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AVERAGE $\mathcal{H}_2$ PERFORMANCE AND MAXIMAL PARAMETER PERTURBATION RADIUS FOR UNCERTAIN SYSTEMS*

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Abstract. In this paper, methods are presented for calculating the maximal parameter perturbation bounds under $\mathcal{H}_2$ performance constraints for a family of uncertain systems and for calculating the average $\mathcal{H}_2$ performance under such parameter variations. The uncertain systems are described by state space models with nonlinear (polynomial) dependencies on real uncertain parameters. All results obtained are based on necessary and sufficient conditions. As a special virtue of the approach, the proposed algorithms for stability analysis and for performance analysis turn out to have exactly the same algebraic structure. An example illustrates the results and the algorithms.

Key words. $\mathcal{H}_2$ performance, stability, robustness, nonlinear perturbation, state space methods

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1. Introduction.

Robust performance analysis for uncertain control systems, which is now receiving a great deal of attention (see [4, 9] and references therein), is a relatively new area in comparison with robust stability analysis. For linear time-invariant systems, the $\mathcal{H}_2$ performance metric arises naturally in a number of different physically meaningful situations; see [4, 6, 3]. The $\mathcal{H}_2$ performance of a linear time-invariant system is measured via the $\mathcal{H}_2$ norm of its transfer matrix. As long as this $\mathcal{H}_2$ norm is less than a given upper bound, the design can stop, and there is usually no need to seek the minimal norm and/or this might not be advisable due to robustness considerations.

Suppose now that the $\mathcal{H}_2$ norm of a nominal (stable) system is less than a given upper bound. Then the question is whether the norm is still less than this upper bound after suffering a parameter perturbation, or alternatively, how to find the maximal domain for perturbation parameters under stability and $\mathcal{H}_2$ norm constraints.

This paper will consider the latter problem and calculate the maximal perturbation interval or radius in perturbation parameter space. The results obtained are not only sufficient but also necessary. The paper is different from most previously published papers which deal with a fixed parameter domain and affine perturbations. One of our motivations comes from [4], which computed the supremum of the $\mathcal{H}_2$ norm in the case of an affine perturbation with perturbation parameter $q \in [0, 1]$. Also in similarity with that paper we shall compute not only the maximal perturbation radii subject to stability and performance constraints but also the average performance over a fixed perturbation set.

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The notation used throughout the paper is as follows. Denote the real number set by $\mathbf{R}$ and the complex plane (the complex open left half plane) by $\mathbf{C}$ ($\mathbf{C}^-$). Let $\mathbf{cs}$: $\mathbf{R^{m\times n}} \to \mathbf{R^{mn}}$ be the column stacking operator on a matrix and $\oplus$: $\mathbf{R^{n\times n}} \times \mathbf{R^{m\times m}} \to \mathbf{R^{mn\times mn}}$ be the standard matrix Kronecker sum defined in [2]. Finally, let $\lambda_k(\cdot)$ be the kth eigenvalue of a square matrix.

2. Problem formulation. Consider a linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= A(q)x(t) + B(q)w(t), \\ z(t) &= C(q)x(t), \end{aligned} \tag{2.1}$$

where $x \in \mathbf{R^n}$, $w \in \mathbf{R^m}$, and $z \in \mathbf{R^p}$ are state, disturbance, and performance vectors, respectively; $A(q)$, $B(q)$, and $C(q)$ are (of compatible dimension) continuous matrix functions of the perturbation parameter vector $q = [q_1, q_2, \ldots, q_l]^T \in \mathbf{R^l}$. The transfer function matrix from w to z can be expressed as $T(s,q) = C(q)(sI - A(q))^{-1}B(q)$. A square constant matrix is called stable if all of its eigenvalues lie in $\mathbf{C}^-$. The corresponding transfer function $T(s,q)$ is said to be stable for a given q if $A(q)$ is stable and its $\mathcal{H}_2$ norm is defined by

$$\|T(s,q)\|_2 \doteq \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{trace}\,[T(j\omega,q)T^*(j\omega,q)]\, d\omega \right\}^{1/2}, \tag{2.2}$$

where $T^*(s,q) \doteq T'(-s,q)$ and $(\cdot)'$ denotes transpose.

We shall make the following standing assumptions on the nominal system—given by $(A(0), B(0), C(0))$—and on the parameter dependence:

AS1. $A(0)$ is stable.

AS2. $\|T(s,0)\|_2^2 < \gamma$.

AS3. The system matrices may be parameterized as

$$\begin{aligned} A(q) &\doteq A_0 + qA_1 + \cdots + q^{m_1}A_{m_1}, \\ B(q) &\doteq B_0 + qB_1 + \cdots + q^{m_2}B_{m_2}, \\ C(q) &\doteq C_0 + qC_1 + \cdots + q^{m_3}C_{m_3}, \end{aligned}$$

where all of A_k, B_k, and C_k are given constant matrices.

Here, γ is a given positive constant which reflects the tolerance of the system as measured by the $\mathcal{H}_2$ performance (for instance, an acceptable output variance of (2.1) to a white noise signal w). The goal is to find the "maximal domain" in $\mathbf{R^l}$ so that $\|T(s,q)\|_2^2 < \gamma$ for every q in the domain. A prerequisite for doing this is that $A(q)$ must be stable for all q in this domain. This means that the robust stability analysis must be completed first (see relevant results in [1, 5, 7, 8]).

The relevant problems will in this paper only be considered for the single parameter case, i.e., $l = 1$. The two parameter case, $l = 2$, can at least in principle be handled by the approach described below applying a line search. However, for medium or large scale problems, computational issues will limit the practical use of this.

To formulate the problem of determining the maximal perturbation radius, first define

$$r_s^- \doteq \inf\{r < 0 : A(q) \text{ is stable } \forall q \in (r, 0)\}, \tag{2.3}$$

$$r_s^+ \doteq \sup\{r > 0 : A(q) \text{ is stable } \forall q \in (0, r)\}, \tag{2.4}$$

$$r_2^- \doteq \inf\{r < 0 : A(q) \text{ is stable and } \|T(s,q)\|_2^2 < \gamma\ \forall q \in (r, 0)\}, \tag{2.5}$$

$$r_2^+ \doteq \sup\{r > 0 : A(q) \text{ is stable and } \|T(s,q)\|_2^2 < \gamma\ \forall q \in (0, r)\}. \tag{2.6}$$

Then (r_s^-, r_s^+) is the maximal perturbation interval of q while keeping the stability of $A(q)$, and (r_2^-, r_2^+) is the maximal perturbation interval of q while keeping $\|T(s,q)\|_2^2 < \gamma$.

PROBLEM 2.1. *Suppose that system* (2.1) *satisfies* AS1, AS2, *and* AS3.

(a) *Find* r_s^- *and* r_s^+.

(b) *Find* r_2^- *and* r_2^+.

Remark 2.2. Obviously, $(r_2^-, r_2^+) \subset (r_s^-, r_s^+)$.

The notion of average performance can be defined in terms of the following problem.

PROBLEM 2.3. *Suppose that system* (2.1) *satisfies* AS1, AS2, *and* AS3 *and that two numbers* $\underline{q}$ *and* $\overline{q}$ *are given, where* $r_s^- < \underline{q} < \overline{q} < r_s^+$.

Find $\dfrac{1}{\overline{q} - \underline{q}} \int_{\underline{q}}^{\overline{q}} \|T(s,q)\|_2^2 \, dq$.

This definition follows the convention in [4]. It can be argued that an alternative problem formulation similar to Problem 2.3 but without the square would be interesting as well. That problem also admits a solution but not one which is as easy to interpret in terms of the problem parameters as the one given for Problem 2.3 below.

Remark 2.4. The integral boundaries of Problem 2.3 have been chosen to be strictly inside the stability interval (not on the closure). This is because, usually, the integral would become unbounded on the stability boundary.

3. Preliminaries. The main idea in this paper is to transform functions that are rational in the independent variable (the uncertain parameter) into a matrix version of the companion form, utilizing the fact that the "denominator" is based on a matrix valued polynomial map. In what follows, we shall provide a matrix result which will prove useful in this respect.

Let $M(r) = M_0 + rM_1 + \cdots + r^m M_m$, where all of the M_k's are $n \times n$ constant matrices, and $|M_0| \neq 0$ ($|\cdot|$ denotes the determinant). Let

$$r^- \doteq \sup\{r < 0 : |M(r)| = 0\}, \tag{3.1}$$

$$r^+ \doteq \inf\{r > 0 : |M(r)| = 0\} \tag{3.2}$$

be the maximal perturbation bounds for nonsingularity of matrices. By simple operations on the matrix and its determinant (see [8]), it can be shown that

$$r^- = \frac{1}{\lambda_{min}^-(\mathbf{M})}, \tag{3.3}$$

$$r^+ = \frac{1}{\lambda_{max}^+(\mathbf{M})}, \tag{3.4}$$

where $\mathbf{M}$ is an $mn \times mn$ matrix given by

$$\mathbf{M} \doteq -\begin{pmatrix} \mathbf{O} & -\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\mathbf{I} \\ M_0^{-1}M_m & M_0^{-1}M_{m-1} & M_0^{-1}M_{m-2} & \cdots & M_0^{-1}M_1 \end{pmatrix}, \tag{3.5}$$

$\lambda_{min}^-(\cdot)$ stands for the minimal value of the negative real eigenvalues (let $\lambda_{min}^-(\cdot) = 0^-$ if there exist no negative real eigenvalues) and $\lambda_{max}^+(\cdot)$ stands for the maximal value

of the positive real eigenvalues (let $\lambda^+_{max}(\cdot) = 0^+$ if no positive real eigenvalues), respectively.

Formulae (3.3) and (3.4) suggest the following algorithm.

ALGORITHM 3.1 (the maximal perturbation bounds for nonsingularity of matrices).

Step 1. *Input* M_k, $k = 0, 1, \ldots, m$, *where* $|M_0| \neq 0$;
Step 2. *Define* $\mathbf{M}$ *as in* (3.5);
Step 3. *Calculate all the eigenvalues of* $\mathbf{M}$;
Step 4. *Find* r^- *and* r^+ *based on* (3.3) *and* (3.4), *then output.*

Algorithm 3.1 will be one of the cornerstones below in solving Problems 2.1 and 2.3. Algorithm 3.1 is conceptually clear and easy to implement, although, admittedly, the numerical aspects can be quite involved for large scale problems, since the relevant matrices will be of very high order. Hence, the main applications for the results below will be in terms of small or medium scale problems.

The following lemma helps us to transform Problem 2.1(a) into that of the maximal perturbation bounds for the nonsingularity of matrices.

LEMMA 3.2. *Suppose that*
(1) Q *is a singly connected domain in* $\mathbf{R^1}$, *and* $0 \in Q$,
(2) $A(0)$ *is stable.*
Then $A(q)$ *are stable* $\forall q \in Q$ *if and only if* $|A(q) \oplus A(q)| \neq 0 \, \forall q \in Q$.

Proof. Recall the continuity of $A(q)$, $B(q)$, $C(q)$ in q and that

$$\lambda_k(A(q) \oplus A(q)) = \lambda_i(A(q)) + \lambda_j(A(q)), \qquad k = 1, \ldots, n^2;\ i, j = 1, \ldots, n.$$

From this observation the lemma becomes obvious. □

By using Lemma 3.2 it follows that

$$r_s^- = \sup\{q < 0 : |A(q) \oplus A(q)| = 0\} \qquad \text{(scalar case)}, \tag{3.6}$$

$$r_s^+ = \inf\{q > 0 : |A(q) \oplus A(q)| = 0\} \qquad \text{(scalar case)}, \tag{3.7}$$

$$r_s = \inf\{r : |A(q) \oplus A(q)| = 0 \quad \text{for some } q,\ \|q\| \leq r\} \qquad \text{(multiparameter case)}. \tag{3.8}$$

Instead of (2.2) in the frequency domain, use the state space approach to compute

$$\|T(s, q)\|_2^2 = \mathbf{trace}\{C'(q)C(q)Q(q)\},$$

where $Q(q) = Q(q)'$ satisfies

$$A(q)Q(q) + Q(q)A(q)' + B(q)B(q)' = 0.$$

It is easy to show the following compact formula (or see [4]):

$$\|T(s, q)\|_2^2 = -\mathbf{cs}[C'(q)C(q)]' \cdot [A(q) \oplus A(q)]^{-1} \cdot \mathbf{cs}[B(q)B'(q)]. \tag{3.9}$$

Going one step from (3.9), the following result is obtained, which helps transform Problem 2.1(b) into that of the maximal perturbation bounds for nonsingularity of matrices.

LEMMA 3.3. *Suppose that*
(1) Q *is a singly connected domain in* $\mathbf{R^1}$, *and* $0 \in Q$,
(2) $A(q)$ *are stable* $\forall q \in Q$,
(3) $\|T(s, 0)\|_2^2 < \gamma$.

Then $\|T(s,q)\|_2^2 < \gamma \ \forall \ q \in Q$ *if and only if* $|\mathbf{M}_\gamma(q)| \neq 0 \, \forall q \in Q$, *where*

$$\mathbf{M}_\gamma(q) \doteq A(q) \oplus A(q) + \frac{1}{\gamma}\mathbf{cs}[B(q)B'(q)] \cdot \mathbf{cs}[(C'(q)C(q)]'. \tag{3.10}$$

Proof. $\|T(s,q)\|_2^2 < \gamma \ \forall q \in Q$
$\Leftrightarrow \gamma + \mathbf{cs}[C'(q)C(q)]' \cdot [A(q) \oplus A(q)]^{-1} \cdot \mathbf{cs}[B(q)B'(q)] > 0 \, \forall q \in Q$ (from (3.9));
$\Leftrightarrow |\gamma I + [A(q) \oplus A(q)]^{-1} \cdot \mathbf{cs}[B(q)B'(q)] \cdot \mathbf{cs}[C'(q)C(q)]'| > 0 \, \forall q \in Q$ (use equality $|\gamma I + XY| = |\gamma I + YX|$);
$\Leftrightarrow |\gamma[A(q) \oplus A(q)]^{-1}| \cdot |\mathbf{M}_\gamma(q)| > 0 \, \forall q \in Q$ (from (3.10));
$\Leftrightarrow |\mathbf{M}_\gamma(q)| \neq 0 \, \forall q \in Q$ (due to the continuity of $A(q), B(q), C(q)$ to q, and Lemma 3.2).
The rest of the proof is trivial and thus omitted. ∎

By using Lemma 3.3 we obtain the following formulae:

$$r_2^- = \sup\{q \in (r_s^-, 0) : |\mathbf{M}_\gamma(q)| = 0\} \qquad \text{(scalar case)}, \tag{3.11}$$

$$r_2^+ = \inf\{q \in (0, r_s^+) : |\mathbf{M}_\gamma(q)| = 0\} \qquad \text{(scalar case)}, \tag{3.12}$$

$$r_2 = \inf\{r : r \leq r_s \text{ and } |\mathbf{M}_\gamma(q)| = 0 \quad \text{for some } q, \|q\| \leq r\} \qquad \text{(multiparameter case)}. \tag{3.13}$$

In section 2 we presented two types of problems. One is the maximal perturbation bounds for system stability; the other is the maximal perturbation bounds for system performance. Lemmas 3.2 and 3.3 help us to transform these two into the maximal perturbation bounds for nonsingularity of matrices. This means that the resulting algorithms will be similar in spirit.

4. Maximal stability and performance radii. This section will describe the main formulae and algorithms.

By using matrix multiplication and the expressions of $A(q)$, $B(q)$, $C(q)$ in Problem 2.1, it can be seen that

$$A(q) \oplus A(q) = \mathbf{A}_0 + q\mathbf{A}_1 + \cdots + q^{m_1}\mathbf{A}_{m_1}, \tag{4.1}$$

$$\mathbf{cs}[B(q)B'(q)] = \mathbf{b}_0 + q\mathbf{b}_1 + \cdots + q^{2m_2}\mathbf{b}_{2m_2}, \tag{4.2}$$

$$\mathbf{cs}[C'(q)C(q)] = \mathbf{c}_0 + q\mathbf{c}_1 + \cdots + q^{2m_3}\mathbf{c}_{2m_3}, \tag{4.3}$$

where

$$\mathbf{A}_k = A_k \oplus A_k, \ k = 0, 1, \ldots, m_1,$$

$$\mathbf{b}_0 = \mathbf{cs}\left[B_0 B_0'\right], \ \ldots, \mathbf{b}_k = \mathbf{cs}\left[\sum_{i+j=k} B_i B_j'\right], \ldots, \mathbf{b}_{2m_2} = \mathbf{cs}\left[B_{m_2} B_{m_2}'\right],$$

$$\mathbf{c}_0 = \mathbf{cs}\left[C_0' C_0\right], \ \ldots, \mathbf{c}_k = \mathbf{cs}\left[\sum_{i+j=k} C_i' C_j\right], \ldots, \mathbf{c}_{2m_3} = \mathbf{cs}\left[C_{m_3}' C_{m_3}\right].$$

Substituting the above expressions for $A(q)$, $B(q)$, and $C(q)$ in (3.10), it can be written then as

$$\mathbf{M}_\gamma(q) = M_{0\gamma} + qM_{1\gamma} + \cdots + q^m M_{m\gamma}, \tag{4.4}$$

where $m = \max\{m_1, 2(m_2 + m_3)\}$ and

$$M_{0\gamma} = (A_0 \oplus A_0) + \frac{1}{\gamma}\mathbf{cs}\left[B_0 B_0'\right] \cdot \mathbf{cs}\left[C_0' C_0\right]', \tag{4.5}$$

and all of other $M_{k\gamma}$ (the detailed expressions are omitted) depend on $\mathbf{A}_i$, $\mathbf{b}_j$, and $\mathbf{c}_k$ in a similar fashion.

By recalling Algorithm 3.1 and using (3.6), (3.7), and (4.1), the following result is obtained.

THEOREM 4.1. *Assume that the system* (2.1) *satisfies* AS1, AS2, *and* AS3. *Then the following two statements are equivalent:*

(1) *system* (2.1) *is stable* $\forall\ |q| < \delta$,

(2) $\min\{-r_s^-, r_s^+\} > \delta$.

To compute the maximal perturbation stability bounds, we can devise the following algorithm from the above results.

ALGORITHM 4.2 (the maximal perturbation bounds for Problem 2.1(a)).

Step 1. *Input* A_k, $k = 0, 1, \ldots, m$, *where* A_0 *must be stable*;

Step 2. *Calculate* $\mathbf{A}_k$, $k = 0, 1, \ldots, m_1$;

Step 3. *Let* $M_k = \mathbf{A}_k$, *recall Algorithm* 3.1, *then compute* r^- *and* r^+;

Step 4. *Let* $r_s^- = r^-$ *and* $r_s^+ = r^+$, *and output.*

From AS2, Lemma 3.3, and (4.5), it can be shown that $|M_{0\gamma}| \neq 0$. By recalling Algorithm 3.1 and using (3.11), (3.12), and (4.4), the following result is obtained.

THEOREM 4.3. *Assume that the system* (2.1) *satisfies* AS1, AS2, *and* AS3. *Then the following two statements are equivalent:*

(1) $\|T(s,q)\|_2 < \gamma\ \forall\ |q| < \delta$,

(2) $\min\{-r_s^-, r_s^+\} > \delta$.

Similarly, to compute the maximal perturbation performance bounds, we can devise the following algorithm from the above results.

ALGORITHM 4.4 (the maximal perturbation bounds for Problem 2.1(b)).

Step 1. *Input* A_i, B_j, *and* C_k, *where we must have AS*1 *and AS*2;

Step 2. *Calculate* $\mathbf{A}_i$, $\mathbf{b}_j$, *and* $\mathbf{C}_k$, *and also* m;

Step 3. *Calculate* $M_{k\gamma}$;

Step 4. *Let* $M_k = M_{k\gamma}$, *and recall Algorithm* 3.1 *to get* r^- *and* r^+;

Step 5. *Output* $r_2^- = \max\{r_s^-, r^-\}$, $r_2^+ = \min\{r_s^+, r^+\}$.

Remark 4.5. Algorithms 4.2 and 4.4 do not need any iteration.

Reference [5] gave the maximal perturbation bounds for Problem 2.1(a) in the simplest case (affinely linear perturbation of a single parameter).

5. Average $\mathcal{H}_2$ performance.

To compute the average performance we follow the line of approach of [4]. In similarity with that approach we shall further assume that $B(q)$ and $C(q)$ are fixed matrices, i.e., we have the following uncertainty structure.

AS4. The system matrices may be parameterized as

$$\begin{aligned} A(q) &\doteq A_0 + qA_1 + \cdots + q^{m_1} A_{m_1}, \\ B(q) &\doteq B_0, \\ C(q) &\doteq C_0. \end{aligned}$$

This assumption can be lifted at the cost of more complicated expressions. These, though, can be obtained easily for a specific application, for instance, by the use of a symbolic algebra package, and the more general result is straightforward, following the idea below.

We define the following matrix:

$$\mathbf{A} \doteq \begin{pmatrix} \mathbf{O} & -\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & -\mathbf{I} \\ \mathbf{A}_0^{-1}\mathbf{A}_{m_1} & \mathbf{A}_0^{-1}\mathbf{A}_{m_1-1} & \mathbf{A}_0^{-1}\mathbf{A}_{m_1-2} & \cdots & \mathbf{A}_0^{-1}\mathbf{A}_1 \end{pmatrix},$$

where, as above, $\mathbf{A}_k = A_k \oplus A_k,\ k = 0, 1, \ldots, m_1$. Note that $\mathbf{A}_0$ is invertible due to assumption AS1. Also define

$$\mathbf{B} \doteq \begin{pmatrix} 0 \\ \vdots \\ \mathbf{A}_0^{-1}\mathbf{cs}(B_0 B_0') \end{pmatrix}.$$

Finally, we need

$$\mathbf{C} \doteq \begin{pmatrix} 0 & \ldots & 0 & \mathbf{cs}\,(C_0' C_0)' \end{pmatrix}.$$

With these definitions, we can obtain the following result for the *average* $\mathcal{H}_2$ performance of the parameter dependent system.

THEOREM 5.1. *Assume $A(q)$, $B(q)$, and $C(q)$ are as described in* AS4 *with $A(0)$ stable. Let $\underline{q}$ and $\overline{q}$ be two real numbers satisfying $r_s^- < \underline{q} < \overline{q} < r_s^+$, where r_s^- and r_s^+ are as defined in* (2.3) *and* (2.4)*. Then*

$$\frac{1}{\overline{q} - \underline{q}} \int_{\underline{q}}^{\overline{q}} \|T(s,q)\|_2^2 \, dq = -\frac{1}{\overline{q} - \underline{q}} \mathbf{C}\mathbf{A}^{-1} \left(\log(I + \overline{q}\mathbf{A}) - \log(I + \underline{q}\mathbf{A}) \right) \mathbf{B},$$

where $\log(\cdot)$ *denotes the matrix logarithm, i.e., the inverse of the matrix exponential.*

Proof. It is straightforward using (3.9) to show that

$$\|T(s,q)\|_2^2 = -\mathbf{C}(I + q\mathbf{A})^{-1}\mathbf{B}.$$

Hence,

$$\int \|T(s,q)\|_2^2 \, dq = -\mathbf{C} \left(\int (I + q\mathbf{A})^{-1} \, dq \right) \mathbf{B} = -\mathbf{C} \left(\mathbf{A}^{-1} \log(I + q\mathbf{A}) \right) \mathbf{B}. \tag{5.1}$$

The last equality holds whenever the argument of the logarithm is a nonsingular matrix. This condition, however, is fulfilled in any open subset of (r_s^-, r_s^+) due to (3.6) and (3.7). ∎

In certain nongeneric cases (where controllability or observability is lost), it might make sense to extend the calculation of average performance to the boundaries of stability. In that case, the integral in (5.1) becomes more involved. Indeed, let T be a nonsingular matrix, such that

$$T^{-1}\mathbf{A}T = \begin{bmatrix} \tilde{\mathbf{A}} & 0 \\ 0 & \mathbf{A}_0 \end{bmatrix},$$

where $\tilde{\mathbf{A}}$ is nonsingular and $\mathbf{A}_0$ is nilpotent of order k. (One possibility is to compose $\tilde{\mathbf{A}}$ and $\mathbf{A}_0$ by Jordan blocks and to choose the columns of T as the corresponding generalized eigenvectors.)

Then it is easy to show that

$$\int \|T(s,q)\|_2^2 \, dq = -\mathbf{C}T \begin{bmatrix} \tilde{\mathbf{A}}^{-1} \log(I + q\tilde{\mathbf{A}}) & 0 \\ 0 & qI + \sum_{i=1}^{k-1} (-1)^i \frac{q^{i+1}}{i+1} \mathbf{A}_0^i \end{bmatrix} T^{-1}\mathbf{B}.$$

6. Example. An example with a single perturbation parameter is cited below. Let

$$A(q) = \begin{bmatrix} -2 & 1 \\ 0 & -1.5 \end{bmatrix} + q \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + q^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + q^3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$B(q) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \; C(q) = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

It is easy to show that

$$A_0 = \begin{bmatrix} -2 & 1 \\ 0 & -1.5 \end{bmatrix}$$

is stable, that

$$T(s,0) = \begin{bmatrix} \frac{1}{s+2} & \frac{s+3}{(s+2)(s+1.5)} \end{bmatrix},$$

and that $\|T(s,0)\|_2^2 \approx 0.8214 < 1 = \gamma$. In this example it may be shown that

$$A(q) \oplus A(q) = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 0 & -3.5 & 0 & 1 \\ 0 & 0 & -3.5 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} + q \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + q^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} + q^3 \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{cs}[B(q)B'(q)] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + q \begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \end{bmatrix} + q^2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix},$$

and

$$\mathbf{cs}[C'(q)C(q)] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Furthermore,

$$\mathbf{M}_\gamma(q) = \begin{bmatrix} -3 & 2 & 2 & 1 \\ 0 & -3.5 & 0 & 1 \\ 0 & 0 & -3.5 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix} + q \begin{bmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 4 & 4 & 4 & 4 \end{bmatrix} + q^2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 5 & 6 & 6 & 5 \end{bmatrix} + q^3 \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Finally, $(r_s^-, r_s^+) = (-1.6710, 0.7683)$ can be calculated, which shows that the family $A(q)$ is stable $\forall\ q \in (-1.6710, 0.7683)$, and $(r_2^-, r_2^+) = (-1.5670, 0.0442)$, meaning that $\|T(s,q)\|_2^2 < 1\ \forall\ q \in (-1.5670, 0.0442)$. These two intervals are furthermore the largest intervals with these properties.

If now, in compliance with Assumption AS4, we fix the input matrix

$$B(q) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we obtain a larger performance interval: $\left(r_2^-, r_2^+\right) = (-1.6668, 0.3182)$, where the $\mathcal{H}_2$ norm is bounded by 1. Moreover, in that interval, the average performance can be expressed in terms of

$$\begin{aligned} \sqrt{\int_{r_2^-}^{r_2^+} \|T(s,q)\|_2^2\, dq} &= \sqrt{-\frac{1}{r_2^+ - r_2^-}\mathbf{C}\mathbf{A}^{-1}\left(\log(I + r_2^+\mathbf{A}) - \log(I + r_2^-\mathbf{A})\right)\mathbf{B}}, \\ &\approx 0.7428 \end{aligned}$$

which for this case is in fact better than the nominal performance, $\|T(s,0)\|_2 \approx 0.9063$!

7. Conclusions. Methods for calculating the maximal parameter-perturbation bounds under $\mathcal{H}_2$ performance constraints for a family of systems described by state space models, with nonlinear dependence on real uncertain parameters, have been presented. The results are not conservative as the information of the system structure is used completely. The algorithms as presented here, for robust performance radii and for stability radii, are algebraically similar in nature. Finally, an explicit expression for average $\mathcal{H}_2$ performance for an uncertainty interval also has been presented.

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