APPLICATION OF AN $\mathcal{H}_\infty$ BASED FDI AND CONTROL SCHEME FOR THE THREE TANK SYSTEM

Jakob Stoustrup [*] **Henrik Niemann** [**]

[*] *Dept. of Control Engineering, Aalborg University, DK-9220 Aalborg, Denmark*
[**] *Dept. of Automation, Technical University of Denmark, DK-2800 Lyngby, Denmark*

Abstract:
The three tank benchmark system is considered in this paper in connection with combined feedback control and fault detection and identification (FDI). The combined design problem is formulated as an $\mathcal{H}_\infty$ design problem by using a standard system setup.

Keywords: Fault identification, H-infinity optimization, benchmark examples, feedback control.

1. INTRODUCTION

The $\mathcal{H}_\infty$ design method has been considered in connection with design of fault detectors for dynamic systems in a number of papers, see e.g. (Chung and Speyer, 1998; Edelmayer *et al.*, 1994; Edelmayer *et al.*, 1996; Edelmayer *et al.*, 1997; Mangoubi *et al.*, 1995; Niemann and Stoustrup, 1996; Qiu and Gertler, 1993; Sauter *et al.*, 1997) to mention a few.

The $\mathcal{H}_\infty$ design method has been applied to obtain feedback controllers as well as filters/observers that are robust against model uncertainties, (Zhou *et al.*, 1996). In the same way, the $\mathcal{H}_\infty$ design method has been used in connection with design of robust fault detectors. By using the $\mathcal{H}_\infty$ design method in connection with design of fault detectors, we get a systematic and well understood method to handle the robustness problem in fault detectors. Furthermore, today a number of numerical algorithms for the $\mathcal{H}_\infty$ design methods are available as e.g. the μ toolbox from MATLAB™, (Balas *et al.*, 1993).

Based on the above, it is naturally to consider a combined design of both feedback controller and residual generator using an $\mathcal{H}_\infty$ method. Integration of feedback controller design and residual generator design has been considered in a number of papers, see e.g. (Nett *et al.*, 1988; Stoustrup *et al.*, 1997; Suzuki and Tomizuka, 1999).

The rest of this paper is organized as follows. In Section 2, the three tank system is described. Both a nonlinear as well as a linear model is given together with a description of a number of different faults in the system. In Section 3, the combined feedback controller and residual generator setup is introduced/formulated as a standard $\mathcal{H}_\infty$ design problem. An $\mathcal{H}_\infty$ design of residual generator/feedback controller for the three tank benchmark problem is given in Section 4 together with some simulation results. A conclusion is given in Section 5.

2. THE THREE TANK SYSTEM

The three tank system consists of three cylindrical tanks with the same diameter, which are interconnected by two pipes. See e.g. (Köppen-Seliger *et al.*, 1999; Ding *et al.*, 1999) for a description of the three tank system. The first and the last tank can be filled with water by two pumps. A nominal outflow is located at the last tank. The aim is to control the water level in the first and the last tank adjusting the flows to the first and the last tank. The mathematical model of this system can be described as

$$
\begin{aligned}
A\frac{dh_1}{dt} &= Q_1 - Q_{13} - Q_{L1} \\
A\frac{dh_2}{dt} &= Q_2 + Q_{32} - Q_{20} - Q_{L2} \\
A\frac{dh_3}{dt} &= Q_{13} - Q_{32} - Q_{L3}
\end{aligned} \tag{1}
$$

where

$$
\begin{aligned}
Q_{13} &= (1 - F_{B13})Q_{13u} \\
Q_{32} &= (1 - F_{B32})Q_{32u} \\
Q_{20} &= (1 - F_{B20})Q_{20u} \\
Q_{13u} &= \mu_1 S_n sign(h_1 - h_3)\sqrt{2g|h_1 - h_3|} \\
Q_{32u} &= \mu_3 S_n sign(h_3 - h_2)\sqrt{2g|h_3 - h_2|} \\
Q_{20u} &= \mu_2 S_n \sqrt{2gh_2} \\
Q_{L1} &= F_{L1}\mu_L S_L \sqrt{2gh_1} \\
Q_{L2} &= F_{L2}\mu_L S_L \sqrt{2gh_2} \\
Q_{L3} &= F_{L3}\mu_L S_L \sqrt{2gh_3}
\end{aligned}
$$

Q_1 and Q_2 denotes the pump inputs to the system, which are the control signals. h_1, h_2 and h_3 denotes the water levels in the three tanks, respectively, which are the measured outputs. A is the cross section area of the tanks. S_n denotes the cross section areas of the connecting pipes between the tanks. S_L means the maximal cross section area of the leakage Q_{Li}. μ_i and μ_L are some constant coefficients. $F_{Li}, i = 1,2,3$ represent leakage faults in tank 1, tank 2 and tank 3, respectively. F_{B13}, F_{B32} and F_{B20} represent clogging faults in the pipes between tank 1 and tank 3, tank 3 and tank 2, and the outlet, respectively.

A linearized model is considered. Let the operating point be given by $h_1 = 0.5$m, $h_2 = 0.2$m and $h_3 = 0.35$m. A state space model is then given by

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + B_f f(t) \\
z(t) &= C_1 x(t) \\
y(t) &= C_2 x(t)
\end{aligned} \tag{2}
$$

where the state $x(t)$ is defined by

$$
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \Delta h_1(t) \\ \Delta h_2(t) \\ \Delta h_3(t) \end{bmatrix}
$$

where $\Delta h_1(t)$, $\Delta h_2(t)$ and $\Delta h_3(t)$ are the level deviations with respect to the operating point of tank 1, tank 2 and tank 3, respectively. $u(t)$ is the control input, $f(t)$ is the fault vector, $z(t)$ are the states to be controlled and $y(t)$ is the measurement output. The matrices are given by

$$
\begin{aligned}
A &= \begin{bmatrix} -\gamma_1 & 0 & \gamma_1 \\ 0 & \gamma_2 & \gamma_1 \\ \gamma_1 & \gamma_1 & \gamma_3 \end{bmatrix} \\
B &= \begin{bmatrix} \gamma_4 & 0 \\ 0 & \gamma_4 \\ 0 & 0 \end{bmatrix} \\
B_f &= \begin{bmatrix} \gamma_5 & 0 & 0 \\ 0 & \gamma_5 & \gamma_6 \\ \gamma_5 & \gamma_5 & 0 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
C_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

The elements in the matrices are given by

$$
\begin{aligned}
\gamma_1 &= 9.28 \times 10^{-3} \\
\gamma_2 &= -1.893 \times 10^{-2} \\
\gamma_3 &= -1.856 \times 10^{-2} \\
\gamma_4 &= 64.935 \\
\gamma_5 &= 2.78 \times 10^{-3} \\
\gamma_6 &= 3.2 \times 10^{-3}
\end{aligned}
$$

The fault vector f given by

$$
f = \begin{bmatrix} F_{B13} \\ F_{B32} \\ F_{B20} \end{bmatrix}
$$

Only clogging faults are included in the above linear model, tank leakage faults are not included. However, in the linear model, it is possible to describe the leakage faults as linear combinations of the clogging faults. Further, it is also possible to include sensor faults in the setup. This means that the maximal number of independent faults in the linear model of the three tank system is 6.

3. CONTROL AND FDI - AN $\mathcal{H}_\infty$ FORMULATION

A combined design of feedback controller and residual generator based on the $\mathcal{H}_\infty$ design method will be described in the following.

Let a general model be given by

$$
\begin{aligned}
z_c(t) &= G_{z_c d} d(t) + G_{z_c u_c} u_c(t) + G_{z_c f} f(t) \\
y(t) &= G_{yd} d(t) + G_{yu_c} u_c(t) + G_{yf} f(t)
\end{aligned}
$$

$d(t) \in \mathcal{R}^m$ is a disturbance signal vector, $u_c(t) \in \mathcal{R}^p$ is the control input vector, $f(t) \in \mathcal{R}^q$ is the fault vector, $z_c(t) \in \mathcal{R}^p$ is the external output vector, and $y(t) \in \mathcal{R}^r$ is the measurement vector. The fault vector f describe both actuator faults, sensor faults and internal faults that can be described as additive faults, see (Stoustrup *et al.*, 1997).

The $\mathcal{H}_\infty$ design problem for a combined feedback controller and residual generator can be formulated in terms of the following two design conditions:

- *The control part*: The $\mathcal{H}_\infty$ norm of the weighted closed loop transfer function from external input d to external output z_c is required to be smaller than or equal to a specified level, i.e. $\|W_{c1}T_{z_cd}W_{c2}\|_\infty \leq \gamma_f$, where T_{z_cd} is the closed loop transfer function from d to z_c and W_{ci} are the two weighting matrices.
- *The FDI residual generator part*: The $\mathcal{H}_\infty$ norm of the weighted transfer function from the vector $v = Vf$ to the residual error $z_e = Vf - r$ is required to be smaller than or equal to a specified level, i.e. $\|W_{e1}T_{z_ef}W_{e2}\|_\infty \leq \gamma_f$, where T_{z_ef} is the closed loop transfer function from the fault vector f to the residual error vector z_e, W_{ei} are the two weighting matrices and V is a weighting matrix of the fault signal.

It is here important to point out that the FDI problem to be solved depends strongly on the selected structure of the weight matrix V. When V is the identity matrix, $V = I$, the design problem is a fault estimation problem. If V has full rank q, but is not the identity matrix, the problem is a fault identification problem. At last, if V is a $1 \times q$ matrix, the problem is a fault detection problem. Furthermore, V is not required to be a static matrix. Typically, dynamics will be included in V in identification problems.

Without loss of generality, it will be assumed in the following that $\gamma_c = \gamma_f =: \gamma$. This can always be obtained by proper scaling of the weighting matrices.

To derive the standard system setup for the combined controller and residual generator design, let us define the following external input and output vectors

$$\begin{aligned} w(t) &= \begin{bmatrix} d(t) \\ f(t) \end{bmatrix} \\ z(t) &= \begin{bmatrix} z_c(t) \\ Vf(t) - r(t) \end{bmatrix} \end{aligned} \tag{3}$$

Based on these two vectors, the system setup for the combined design is given by:

$$\begin{aligned} z(t) &= \begin{pmatrix} G_{z_cd}(t) & G_{z_cf}(t) \\ 0 & V \end{pmatrix} w(t) \\ &\quad + \begin{pmatrix} G_{z_cu_c}(t) & 0 \\ 0 & -I \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} G_{yd}(t) & G_{yf}(t) \end{pmatrix} w(t) \\ &\quad + \begin{pmatrix} G_{yu_c}(t) \\ 0 \end{pmatrix} u(t) \end{aligned} \tag{4}$$

Including weighting matrices in the setup as described in the design conditions given above, the system setup turns out to be given by

$$\begin{aligned} \tilde{z}(t) &= \begin{pmatrix} W_{c1}G_{z_cd}W_{c2} & W_{c1}G_{z_cf}W_{e2} \\ 0 & W_{e1}VW_{e2} \end{pmatrix} \tilde{w}(t) \\ &\quad + \begin{pmatrix} W_{c1}G_{z_cu_c} & 0 \\ 0 & -W_{e1} \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} G_{yd}W_{c2} & G_{yf}W_{e2} \end{pmatrix} \tilde{w}(t) \\ &\quad + \begin{pmatrix} G_{yu_c} \\ 0 \end{pmatrix} u(t) \end{aligned}$$

Designing a controller for the above system in (4) gives, directly provides a abstract control vector $u(t)$ of the form:

$$u(t) = \begin{pmatrix} u_c(t) \\ r(t) \end{pmatrix}$$

where $u_c(t)$ is the real control signal and $r(t)$ is the residual vector. This controller will therefore handle both the feedback control problem as well as the FDI problem.

Using an $\mathcal{H}_\infty$ design method for the design of a controller for the weighted system will not only give a controller that will satisfy the closed-loop conditions given above, if such is possible. The $\mathcal{H}_\infty$ norm of the two cross coupling closed-loop transfer functions will also be bounded by γ. An $\mathcal{H}_\infty$ design will give:

- Closed-loop transfer function from disturbance vector d to z_c denoted T_{z_cd}: $\|T_{z_cd}\|$ small implies good disturbance rejection.
- Closed-loop transfer function from fault vector f to z_c denoted T_{z_cf}: $\|T_{z_cf}\|$ small implies that small, undetected faults will not influence the output.
- Closed-loop transfer function from disturbance vector d to z_e denoted T_{z_ed}: $\|T_{z_ed}\|$ small reduces the number of false alarms.
- Closed-loop transfer function from fault vector f to z_e denoted T_{z_ef}: $\|T_{z_ef}\|$ small gives good fault detection/identification/estimation of the fault vector.

It has been pointed out in (Stoustrup *et al.*, 1997), that a combined feedback controller and fault estimator design problem can be separated into a feedback controller design problem and a fault estimator design problem. However, it is not necessary obviously that two separate design will give the best controller and residual generator. Using a standard $\mathcal{H}_\infty$ design method will give a controller/residual generator of order n in the combined design and $2n$ in the separate design. Furthermore, it is also clearer how to handle the cross coupling terms in the combined design.

In most FDI schemes the direct influence of faults on the outputs is not of significant importance, since the operator or a supervisor is intended to take action in the event of faults.

However, it should be noted that since the proposed method is able to design a controller that minimizes the cross coupling transfer function from fault vector to external output z_c, T_{z_cf}, implies that the method also comprises an approach to *fault tolerant* systems. At least in the case where some faults might not be detectable by the sensor signals, it is highly important to design the closed loop system to be fault tolerant against such faults.

A block diagram for the combined control/fault estimation problem is given in Figure 1.

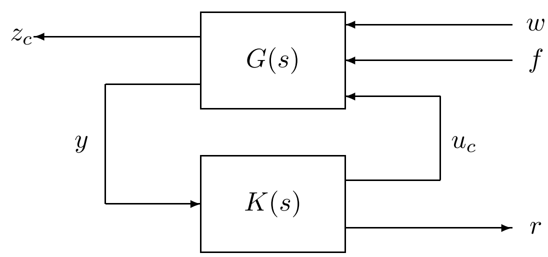

Fig. 1. Generalized setup for control and fault detection

4. APPLICATION TO THE THREE TANK SYSTEM

The above described design method for feedback controller/residual generator is applied on the three tank benchmark problem. In this paper, the design of a feedback controller/fault estimator is considered.

The design derived for the three tank benchmark problem is not optimized with respect to the robustness of the feedback loop. The main idea with the example is to show how simple it is to apply the the derived design method. For obtaining a robust design, first we need to calculate the model uncertainties for the linear model based on the nonlinear model given in Section 2. Then, based on this model description, dynamic weighting matrices can be calculated for the following $\mathcal{H}_\infty$ design.

Another important thing in connection with the benchmark problem is the fact that the fault signals enter all the states in the model. As a result of this, it will not in general be possible to reject disturbance in the residual signals (fault estimates) and at the same time get good fault estimates. Therefor, threshold values need to be derived for the residual signals with respect to the disturbance inputs. With respect to the feedback controller, rejection of the disturbance inputs will also at the same time reject/minimize the effect from the fault signals on the system. This mean that we get a more robust feedback system with respect to the fault inputs. In this example, disturbance inputs will not be considered.

In connection with the design of the $\mathcal{H}_\infty$ feedback controller, we have full state information. This mean that it is possible to design a static state feedback controller. Further, it is also possible to include weighting matrices at the output in the system and still use the $\mathcal{H}_\infty$ state feedback design method. This has been considered in (Stoustrup and Niemann, 1993). In this example, however, a normal full order $\mathcal{H}_\infty$ controller will be applied.

The important part of the design is the selection of the weighting matrices. As said above, the feedback design has not been optimized, the focus has only been on the fault estimation part of the controller. The weighting matrices that need to be applied in connection with the fault estimation part need to be selected such that the fault is scaled to the same level. In this case, it was enough to scale the fault estimation errors individual by constant weights. The weight used in this design is

$$W_{e1} = \begin{bmatrix} 600 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

which will give fault estimation error in the same level.

The standard $\mathcal{H}_\infty$ design method has been applied for the design of the feedback controller/fault estimator, see e.g. (Zhou *et al.*, 1996), though this has required some care. The regular/standard $\mathcal{H}_\infty$ design method can not be applied directly on the three tank system, since the rank conditions for the direct feed through matrices are not satisfied. This problem has been handled by using the cheap control formulation of the problem, where the direct feed through matrices are perturbed, such that the rank conditions are satisfied, see e.g. (Niemann and Stoustrup, 1993; Saberi and Sannuti, 1987). Another problem in connection with using an $\mathcal{H}_\infty$ design methods is the formulation of the fault estimation problem. As shown in Section 3, the transfer function from the fault vector f to the estimation error z_e will include a direct term, i.e. D_{11} is non zero. As a result of this, a lower bond on the $\mathcal{H}_\infty$ norm of the closed loop transfer function from external input to external output is given by:

$$\gamma_{min} \geq \|D_{11}\|_\infty$$

In this case, $D_{11} = I$. This implies that γ will be larger than or equal to 1. This will give a fault estimator where the estimation error can be more than 100%. This problem can be handled in two ways. Either, the fault estimation error can be filtered by a low-pass filter, i.e.

$$z_{ef} = W_{e1}(s) z_e$$

where W_{e1} is a low pass wighting matrix, see Section 3. This will remove the direct feed through term D_{11}. In this example, a constant W_{e1} has been selected as shown above. The other method is to model the fault signal as the output signal from a low pass dynamic system, i.e.

$$f = W_{e2}\tilde{f}$$

In this case, W_{e2} has been selected as a diagonal weighting matrix with three first order low pass transfer functions in the diagonal. The weight matrix is given by

$$W_{e2} = diag(\frac{0.001}{s+0.001}, \frac{0.001}{s+0.001}, \frac{0.001}{s+0.001})$$

This will result in a design problem of order 5.

The results of the design is shown in Figure 2 and 3. A simulation of the fault estimates are given in Figure 2 for the three faults in the system and the water levels in Tank no. 1 and Tank no. 2 are shown in Figure 3.

The fault signals to the system has been described as step functions with amplitude 1. Fault signal no. 1 enter the system from $t = 1$ sec. to $t = 3$ sec. Fault no. 2 enter the system from $t = 4$ sec. to $t = 6$ sec. and the last fault signal enter the system from $t = 7$ sec. to $t = 9$ sec., see Figure 2. As it can be seen from Figure 2, the estimates of the fault signals are quite fast and accurate. The estimate of fault no. 1 is independent of the other fault signals, whereas some effects from the other fault signals appear in the estimates of fault no. 2 and 3. Especially the effect from the first fault on estimate of fault no. 2 is noticeable, see Figure 2.

In spite of the fact that the feedback controller has not been optimized carefully, it turns out that the controller works rather well, see 3. The water levels in Tank nos. 1 and 2 exhibits almost no change from the nominal levels, due to a fast feedback controller.

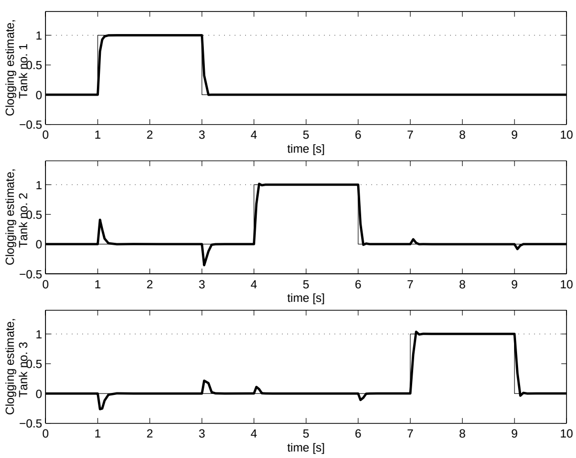

Fig. 2. The fault estimates of the three clogging faults.

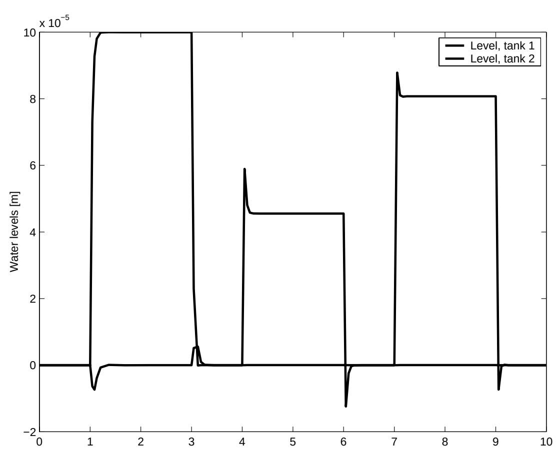

Fig. 3. Tank levels. Solid line - water level in tank no. 1, dashed line - water level in tank no. 2.

5. CONCLUSION

A combined design of a feedback controller and a fault estimator for the three tank benchmark problem has been considered in this paper. The standard $\mathcal{H}_\infty$ design method has been applied for the design of the combined feedback controller/fault estimator. It has been verified by simulation that the derived controller/fault estimator works very well.

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