m M_m$ where all of M_k are $n \times n$ constant matrices, and $|M_0| \ne 0$ ($|\cdot|$ denotes the determinant). Define*

$$
r^- \doteq \sup \{ r < 0 : |M(r)| = 0 \} \tag{3}
$$

$$
r^+ \doteq \inf \{ r > 0 : |M(r)| = 0 \} \tag{4}
$$

Then

$$
r^- = \frac{1}{\lambda_{\min}^-(\mathbf{M})} \tag{5}
$$

$$
r^+ = \frac{1}{\lambda_{\max}^+(\mathbf{M})} \tag{6}
$$

where **M** *is an mnth-order square matrix given by*

$$\mathbf{M} \doteq - \begin{pmatrix} O & -I & O & \cdots & O \\ O & O & -I & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & -I \\ M_0^{-1}M_m & M_0^{-1}M_{m-1} & M_0^{-1}M_{m-2} & \cdots & M_0^{-1}M_1 \end{pmatrix} \tag{7}$$

where **O** *and* **I** *are the nth-order zero matrix and identity matrix, respectively, and $\lambda_{\min}^-(\cdot)$ stands for the minimal value of the negative real eigenvalues (let $\lambda_{\min}^-(\cdot) = 0^-$ if there exist no negative real eigenvalues), and $\lambda_{\max}^+(\cdot)$ the maximal value of the positive real eigenvalues (let $\lambda_{\max}^+(\cdot) = 0^+$ if no positive real eigenvalues), respectively.*

The following lemma helps us to transform Problems 1(a) and 2(a) into that of the maximal perturbation bounds for non-singularity of matrices.

Lemma 5: *Suppose that*

(i) *Q is a single connected domain in $\mathbb{R}^l$, and $0 \in Q$,*

(ii) $_A(0)$ *is stable.*

Then $A(q)$ are stable for all $q \in Q$ if and only if $|A(q) \otimes A(q) - I \otimes I| \neq 0$ for all $q \in Q$, where I is the nth-order identity matrix.

Proof: Recall the continuity of $A(q)$ in q, that the eigenvalues of a matrix are continuous functions of its entries, and that

$$\lambda_k(A(q) \otimes A(q)) = \lambda_i(A(q))\lambda_j(A(q))$$

$$k = 1, \ldots, nn; \; i, j = 1, \ldots, n.$$

then the result is immediate. □

By using Lemma 5 we can show that

$$r_s^- = \sup \{q < 0 : |A(q) \otimes A(q) - I \otimes I| = 0\} \tag{8}$$

$$r_s^+ = \inf \{q > 0 : |A(q) \otimes A(q) - I \otimes I| = 0\} \tag{9}$$

$$r_s = \inf \{r : |A(q) \otimes A(q) - I \otimes I| = 0 \text{ for some } q \in \partial U(r)\} \tag{10}$$

Instead of (2) in the frequency domain, we use here the state-space approach to compute

$$\|G(z, q)\|_2^2 = \text{Trace} \{C'(q)C(q)Q(q)\}$$

where $Q(q) = Q(q)'$ satisfies

$$A(q)Q(q)A'(q) - Q(q) + B(q)B(q)' = 0$$

By using the column stacking operation we can give a more compact formula

$$\|G(z, q)\|_2^2 = - cs\left[C'(q)C(q')\right]' \cdot (A(q) \otimes A(q) - I \otimes I)^{-1} \cdot cs\left[B(q)B'(q)\right] \tag{11}$$

Going one step from (11), we get the following result which helps us to transform Problems 1(b) and 2(b) into that of the maximal perturbation bounds for the non-singularity of matrices.

Lemma 6: *Suppose that*

   (i) *Q is a single connected domain in $\mathbb{R}^l$. and $0 \in Q$,*

   (ii) *$_A(q)$ are Schur-stable $\forall q \in Q$,*

   (iii) *$\|G(z,0)\|_2^2 < \gamma$.*

Then $\|G(z,q)\|_2^2 < \gamma \;\forall q \in Q$ if and only if $|M_\gamma(q)| \neq 0$ for all $q \in Q$, where

$$M_\gamma(q) \doteq (A(q) \otimes A(q) - I \otimes I) + \frac{1}{\gamma} cs\big[B(q)B\langle q\rangle\big] \cdot cs\big[C\langle q\rangle C(q)\big] \tag{12}$$

Proof: $\|G(z,q)\|_2^2 < \gamma$ for all $q \in Q$

$\Leftrightarrow$ $\gamma + cs\big[C\langle q\rangle C(q)\big] \cdot (A(q) \otimes A(q) - I \otimes I)^{-1} \cdot cs\big[B(q)B\langle q\rangle\big] > 0 \;\forall q \in Q.$
$$\text{(from (11))}$$

$\Leftrightarrow$ $\big|\gamma I + (A(q) \otimes A(q) - I \otimes I)^{-1} \cdot cs\big[B(q)B\langle q\rangle\big] \cdot cs\big[C\langle q\rangle C(q)\big]\big| > 0 \;\forall q \in Q.$
$$\text{(use equality } |\gamma I + XY| = |\gamma I + YX|)$$

$\Leftrightarrow$ $\big|\gamma(A(q) \otimes A(q) - I \otimes I)^{-1} \cdot M_\gamma(q)\big| > 0 \;\forall q \in Q$ (from (12))

$\Leftrightarrow$ $|M_\gamma(q)| \neq 0$ for all $q \in Q$ (due to the continuity of $A(q), B(q), C(q)$ to q, and Lemma 5)

The remaining part of the proof is trivial and omitted.   □

    By using Lemma 6 we obtain the following formulae being suited for calculations.

$$r_2^- = \sup \big\{q \in (r_s^-, 0) : |M_\gamma(q)| = 0\big\} \tag{13}$$

$$r_2^+ = \inf \big\{q \in (0, r_s^+) : |M_\gamma(q)| = 0\big\} \tag{14}$$

$$r_2 = \inf \big\{r : r < r_s \text{ and } |M_\gamma(q)| = 0 \text{ for some } q \in \partial U(r)\big\} \tag{15}$$

In §2 we presented two types of problems. One is the maximal perturbation bounds for system stability; the other is the maximal perturbation bounds for system performance. Lemmas 5 and 6 help us to transform these two into the maximal perturbation bounds for the non-singularity of matrices, so the computational schemes become similar in nature for these two rather different problems.

4.   Main results

    In this section we shall combine the preliminary results in order to provide answers to Problem 1 and Problem 2.

4.1. *Single parameter case*

    By using matrix multiplication and the expressions of $A(q), B(q), C(q)$ in Problem 1, we then have

$$(A(q) \otimes A(q) - I \otimes I) = \mathbf{A}_0 + q\mathbf{A}_1 + \cdots + q^{2m_1}\mathbf{A}_{2m_1} \tag{16}$$

$$cs\big[B(q)B\langle q\rangle\big] = \mathbf{b}_0 + q\mathbf{b}_1 + \cdots + q^{2m_2}\mathbf{b}_{2m_2} \tag{17}$$

$$cs\big[C\langle q\rangle C(q)\big] = \mathbf{c}_0 + q\mathbf{c}_1 + \cdots + q^{2m_3}\mathbf{c}_{2m_3} \tag{18}$$

where

$$\mathbf{A}_0 = (A_0 \otimes A_0 - I \otimes I), \ldots, \mathbf{A}_i = \sum_{j+k=i} A_j \otimes A_k, \ldots, \mathbf{A}_{2m_1} = A_{m_1} \otimes A_{m_1}$$

$$\mathbf{b}_0 = cs\left[B_0 B_0'\right] \ldots, \mathbf{b}_i = cs\left[\sum_{j+k=i} B_j B_k'\right], \ldots, \mathbf{b}_{2m_2} = cs\left[B_{m_2} B_{m_2}'\right]$$

$$\mathbf{c}_0 = cs\left[C_0' C_0\right] \ldots, \mathbf{c}_i = cs\left[\sum_{j+k=i} C_j' C_k\right], \ldots, \mathbf{c}_{2m_3} = cs\left[C_{m_3}' C_{m_3}\right]$$

Substituting the above expressions for $A(q), B(q), C(q)$ in (12), it can then be rewritten as

$$M_\gamma(q) = \mathbf{M}_{0\gamma} + q\mathbf{M}_{1\gamma} + \cdots + q^m \mathbf{M}_{m\gamma} \tag{19}$$

where $m = \max\{2m_1, 2(m_2 + m_3)\}$, and

$$\mathbf{M}_{0\gamma} = (A_0 \otimes A_0 - I \otimes I) + \frac{1}{\gamma} cs\left[B_0 B_0'\right] \cdot cs\left[C_0' C_0\right]' \tag{20}$$

and all of the other $\mathbf{M}_{k\gamma}$ depend on $\mathbf{A}_i$, $\mathbf{b}_j$, and $\mathbf{c}_k$ (the detailed expressions are omitted here).

By recalling Lemma 4, and using (8), (9) and (16), we can then formulate Theorem 7.

Theorem 7—Maximal perturbation bounds for Problem 1(a): *Splitting $A(q) \otimes A(q) - I \otimes I$ as (16) gives us the coefficient matrices $\mathbf{A}_k$, $k = 0, \ldots, 2m_1$. Define the following $2m_1$nth-order square matrix*

$$\mathscr{A} \doteq - \begin{pmatrix} \mathbf{O} & -\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\mathbf{I} \\ \mathbf{A}_0^{-1}\mathbf{A}_m & \mathbf{A}_0^{-1}\mathbf{A}_{m-1} & \mathbf{A}_0^{-1}\mathbf{A}_{m-2} & \cdots & \mathbf{A}_0^{-1}\mathbf{A}_1 \end{pmatrix} \tag{21}$$

where $\mathbf{O}$, and $\mathbf{I}$ are the nth-order zero matrix and identity matrix, respectively. Then

$$r_s^- = \frac{1}{\lambda_{\min}^-(\mathscr{A})} \tag{22}$$

$$r_s^+ = \frac{1}{\lambda_{\max}^+(\mathscr{A})} \tag{23}$$

where $\lambda_{\min}^-(\cdot)$ stands for the minimal value of the negative real eigenvalues (let $\lambda_{\min}^-(\cdot) = 0^-$ if there exist no negative real eigenvalues), and $\lambda_{\max}^+(\cdot)$ the maximal value of the positive real eigenvalues (let $\lambda_{\max}^+(\cdot) = 0^+$ if no positive real eigenvalues), respectively.

From AS2, Lemma 6, and (20), it can be shown that $|M_{0\gamma}| \neq 0$. By recalling Lemma 4, and using (13), (14) and (19), we can then formulate Theorem 8.

Theorem 8—Maximal perturbation bounds for Problem 1(b): *Splitting $M_\gamma(q)$ as (19) gives us the coefficient matrices $M_{k\gamma}$, $k = 0, \ldots, m$ where*

$m = \max\{2m_1, 2(m_2 + m_3)\}$. *Define the following 2mn-order square matrix*

$$\mathscr{M}_\gamma \doteq - \begin{pmatrix} \mathbf{O} & -\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\mathbf{I} \\ \mathbf{M}_{0\gamma}^{-1}\mathbf{M}_{m\gamma} & \mathbf{M}_{0\gamma}^{-1}\mathbf{M}_{(m-1)\gamma} & \mathbf{M}_{0\gamma}^{-1}\mathbf{M}_{(m-2)\gamma} & \cdots & \mathbf{M}_{0\gamma}^{-1}\mathbf{M}_{1\gamma} \end{pmatrix} \tag{24}$$

where $\mathbf{O}$, *and* $\mathbf{I}$ *are an n-order zero matrix and an identity matrix, respectively. Then*

$$r_2^- = \max\left\{r_s^-, \frac{1}{\lambda_{\min}^-(\mathscr{M}_\gamma)}\right\} \tag{25}$$

$$r_2^+ = \min\left\{r_s^+, \frac{1}{\lambda_{\max}^+(\mathscr{M}_\gamma)}\right\} \tag{26}$$

where $\lambda_{\min}^-(\cdot)$ *stands for the minimal value of the negative real eigenvalues* (*let* $\lambda_{\min}^-(\cdot) = 0^-$ *if there exist no negative real eigenvalues*), *and* $\lambda_{\max}^+(\cdot)$ *the maximal value of the positive real eigenvalues* (*let* $\lambda_{\max}^+(\cdot) = 0^+$ *if no positive real eigenvalues*), *respectively.*

Remark 9: The algorithms corresponding to Theorems 7 and 8 do not need any iteration. Ackermann and Barmish (1988) first gave the maximal perturbation bounds for Problem 1(a) in the simplest case (affinely linear perturbation of a single parameter). □

4.2. *Two parameter case*

In order to solve Problem 2, we need to introduce polar coordinates, namely, $q_1 = r\cos\theta$, $q_2 = r\cos\theta$, thus

$$A(q) = A(r,\theta) = A_0 + rA_1(\theta) + \cdots + r^{m_1}A_{m_1}(\theta)$$
$$B(q) = B(r,\theta) = B_0 + rB_1(\theta) + \cdots + r^{m_2}B_{m_2}(\theta)$$
$$C(q) = C(r,\theta) = C_0 + rC_1(\theta) + \cdots + r^{m_3}C_{m_3}(\theta)$$

where

$$A_k(\theta) \doteq \sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j A_{ij}, \quad k = 1,\ldots,m_1$$

$$B_k(\theta) \doteq \sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j B_{ij}, \quad k = 1,\ldots,m_2$$

$$C_k(\theta) \doteq \sum_{i+j=k}(\cos\theta)^i(\sin\theta)^j C_{ij}, \quad k = 1,\ldots,m_3$$

Obviously, for a fixed θ, Problem 2 is fully transformed into Problem 1. But now we need a grid for the interval $[0, 2\pi)$, finally

$$r_s = \inf\{r_s^+(\theta), \theta \in [0, 2\pi)\}$$
$$r_2 = \inf\{r_2^+(\theta), \theta \in [0, 2\pi)\}$$

The algorithms corresponding to Problems 2(a) and 2(b) are briefly listed below.

Algorithm 1—Maximal stability radius for Problem 2(a):

Step 1. Select a large natural number p, and let $\theta_j = 2j\pi/p$, $j = 0, 1, \ldots, p - 1$;

Step 2. Let $A_k = A_k(\theta_j)$, repeatedly recall Theorem 7 to get r_{sj}^+, $j = 0, 1, \ldots, p - 1$;

Step 3. Find $r_s = \min\{r_{sj}^+, j = 0, 1, \ldots, p - 1\}$, then output it.

Algorithm 2—Maximal stability radius for Problem 2(b):

Step 1. Select a large natural number p, and let $\theta_j = 2j\pi/p$, $j = 0, 1, \ldots, p - 1$;

Step 2. Let $A_k = A_k(\theta_j)$, $B_k = C_k(\theta_j)$ and $A_k = C_k(\theta_j)$, repeatedly recall Theorem 8 to get r_{2j}^+, $j = 0, 1, \ldots, p - 1$;

Step 3. Find $r_2 = \min\{r_{2j}^+, j = 0, 1, \ldots, p - 1\}$, then output it.

Remark 10: Solving Problem 2 involves a one-dimensional search in contrast to Problem 1 which can be solved non-iteratively. □

5. Example

An example with a single perturbation parameter is cited below. Let

$$A(q) = \begin{bmatrix} 0\cdot1 & 1 \\ 0 & 0\cdot5 \end{bmatrix} + q\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + q^2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$B(q) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad C(q) = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

It is easy to show that

$$A_0 = \begin{bmatrix} 0\cdot1 & 1 \\ 0 & 0\cdot5 \end{bmatrix}$$

is Schur stable, and

$$A(q) \otimes A(q) - I \otimes I$$

$$= \begin{bmatrix} -0\cdot9900 & 0\cdot1000 & 0\cdot1000 & 1\cdot000 \\ 0 & -0\cdot9500 & 0 & 0\cdot5000 \\ 0 & 0 & -0\cdot9500 & 0\cdot5000 \\ 0 & 0 & 0 & -0\cdot7500 \end{bmatrix} + q\begin{bmatrix} 0 & 0\cdot1000 & 0\cdot1000 & 2\cdot000 \\ 0 & 0 & 0 & 0\cdot5000 \\ 0 & 0 & 0 & 0\cdot5000 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ q^2\begin{bmatrix} 0 & 0 & 0 & 1\cdot0000 \\ 0\cdot1000 & 0 & 1\cdot0000 & 0 \\ 0\cdot1000 & 1\cdot0000 & 0 & 0 \\ 0 & 0\cdot5000 & 0\cdot5000 & 0 \end{bmatrix} + q^3\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + q^4\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

after calculating $\mathscr{A}$ and all its eigenvalues, we get $(r_s^-, r_s^+) = (-1\cdot6711, 0\cdot7683)$. In this example we can show

$$G(z, 0) = \begin{bmatrix} \dfrac{1}{z - 0\cdot1} & \dfrac{z + 0\cdot9}{(z - 0\cdot1)(z - 0\cdot5)} \end{bmatrix}, \quad \text{and} \quad \|G(z,0)\|_2^2 \approx 2\cdot0162$$

Now we select the upper bound of $\mathcal{H}_2$ performance as $\gamma = 2 \cdot 1$.

$$cs\big[B(q)B'(q)\big] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + q\begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \end{bmatrix} + q^2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix}$$

and $cs\big[C'(q)C(q)\big] = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ furthermore

$$M_\gamma(q) = (A(q) \otimes A(q) - I \otimes I) + \frac{1}{\gamma} cs\big[B(q)B'(q)\big] \cdot cs\big[C'(q)C(q)\big]$$

$$= \begin{bmatrix} -0\cdot5138 & 0\cdot5762 & 0\cdot5762 & 1\cdot4762 \\ 0 & -0\cdot9500 & 0 & 0\cdot5000 \\ 0 & 0 & -0\cdot9500 & 0\cdot5000 \\ 0\cdot4762 & 0\cdot4762 & 0\cdot4762 & -0\cdot2738 \end{bmatrix}$$

$$+ q\begin{bmatrix} 0\cdot9624 & 1\cdot0524 & 1\cdot0524 & 2\cdot9524 \\ 0\cdot4762 & 0\cdot4762 & 0\cdot4762 & 0\cdot9762 \\ 0\cdot4762 & 0\cdot4762 & 0\cdot4762 & 0\cdot9762 \\ 1\cdot9048 & 1\cdot9048 & 1\cdot9048 & 1\cdot9048 \end{bmatrix}$$

$$+ q^2\begin{bmatrix} 0\cdot4762 & 0\cdot4762 & 0\cdot4762 & 1\cdot4762 \\ 0\cdot5762 & 0\cdot4762 & 1\cdot4762 & 0\cdot4762 \\ 0\cdot5762 & 1\cdot4762 & 0\cdot4762 & 0\cdot4762 \\ 2\cdot3810 & 2\cdot8810 & 2\cdot8810 & 2\cdot3810 \end{bmatrix} + q^3\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ q^4\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

After calculating $\mathcal{M}_\gamma$ and all its eigenvalues, finally we get $(r_2^-, r_2^+) = (-1\cdot6711, 0\cdot0433)$. □

6. Conclusions

In this paper we have investigated stability robustness and $\mathcal{H}_2$ performance robustness of discrete time systems with nonlinear parametric uncertainties.

We restricted ourselves to the class of polynomial uncertainty descriptions, since this class is dense in the set of continuous matrix valued functions defined on compact sets of parameters equipped with the topology of pointwise convergence.

For this class we obtained explicit formulae both for the stability robustness perturbation radius and for the $\mathcal{H}_2$ performance robustness perturbation radius in the case of a single parameter.

In the two parameter cases, we described line search algorithms as the natural extensions of the explicit formulae for the one parameter cases. More parameters

could easily be included in the framework, but the computational cost involved would be quite considerable.

Further research could address $\mathscr{H}_\infty$ performance robustness, and possibly mixed $\mathscr{H}_2/\mathscr{H}_\infty$ problems under structured perturbations.

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