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Stability of rotor systems: A complex modelling approach

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Abstract. The dynamics of a large class of rotor systems can be modelled by a linearized complex matrix differential equation of second order, $M\ddot{z} + (D + iG)\dot{z} + (K + iN)z = 0$, where the system matrices M, D, G, K and N are real symmetric. Moreover M and K are assumed to be positive definite and D, G and N to be positive semidefinite. The complex setting is equivalent to twice as large a system of second order with real matrices. It is well known that rotor systems can exhibit instability for large angular velocities due to internal damping, unsymmetrical steam flow in turbines, or imperfect lubrication in the rotor bearings. Theoretically, all information on the stability of the system can be obtained by applying the Routh-Hurwitz criterion. From a practical point of view, however, it is interesting to find stability criteria which are related in a simple way to the properties of the system matrices in order to describe the effect of parameters on stability. In this paper we apply the Lyapunov matrix equation in a complex setting to an equivalent system of first order and prove in this way two new stability results. We then compare the usefulness of these results with the more classical approach applying bounds of appropriate Rayleigh quotients. The rotor systems tested are: a simple Laval rotor, a Laval rotor with additional elasticity and damping in the bearings, and a number of rotor systems with complex symmetric 4×4 randomly generated matrices.

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1. Introduction

When we speed up a rotor we can observe a resonance phenomenon if the rotation frequency of the rotor equals one of the eigenfrequencies. These frequencies are called the critical frequencies and should, of course, be avoided. But besides the resonances we may also observe linear parametric excitation and self-excited vibrations of the rotor. As sources for the parametric excitation we can mention e.g. rotating unisotropy and the effect of grooves. This type of excitation leads to linear differential equations with nonconstant coefficients and shall not be treated in the present paper.

The unstable self-excited lateral vibrations of the rotor which we are dealing with in this work have frequencies quite different from the rotation frequency.

There are three main sources to these kind of instabilities. First the internal friction or damping of a flexible rotor shaft can lead to unstable lateral vibrations. This is surprising, because normally friction applied to a system will damp the free vibrations. But the effect has been well known since the twenties, see Kimball [1]. Internal friction is due to structural damping or caused by rotor parts sliding against each other. The second source of instabilities is the unsymmetrical forces on the rotor blades due to the unsymmetrical steam flow in turbines. The third source is the lubrication mechanism in the rotor bearings which also gives rise to an unsymmetrical distribution of forces applied to the rotor system.

The mathematical model that describes the lateral vibrations of a rotor system consists of a set of second order linearized differential equations and is known as a non-classical or non-conservative system. The equations are characterized by a mass, a damping and a stiffness matrix. The internal damping and the unsymmetrical forces mentioned above are responsible for a skew-symmetric part in the stiffness matrix. This skew-symmetric part is proportional to the rotor speed, and if the speed exceeds a certain limit, the solution to the non-classical system may show an exponentially growing flutter behaviour which means instability.

Investigations of rotor systems have received considerable attention in the literature. The classical books by Gasch and Pfützner [2], by Müller [3] and by Huseyin [4], proceedings of conferences and symposia like Euromech 38 [5] and IUTAM 1994 [6] as well as papers by Kellenberger [7], Pedersen [8], Schweitzer et al. [9], A.A. Müller and P.C. Müller [10], P.C. Müller [11] and many others deal with critical speed, the influence of external and internal damping, bearing characteristics, asymmetries, forced vibrations, steam flow in turbines, and stability.

In the present paper we will concentrate on the stability of rotor systems in the sense of Lyapunov. In this connection, the classical Routh-Hurwitz criterion is somewhat cumbersome to use if the number of degrees of freedom ($2n$) is large, but it can be applied successfully for $2n \leq 4$, see e.g. Kliem [12]. The Lienard-Chipart criterion can possibly give some simplification here, see e.g. Müller [3]. Another way to investigate stability for a non-classical system was shown by Metelitsyn [13] and by Frik [14]. This result can be slightly improved for rotor systems and will be commented on below. An essential contribution to clarifying stability conditions for weak damping was made by means of a perturbation analysis by Müller [11]. Gershgorin circles were applied by Kliem [15], an energy criterion was developed of Kliem and Pommer [16], and Lyapunov functions were used by Ahmadian and Inman [17] and recently by Junfeng Li [18], just to mention some of the interesting results in the comprehensive literature on this subject. Nevertheless, to the authors' knowledge necessary and sufficient conditions for the stability of rotor system models expressed by the system matrices are not known. And more generally – there is still a lack of simple and applicable stability theorems for non-classical systems.

In the following we will derive sufficient conditions – expressed by the properties of the system matrices – for the stability of a non-classical rotor system model.

The main tool of the investigation is the Lyapunov matrix equation in a complex setting. With help of two new criteria we can improve stability limits for a large class of rotor systems. We will illustrate this by examples and compare to classical methods.

2. Mathematical model

Free lateral vibrations of a large class of rotor systems and centrifuges, where the rotating elements are symmetrical with respect to a rotor axis and the bearings are isotropic, can be described by linearized differential equations in the form of (see e.g. Schweitzer et al. [9])

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \ddot{x} + \begin{bmatrix} D & G \\ -G & D \end{bmatrix} \dot{x} + \begin{bmatrix} K & N \\ -N & K \end{bmatrix} x = 0. \tag{1}$$

The vector x denotes the $2n$ generalized coordinates of the rotor system. The $n \times n$ matrices M, D, G, K and N are all symmetric. Moreover the mass matrix M and the stiffness matrix K are both positive definite (> 0) while the damping matrix D, the gyroscopic matrix G and the circulatory matrix N are all assumed to be positive semidefinite (≥ 0).

Let us for example look at the case where internal friction is responsible for the instability of the rotor system. Friction or damping is often modelled viscously and can be split into external and internal damping

$$D = D_e + D_i. \tag{2}$$

Both G and N are linear matrix functions of the angular velocity Ω of the rotor system, see e.g. Müller [3],

$$G = \Omega G_0, \quad N = \Omega D_i \tag{3}$$

where G_0 is a constant matrix. The structure of N assumes coordinates x with respect to an inertial frame. (Using a frame rotating with Ω, external damping D_e will appear in N and K will be dependent on Ω).

Equation (1) is a special example of the non-classical (or non-conservative) system

$$M_1 \ddot{x} + (D_1 + G_1)\dot{x} + (K_1 + N_1)x = 0, \tag{4}$$

where M_1, D_1 and K_1 are symmetric and G_1 and N_1 skew-symmetric matrices. For rotor systems we can identify

$$M_1 = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \quad D_1 = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & N \\ -N & 0 \end{bmatrix}.$$

General results derived from equation (4) can also be used for (1).

A convenient rewriting of system (1) results in the complex equation

$$M\ddot{z} + (D + iG)\dot{z} + (K + iN)z = 0, \tag{5}$$

where $z = x_1 - ix_2$ and $x = [x_1^T \; x_2^T]^T$.

In this setting we use the fact that every real matrix $\begin{bmatrix} B & C \\ -C & B \end{bmatrix}$ can be associated with a complex matrix $B + iC$ and vice versa. If λ_i are the eigenvalues of the complex matrix, the eigenvalues of the real matrix are λ_i and the complex conjugate $\bar{\lambda}_i$. One of the advantages of the complex setting is that the size of the matrices is halved. Notice that in equation (5) the system matrices $D + iG$ and $K + iN$ are complex symmetric and are divided into their Hermitian parts $D \geq 0$ and $K > 0$ and skew-Hermitian parts iG and iN with $G \geq 0$ and $N \geq 0$.

Physically, it seems obvious that K, D_e and $G = \Omega G_0$ tend to stabilize a rotor system, while $N = \Omega D_i$ tends to destabilize it. Therefore, we use Ω as parameter and try to find Ω_{crit} so that the system is stable for $\Omega < \Omega_{\text{crit}}$ and unstable for $\Omega > \Omega_{\text{crit}}$. Since Kimball [1] it has been well known that internal friction can induce self-excited vibrations in rotor systems if Ω is large enough. Our aim now is to discuss stability expressed by the properties of the system matrices D, G, K and N and apply the Lyapunov matrix equation. For this purpose the representation (5) of a rotor system is of advantage.

3. Stability analysis

Assume that the mass matrix M is nonsingular and consider system (5) in the form

$$I\ddot{\nu} + (D + iG)\dot{\nu} + (K + iN)\nu = 0. \tag{6}$$

Here $M = I$ (identity matrix) has easily been established from (5) by means of the transformation $z = M^{-1/2}\nu$ and premultiplying by $M^{-1/2}$. Then the symmetry and sign properties of $D, G \ldots$ etc. are preserved for the new system matrices $M^{-1/2}DM^{-1/2}$, $M^{-1/2}GM^{-1/2}$ etc. Calling these new system matrices again D, G, K and N, results in equation (6). We now formulate the main result.

Theorem 3.1. *System (4) is asymptotically stable if the following three conditions are all satisfied:*

$$\begin{aligned} &a) \quad K > 0, \\ &b) \quad D > 0, \\ &c) \quad (DK + KD + GN + NG - 2ND^{-1}N) + i(KG - GK + DN - ND) > 0. \end{aligned} \tag{7}$$

Condition (7c) can be used in the following way to obtain an estimate for Ω_{crit} if $DK + KD > 0$. If we insert relations (3) into (7c), then this condition reads

$$Q(\lambda) = \lambda^2 C + \lambda B + A > 0, \quad \lambda = i\Omega \tag{8}$$

with symmetric system matrices A and C and skew-symmetric matrix B defined by

$$\begin{aligned} A &= DK + KD, \\ B &= KG_0 - G_0 K + DD_i - D_i D, \\ C &= 2D_i D^{-1} D_i - G_0 D_i - D_i G_0. \end{aligned} \tag{9}$$

$Q(\lambda)$ is a Hermitian matrix of the parameter $\lambda = i\Omega$ and for $\Omega = 0$ is $Q(0) > 0$ (i.e. stability condition (8) is satisfied) if $DK + KD > 0$. For increasing Ω it can happen that $Q(\lambda)$ loses its positive definite property because an eigenvalue of $Q(\lambda)$ becomes zero. This means

$$\det\big(Q(\lambda)\big) = 0, \tag{10}$$

such that the *quadratic conservative gyroscopic eigenvalue problem*

$$Q(\lambda)u = (\lambda^2 C + \lambda B + A)u = 0, \quad u \neq 0 \tag{11}$$

has a purely imaginary eigenvalue $\lambda = i\Omega$. Therefore, stability of system (6) is guaranteed if, besides $K > 0$, $D > 0$, $DK + KD > 0$, the relation

$$\Omega < \Omega_{\text{crit}} = |\lambda|_{\min} \tag{12}$$

is satisfied, where $|\lambda|_{\min}$ is the smallest absolute value of the *purely imaginary* eigenvalues of problem (11). If (11) does not possess purely imaginary eigenvalues, the stability condition (8) is satisfied for all real values of Ω, i.e. system (6) remains stable for all Ω.

Investigations of conservative gyroscopic eigenvalue problems (11) can e.g. be found in Barkwell and Lancaster [19] and in Seyranian et al. [20].

Numerically it can be convenient to find $|\mu|_{\max} = 1/|\lambda|_{\min}$, where $\mu_{\max}$ is the largest absolute value of the purely imaginary eigenvalues of the conservative gyroscopic problem $(\mu^2 A + \mu B + C)u = 0$.

Conditions (7c) and (8) have the disadvantage that the combination $DK+KD$ does not need to be positive definite although K and D individually do possess this property. In this case Theorem 3.1 is of no practical use, since even for $\Omega = 0$ it does not imply stability. That means if e.g. D is diagonal with elements d_i, we should confine ourselves to cases where

$$\sum_{j=1, j\neq i}^{n} |k_{ij}|(d_i + d_j) < 2k_{ii}d_i, \quad i = 1, \ldots, n.$$

According to Gershgorin's theorem, this condition is sufficient for $DK + KD$ to be positive definite.

Otherwise it may be possible to use the next result if we know the smallest eigenvalue $d_{\min}$ of D.

Theorem 3.2. *System (4) is asymptotically stable if the following three conditions are all satisfied:*

$$
\begin{aligned}
&a) \quad K > 0,\\
&b) \quad D > 0, \\
&c) \quad 2dK + GN + NG + i(KG - GK) - \big((dD - d^2 I) - i(2N + GD - dG)\big)\\
&\qquad \cdot (4D - 2dI)^{-1}\big((dD - d^2 I) + i(2N + DG - dG)\big) > 0,
\end{aligned}
\tag{13}
$$

where $0 < d < 2d_{\min}$*, and* $d_{\min}$ *is the smallest eigenvalue of* D.

Like Theorem 3.1, Theorem 3.2 can be interpreted as a problem of determining the smallest absolute value of all the imaginary eigenvalues of a conservative gyroscopic problem $(\lambda^2 C + \lambda B + A)u = 0$ with similar matrices as given by (9).

Proof of Theorems 3.1 and 3.2. System (6) can be written as $\dot{Y} = AY$ with

$$
A = \begin{bmatrix} 0 & I \\ -K - iN & -D - iG \end{bmatrix}, \quad Y = \begin{bmatrix} \nu \\ \dot{\nu} \end{bmatrix}.
$$

According to Lyapunov's theorem, see e.g. Lancaster and Tismenetsky [21], system $\dot{Y} = AY$ is asymptotically stable if there exist Hermitian matrices $P > 0$ and $Q > 0$ which satisfy the matrix equation

$$
A^* P + PA = -Q. \tag{14}
$$

Here A^* denotes the conjugate transpose of A. Notice that the Lyapunov equation (14) normally is written in its real form, but to our purpose it is convenient to use the complex setting.

To investigate positive definite property of matrices we make use of *Schur's condition*, see e.g. Müller [3]:

Lemma 3.1. *A Matrix* $R = \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix}$ *with Hermitian submatrices* R_1 *and* R_3 *is positive definite, if and only if both* R_3 *and* $R_1 - R_2 R_3^{-1} R_2^*$ *are positive definite.*

We shall now postulate the following Lyapunov matrix

$$
P = \begin{bmatrix} 2K + (D - iG)(D + iG) & D - iG \\ D + iG & 2I \end{bmatrix}. \tag{15}
$$

There are two reasons for this particular choice. Firstly, it makes it easy to check whether the matrices P and Q are positive definite by Schur's condition. Secondly, this choice leads to the correct stability limit for the most simple rotor system, the so-called Laval rotor (see Example 1).

Applying Schur's condition to the Hermitian matrix P we get $2I > 0$, which is trivially fulfilled, and $2K + \frac{1}{2}(D - iG)(D + iG) > 0$. This condition is also fulfilled since $K > 0$ was assumed in (7a) and $(D - iG)(D + iG) \geq 0$. The matrix Q is now determined by the Lyapunov equation (14) as

$$Q = \begin{bmatrix} DK + KD + GN + NG + i(KG - GK + DN - ND) & -i2N \\ i2N & 2D \end{bmatrix}. \tag{16}$$

Then the check of $Q > 0$ by means of Schur's condition results immediately in $D > 0$ and $(DK + KD + GN + NG - 2ND^{-1}N) + i(KG - GK + DN - ND) > 0$.

This establishes the proof of Theorem 3.1.

The proof of Theorem 3.2 is quite similar. In (15) we then have to make slight changes to

$$P = \begin{bmatrix} 2K + (dI - iG)(dI + iG) & dI - iG \\ dI + iG & 2I \end{bmatrix}, \tag{17}$$

where d is a real positive number. This matrix P is always positive definite if $K > 0$. The matrix Q, computed by equation (14), becomes

$$Q = \begin{bmatrix} 2dK + GN + NG + i(KG - GK) & (dD - d^2I) - i(2N + GD - dG) - i2N \\ (dD - d^2I) + i(2N + DG - dG) & 4D - 2dI \end{bmatrix}.$$

According to Schur's condition this matrix Q is positive definite if d is chosen as $0 < d < 2d_{\min}$ and conditions b) and c) of theorem 2 are satisfied.

Example 1. The simplest rotor system is the Laval rotor. A massless shaft with elastic coefficient $k > 0$ carries a single unit mass and rotates with constant angular velocity Ω. External and internal damping are called d_e and $d_i > 0$ respectively and the gyroscopic force is $g\Omega$. The equation of motion for the centre of mass moving in a plane perpendicular to the shaft is

$$\ddot{\nu} + \big[(d_e + d_i) + ig\Omega\big]\dot{\nu} + \big[k + id_i\Omega\big]\nu = 0. \tag{18}$$

In this setting we have $D = d_e + d_i$, $G = g\Omega$, $K = k$, $N = d_i\Omega$. Then the stability conditions in Theorem 3.1 as well as in Theorem 3.2 (with the choice $d = d_e + d_i$) leads to

$$\Omega^2 d_i\big(d_i - (d_e + d_i)g\big) < k(d_e + d_i)^2 \tag{19}$$

which gives the exact stability limit of system (18). Applying the theorems in [18] for comparison leads to results quite far from the exact stability limit.

We will now compare the results achieved by the stability Theorems 1 and 2 with a classical approach.

For this purpose we rewrite condition (7c) in its real form:

$$\begin{bmatrix} DK+KD+GN+NG-2ND^{-1}N & KG-GK+DN-ND \\ GK-KG+ND-DN & DK+KD+GN+NG-2ND^{-1}N \end{bmatrix} > 0.$$

According to Schur's condition this means

$$\begin{aligned} &a) \quad B = DK+KD+GN+NG-2ND^{-1}N > 0, \\ &b) \quad B - \big[(KG-GK)+(DN-ND)\big]B^{-1}\big[(GK-KG)+(ND-DN)\big] > 0. \end{aligned} \tag{20}$$

Condition (20a) can be interpreted as a matrix generalization of classical scalar results using Rayleigh quotients. These results are known at least since Metelitsyn [13] and Frik [14], but only formulated for *real* systems of form (4). A later reference in English is Huseyin [4]. For our purpose we need an extension to *complex* systems of form (5) or (6).

Introducing in equation (6) a solution of form $\nu = u\exp(\lambda t)$ with $u^*u = 1$ (u^* is the conjugate transpose of u) one obtains

$$\big(\lambda^2 I + \lambda(D+iG) + K + iN\big)u = 0. \tag{21}$$

Premultiplying by u^* yields

$$\lambda^2 + (d+ig)\lambda + k + in = 0 \tag{22}$$

where $d > 0$, $g \geq 0$, $k > 0$ and $n \geq 0$ are the Rayleigh quotients of the symmetric matrices $D > 0$, $G \geq 0$, $K > 0$ and $N \geq 0$ respectively. Requiring that both roots λ of equation (22) satisfy $\mathrm{Re}(\lambda) < 0$ (at least one of these roots is an eigenvalue of (21)), results in the following

Lemma 3.2. *System (6) is asymptotically stable if $K > 0$ and $D > 0$ and*

$$kd^2 + dgn - n^2 > 0. \tag{23}$$

Since we are dealing with rotor systems of the form

$$I\ddot{\nu} + (D + i\Omega G_0)\dot{\nu} + (K + i\Omega D_i)\nu = 0,$$

both g and n are proportional to Ω,

$$g = \Omega g_0, \quad n = \Omega d_i. \tag{25}$$

Here $g_0 \geq 0$ and $d_i \geq 0$ are new Rayleigh quotients of the matrices G_0 and D_i. Then (23) implies immediately the following sufficient stability condition for system (24):

$$\Omega^2 d_i(d_i - dg_0) < kd^2. \tag{26}$$

Inequality (26) is a generalization of the stability condition (19) for a simple Laval rotor. Thus we have

Lemma 3.3. *If* $\frac{\max(d_i)}{\min(d)} \leq \min(g_0)$*, then the system (24) is stable for all values of* Ω *(Gyroscopic stabilization).*
If $\frac{\max(d_i)}{\min(d)} > \min(g_0)$*, then the system (24) is stable for all values of* Ω *satisfying*

$$\Omega^2 < \frac{\min(k)\min(d)^2}{\max(d_i)\big(\max(d_i) - \min(d)\min(g_0)\big)}. \tag{27}$$

Putting $\min(g_0) = 0$ in (27), leads to a simplified stability condition for rotor systems

$$\Omega^2 < \frac{\min(k)\min(d)^2}{\max(d_i)^2}. \tag{28}$$

This condition could also be achieved by means of an energy approach, see Kliem and Pommer [16].

Lemma 3.3 has to our knowledge not been mentioned previously in the literature, although it follows straightforward from the classical Rayleigh quotient approach. Notice also that for *real* non-classical systems of form (4) the matrices G_1 and N_1 are skew-symmetric and therefore the Rayleigh quotients ig and in in (22) are limited by the eigenvalues $ig_{\max}, -ig_{\max}$ and $in_{\max}, -in_{\max}$ of maximal absolute value. This makes the use of (23) for real systems less favourable than for rotor systems for which $0 \leq g \leq g_{\max}$ and $0 \leq n \leq n_{\max}$ is valid.

For rotor systems with several degrees of freedom, the two new Theorems 3.1 and 3.2 can yield better stability results than the Lemma 3.3, as we will show in the next examples. Moreover, we will demonstrate by statistical methods that they often do so. The results are listed in table 1 below.

Example 2. Consider the Laval rotor from example 1, not subjected to gyroscopic forces but now additionally with mass m_b, damping d_b and elasticity k_b in the bearings, see Fig. 1.

A linear model is described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\nu} + \begin{bmatrix} (d_e + d_i)/m & -d_i/\sqrt{mm_b} \\ -d_i/\sqrt{mm_b} & (d_b + d_i)/m_b \end{bmatrix} \dot{\nu}$$
$$+ \left\{ \begin{bmatrix} k/m & -k\sqrt{mm_b} \\ -k/\sqrt{mm_b} & (k + k_b)/m_b \end{bmatrix} + i \begin{bmatrix} d_i\Omega/m & -d_i\Omega/\sqrt{mm_b} \\ -d_i\Omega/\sqrt{mm_b} & d_i\Omega/m_b \end{bmatrix} \right\} \nu = 0. \tag{29}$$

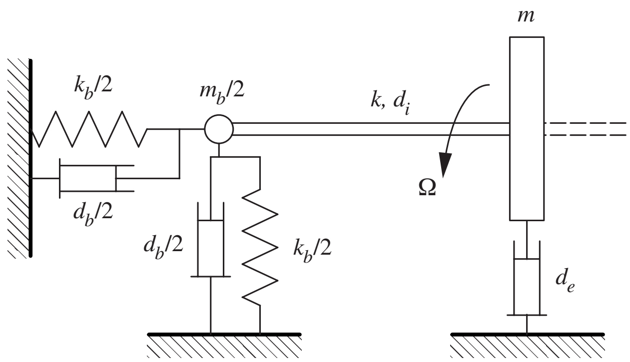

Figure 1.
Physical model of a Laval rotor including mass, elasticity and damping in the bearings.

To compare the stability limits derived by Theorem 3.1 and by Theorem 3.2 where we use $d = d_{\min}$, with that derived by Lemma 3.3, we consider the specific case

$$m = 1, \quad d_i = 1, \quad d_e = 5, \quad d_b = 10, \quad k = 100, \quad k_b = 400$$

and for different bearing masses $m_b = 1$, $m_b = 0.1$ and $m_b = 0.01$. The results are listed in table 1.

Example 3. We tested 1000 systems of form (6) with 4×4 randomly generated matrices, where $D > 0$, $K > 0$ and K was weakly diagonal dominant. Theorem 3.1 was only applicable in 66% of all 1000 cases, but in these cases it gave the considerably better average ratio $\Omega/\Omega_{\text{crit}} < 0.25$ than Theorem 3.2 (with $d = d_{\min}$) or Lemma 3.3. (see table 1).

Finally we have to mention that Theorem 3.1 (in a setting with only real system matrices) can also be deduced from a result for non-classical real systems by Ahmadian and Inman [17]. In this case an equivalent form of the Schur condition is required for the application to rotor systems.

4. Conclusions

Using a complex modelling approach, we have presented sufficient conditions – expressed by properties of the system matrices – for asymptotical stability of a class of dynamical systems (Theorem 3.1 and Theorem 3.2). With help of statistical methods we have demonstrated that the obtained stability limits are significantly better in average than those obtained from classical Rayleigh quotient methods (Lemma 3.3) .

Table 1. Obtained stability limits in fraction of the exact critical frequency Ω_{crit} from example 2 and example 3

System	Theorem 3.1.	Theorem 3.2	Lemma 3.3
Laval rotor $m_b = 1$	$0.48 \cdot \Omega_{\text{crit}}$	$0.50 \cdot \Omega_{\text{crit}}$	$0.23 \cdot \Omega_{\text{crit}}$
Laval rotor $m_b = 0.1$	$0.44 \cdot \Omega_{\text{crit}}$	$0.61 \cdot \Omega_{\text{crit}}$	$0.05 \cdot \Omega_{\text{crit}}$
Laval rotor $m_b = 0.01$	failed $(DK + KD) > 0$ not satisfied	$0.62\Omega_{\text{crit}}$	$0.01\Omega_{\text{crit}}$
Random 4×4 matrices	$0.25 \cdot \Omega_{\text{crit}}$ Applicable in 66% of examples generated	$0.08 \cdot \Omega_{\text{crit}}$	$0.07 \cdot \Omega_{\text{crit}}$

For rotor systems it is frequently the case that the stiffness and damping matrices are diagonally dominant. Then the basic assumption for our main Theorem 3.1 - which involves positive definite property of a combination of these two matrices - is automatically satisfied.

However, if this combination is not positive definite, a slightly modified version of the results still holds (Theorem 3.2). And this result in itself turns out to be quite powerful.

In the case of a simple Laval rotor, the established results yields the exact stability limit.

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