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Generalized H_2 Control Synthesis for Periodic Systems

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Abstract

A control synthesis of periodic processes is addressed in this paper. A class of linear discrete time periodic systems with performance specified by the generalized H_2 operator norm is considered. The paper proposes a LMI solution to this problem, the sufficient and necessary conditions for solvability of H_2 suboptimal control are stated. An algorithm for state feedback control synthesis is provided.

1 Introduction

Over the past two decades the periodic control theory has gained considerable attention. Its applications range from biology to engineering and from chemical process to aerospace. A broad spectrum of results on periodic systems are available in the literature. The topics of structural properties, stability, quadratic optimal control and their relations to the periodic Lyapunov and Riccati equations were reported in [1], [2], [3], [4], [5]. An impetuous development took place after introducing the lift operator [6]. The results known from the control theory of linear time invariant systems became generalized to periodic systems, the techniques like pole placement [7], linear quadratic control [2], H_∞ [8] became available for the periodic systems. Only recently a solution to H_∞ synthesis problem for time varying, thereby to periodic systems, has been established in [9].

The contribution of this study is a Linear Matrix Inequality (LMI) formulation of the H_2 control synthesis problem. An important issue considered is the causality of the controller. The lifted counterpart of the control must possess the block Toeplitz structure. The paper considers a periodic discrete time system which performance is specified by the generalized H_2 operator norm as provided in [10]. The sufficient and necessary conditions for solvability of a suboptimal control synthesis problem are formulated, and an algorithm for state feedback control synthesis is developed. The result of this paper is a generalization of the H_2 control synthesis for linear discrete time invariant systems [11] and [12] to periodic processes.

The paper is organized as follows. The performance specification and stability issues for periodic systems are reviewed in Section 2. The main result of this work is presented in Section 3. The periodic H_2 control design is converted to a solution of certain LMIs, the conditions for solvability of this problem are formulated.

Notations

The following symbols are used throughout the paper:

$\mathbb{Z}_+$	set of all positive integers and zero,
$\mathbb{R}^{m\times p}$	all matrices m by p with real components,
w	discrete time lift operator,
W	z-transform of lift operator,
λ	shift operator,
Λ	z-transform of shift operator,
tr$\mathbf{A}$	trace of $\mathbf{A}$,
im$\mathbf{A}$	image of $\mathbf{A}$,
ker$\mathbf{A}$	null space of $\mathbf{A}$,
$\mathbf{I}$	identity matrix.

2 Periodic Systems

2.1 Properties

For the consistency of the presentation, the definitions of l_2 and H_2 spaces are briefly stated. The section is concluded with the stability lemma for periodic systems.

Consider a discrete signal $\mathbf{u} = \{\mathbf{u}(t)\}$, $t \in \mathbb{Z}_+$, where $\mathbf{u}(t) \in \mathbb{R}^m$. The space of all sequences $\mathbf{u}$ such that

$$||\mathbf{u}||^2 \equiv \sum_{t\in\mathbb{Z}_+} \mathbf{u}(t)^{\mathrm{T}}\mathbf{u}(t) < \infty, \tag{1}$$

is denoted by l^2. The space l^2 with the definition of the norm given by $||\mathbf{u}||$ becomes a Hilbert space with the inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{t\in\mathbb{Z}_+} \mathbf{u}(t)^{\mathrm{T}}\mathbf{v}(t). \tag{2}$$

Let H^2 denote the class of functions

$$\mathbf{u}(z) = \sum_{t \in \mathbf{Z}_+} u(t) z^{-t}, \tag{3}$$

such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{u}(re^{i\tau})^{\mathrm{T}} \mathbf{u}(re^{i\tau}) d\tau, \text{ where } 0 \leq r < 1, \tag{4}$$

has an upper bound independent of r. If the definition of the norm in H^2 is

$$||\mathbf{u}(z)|| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{u}(e^{i\tau})^{\mathrm{T}} \mathbf{u}(e^{i\tau}) d\tau \right)^{\frac{1}{2}}, \tag{5}$$

then the z-transform $\mathbf{u} \mapsto \mathbf{u}(z)$ is an isometric isomorphism.

The right shift operator $\Lambda : H^2 \to H^2$ is defined as

$$\Lambda : \mathbf{u}(z) \mapsto z^{-1} \mathbf{u}(z), \tag{6}$$

or in the discrete time domain, $\lambda : l^2 \to l^2$

$$\lambda : \{\mathbf{u}(t)\} \mapsto \{\mathbf{u}(t-1)\}. \tag{7}$$

Consider a dynamic system, a linear operator $s : l^2 \to l^2$. The system s is N-periodic if and only if

$$s\lambda^N = \lambda^N s, \tag{8}$$

where λ denotes the right shift operator in the discrete time domain. Notice that a time invariant operator is $N = 1$ periodic.

In the paper a state space representation of the periodic system is used. The following three periodic systems are considered:

- A system of specifications used for the standard H_2 synthesis

$$s_1 : \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix},$$
$$\begin{array}{rcl} \mathbf{x}(t+1) & = & \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}_1(t)\mathbf{w}(t) + \mathbf{B}_2(t)\mathbf{u}(t) \\ \mathbf{z}(t) & = & \mathbf{C}_1(t)\mathbf{x}(t) + \mathbf{D}_{12}(t)\mathbf{u}(t) \\ \mathbf{y}(t) & = & \mathbf{C}_2(t)\mathbf{x}(t) + \mathbf{D}_{21}(t)\mathbf{w}(t), \end{array} \tag{9}$$

where the system matrices are periodic $\mathbf{B}_1(t+N) = \mathbf{B}_1(t) \in \mathbb{R}^{s \times n}$, $\mathbf{B}_2(t+N) = \mathbf{B}_2(t) \in \mathbb{R}^{m \times n}$, $\mathbf{C}_1(t+N) = \mathbf{C}_1(t) \in \mathbb{R}^{n \times r}$, $\mathbf{C}_2(t+N) = \mathbf{C}_2(t) \in \mathbb{R}^{n \times p}$, $\mathbf{D}_{12}(t+N) = \mathbf{D}_{12}(t) \in \mathbb{R}^{m \times r}$, and $\mathbf{D}_{21}(t+N) = \mathbf{D}_{21}(t) \in \mathbb{R}^{s \times p}$.

- A simplified input output system used for the analysis

$$s_2 : \mathbf{u} \mapsto \mathbf{z},$$
$$\begin{array}{rcl} \mathbf{x}(t+1) & = & \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}_2(t)\mathbf{u}(t) \\ \mathbf{z}(t) & = & \mathbf{C}_1(t)\mathbf{x}(t) + \mathbf{D}_{12}(t)\mathbf{u}(t). \end{array} \tag{10}$$

The general results will be derived for a periodic control and extended by duality to estimation.

Stability of a periodic system can be analyzed using the Periodic Lyapunov Lemma [2]

Lemma 1 *Consider the system s_1 in Eq. (10) and the Lyapunov equation*

$$\lambda \mathbf{Q}(t) = \mathbf{A}(t)^{\mathrm{T}} \mathbf{Q}(t) \mathbf{A}(t) + \mathbf{C}(t)^{\mathrm{T}} \mathbf{C}(t). \tag{11}$$

$\mathbf{A}(t)$ is stable if and only if, for any periodic $\mathbf{C}(t)$ such that $(\mathbf{A}(t), \mathbf{C}(t))$ is detectable there exists a symmetric periodic, positive semidefinite solution $\mathbf{Q}(t)$ of (11).

A direct conclusion from the Periodic Lyapunov Lemma is the following corollary, [13].

Corollary 1 *The system s_1 is stable if and only if there exists a periodic, positive definite matrix $\mathbf{Q}(t)$ satisfying*

$$\lambda \mathbf{Q}(t) > \mathbf{A}(t)^{\mathrm{T}} \mathbf{Q}(t) \mathbf{A}(t) \ \forall t \in \{0, 1, \cdots, N-1\}. \tag{12}$$

To find the steady state solution of the discrete periodic Lyapunov equation (11) it is desired to find the periodic generator $\tilde{\mathbf{Q}}$ by solving the following discrete algebraic Lyapunov equation [14], one for each $t \in \{0, ..., N-1\}$

$$\tilde{\mathbf{Q}}(t) = \mathbf{\Phi}(t+N, t)^{\mathrm{T}} \tilde{\mathbf{Q}}(t) \mathbf{\Phi}(t+N, t) \tag{13}$$
$$+ \sum_{j=t}^{t+N-1} \mathbf{\Phi}(t+N, j+1)^{\mathrm{T}} \mathbf{C}(j)^{\mathrm{T}} \mathbf{C}(j) \mathbf{\Phi}(t+N, j+1),$$

where $\mathbf{\Phi}(t, t_i)$ is the state transition matrix at sample t with the initial time t_i.

2.2 Lifted system

This subsection introduces a lifting operator, an isometric isomorphism which transforms a linear periodic system to a time invariant representation. Following [6] the lift operator is defined

$$w : \{\mathbf{u}(t)\} = (\mathbf{u}_0, \mathbf{u}_1, \cdots) \mapsto \begin{bmatrix} p_0(\{\mathbf{u}(t)\}) \\ p_1(\{\mathbf{u}(t)\}) \\ \vdots \\ p_{N-1}(\{\mathbf{u}(t)\}) \end{bmatrix}, \tag{14}$$

where $p_i : l^2 \to l^2$ is a projection operator

$$p_i(\{\mathbf{u}(t)\}) \equiv \{\mathbf{u}_{Nt+i}\} = (\mathbf{u}_i, \mathbf{u}_{N+i}, \mathbf{u}_{2N+i}, \cdots). \quad (15)$$

The following property is valid for the lift operator defined in Eq. (14)

$$w\lambda^N = \lambda w. \quad (16)$$

The lifted system $\tilde{s} \equiv wsw^{-1}$ is time invariant. This can be shown using the property (16) and

$$\tilde{s}\lambda = wsw^{-1}\lambda = w\lambda^N sw^{-1} = \lambda wsw^{-1} = \lambda\tilde{s}, \quad (17)$$

hence then the system $\tilde{T}$ is 1-periodic thereby time invariant.

The z-transform of the projection operator $P_i : H^2 \to H^2$ is

$$P_i(z)(y) = \sum_{n\in Z_+} \hat{y}(i+Nn)z^{-n}. \quad (18)$$

Notice that the sum $\sum_{i=0}^{N-1} P_i$ gives the identity operator, I. Now the z-transformed lift operator $W : H^2 \to (H^2)^N$ is

$$Wu(z) = \begin{bmatrix} P_0 & P_1 & \dots & P_{N-1} \end{bmatrix}^T u(z), \quad (19)$$

and its inverse

$$W^{-1} = \sum_{i=0}^{N-1} z^{-i} P_i(z^N). \quad (20)$$

The z-transform of the lifted system is then $\tilde{S} = WSW^{-1}$. This operator is linear and time invariant, hence can be treated as the generalization of the transfer function for periodic systems. Along these lines the performance will be specified for the lifted system in the next section.

The explicit formula for the lift of s_2 in Eq. (10) is

$$\begin{aligned}
\mathbf{x}(t+N) &= \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{B}}_1\mathbf{u}(t) + \tilde{\mathbf{B}}_2\mathbf{u}(t+1) + \dots \\
&+ \tilde{\mathbf{B}}_N\mathbf{u}(t+N-1) \\
\mathbf{y}(t) &= \tilde{\mathbf{C}}_N\mathbf{x}(t) + \tilde{\mathbf{D}}_{1,1}\mathbf{u}(t) \\
\mathbf{y}(t+1) &= \tilde{\mathbf{C}}_{N-1}\mathbf{x}(t) + \tilde{\mathbf{D}}_{2,1}\mathbf{u}(t) + \dots \\
&+ \tilde{\mathbf{D}}_{N-1,N-1}\mathbf{u}(t+N-2) \\
&\dots \\
\mathbf{y}(t+N-1) &= \tilde{\mathbf{C}}_1\mathbf{x}(t) + \tilde{\mathbf{D}}_{N,1}\mathbf{u}(t) + \dots \\
&+ \tilde{\mathbf{D}}_{N,N}\mathbf{u}(t+N-1), \quad (21)
\end{aligned}$$

where for $i \neq j$

$$\begin{aligned}
\tilde{\mathbf{A}} &= \mathbf{\Phi}(t+N,t), \\
\tilde{\mathbf{B}}_i(t) &= \mathbf{\Phi}(t+N,t+i)\mathbf{B}_2(t+i-1) \\
\tilde{\mathbf{C}}_i(t) &= \mathbf{C}_1(t+N-i)\mathbf{\Phi}(t+N-i,t), \\
\tilde{\mathbf{D}}_{i,j}(t) &= \mathbf{C}_1(t+i-1)\mathbf{\Phi}(t+i-1,t+j)\mathbf{B}_2(t+j-1) \\
\tilde{\mathbf{D}}_{i,i}(t) &= \mathbf{D}_{12}(t+i-1).
\end{aligned}$$

Notice that the matrix $\tilde{\mathbf{A}}$ is the monodromy matrix [3], which is time independent. The generalized transfer function of the system in Eq. (21) is

$$\begin{aligned}
\mathbf{S}(z) &= \begin{bmatrix} \tilde{\mathbf{D}}_{1,1} & 0 & \dots & 0 \\ \tilde{\mathbf{D}}_{2,1} & \tilde{\mathbf{D}}_{2,2} & \dots & 0 \\ \dots & & & \\ \tilde{\mathbf{D}}_{N-1,1} & \dots & \tilde{\mathbf{D}}_{N-1,N-1} & 0 \\ \tilde{\mathbf{D}}_{N,1} & \tilde{\mathbf{D}}_{N,2} & \dots & \tilde{\mathbf{D}}_{N,N} \end{bmatrix} \\
&+ \begin{bmatrix} \tilde{\mathbf{C}}_N & \dots & \tilde{\mathbf{C}}_1 \end{bmatrix}^T (z\mathbf{I} - \tilde{\mathbf{A}})^{-1} \begin{bmatrix} \tilde{\mathbf{B}}_1 & \dots & \tilde{\mathbf{B}}_N \end{bmatrix}.
\end{aligned} \quad (22)$$

The Periodic Lyapunov Lemma stated in Subsection 2.1 is the generalization of the well known Lyapunov Lemma for discrete time invariant systems. It relates the solution of the algebraic Lyapunov equation

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{A}}^T\tilde{\mathbf{Q}}\tilde{\mathbf{A}} + \tilde{\mathbf{C}}^T\tilde{\mathbf{C}}, \quad (23)$$

with stability of the $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$ detectable system. Observe that if the lift in Eq. (21) of the system $(\mathbf{A}(t), \mathbf{C}(t))$ gives $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$ then Eqs. (11) and (23) become equivalent.

2.3 Performance Specification

The H_2 operator norm for a discrete, time invariant, stable, casual system $R : (H^2)^m \to (H^2)^p$ is defined [15] by

$$||R||_2 \equiv \left(\frac{1}{2\pi}\text{tr}\int_{-\pi}^{\pi} R(e^{i\tau})R^*(e^{i\tau})d\tau\right)^{\frac{1}{2}}, \quad (24)$$

or equivalently using the Parseval's relation between H_2 and l_2

$$||R||_2 = ||r||_2 \equiv \left(\sum_{i=1}^{m} ||r\delta(t)\mathbf{e}_i||^2\right)^{\frac{1}{2}}, \quad (25)$$

where $r : (l^2)^m \to (l^2)^p$; $\mathbf{e}_i$ is the standard basis of the input space $\mathbb{R}^m$, thus $\delta\mathbf{e}_i$ is the impulse applied to the i-th input.

This standard definition indicates that the H_2 norm is characterized by the l_2 norm of the impulse response, on the other hand the response of the system is dependent on the time when the impulse is initiated. Following [10] the H_2 norm for a periodic system s is defined by

$$||s||_2 \equiv \left(\frac{1}{N}\sum_{j=0}^{N-1}\sum_{i=1}^{m} ||s\delta(t-j)\mathbf{e}_i||\right)^{\frac{1}{2}}. \quad (26)$$

Definition in (26) corresponds to the standard H_2 norm if the system s is time invariant. Furthermore, the H_2 norm for a periodic system is equivalent to $1/\sqrt{N}$ of the

H_2 norm of its lift. The generalized H_2 norm for the system s_2 ($\mathbf{D}_{12} = \mathbf{0}$) takes the following form

$$
\begin{aligned}
&||s_2||_2 \qquad (27)\\
&= \Big(\frac{1}{N}\mathrm{tr}\sum_{i\in\mathbf{Z}_+}\begin{bmatrix}\mathbf{B}_2(t)^{\mathrm{T}}\mathbf{\Phi}(t+N,t+1)^{\mathrm{T}}\\ \cdots \\ \mathbf{B}_2(t+N-1)^{\mathrm{T}}\end{bmatrix}\mathbf{\Phi}^{\mathrm{i}}(\mathrm{t}+\mathrm{N},\mathrm{t})^{\mathrm{T}}\\
&\times \left[\mathbf{\Phi}(t+N-1,t)^{\mathrm{T}}\mathbf{C}_2(t+N-1)^{\mathrm{T}} \ \ \cdots \ \ \mathbf{C}_2(t)^{\mathrm{T}}\right]^{\mathrm{T}}\\
&\times \begin{bmatrix}\mathbf{C}_2(t+N-1)\mathbf{\Phi}^{i}(t+N-1,t)\\ \cdots \\ \mathbf{C}_2(t)\end{bmatrix}\mathbf{\Phi}(t+N,t)\\
&\times \left[\mathbf{\Phi}(t+N,t+1)\mathbf{B}_2(t) \ \ \cdots \ \ \mathbf{B}_2(t+N-1)\right]\Big)^{\frac{1}{2}}.
\end{aligned}
$$

By grouping the terms containing the matrix $\mathbf{B}_2(\cdot)$ Eq. (27) is simplified to the following expression

$$
\begin{aligned}
&||s_2||_2\\
&= \Big(\frac{1}{N}\mathrm{tr}\sum_{\mathrm{t}=0}^{\mathrm{N}-1}\mathbf{B}_2(t)^{\mathrm{T}}\Big(\sum_{\mathrm{j}=\mathrm{t}}^{\mathrm{t}+\mathrm{N}-1}\sum_{\mathrm{i}\in\mathbf{Z}_+}\mathbf{\Phi}(\mathrm{t}+\mathrm{N},\mathrm{t})^{\mathrm{T}}\\
&\times \mathbf{\Phi}(t+N,j+1)^{\mathrm{T}}\mathbf{C}_2(j)^{\mathrm{T}}\mathbf{C}_2(j)\mathbf{\Phi}(t+N,j+1)\\
&\times \mathbf{\Phi}(t+N,t)\Big)\mathbf{B}_2(t)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

The sum in the inner bracket is the solution of the algebraic Lyapunov equation (13), which as mentioned before is equivalent to the periodic solution of the periodic Lyapunov equation (11)

$$||s_2||_2 = \left(\frac{1}{N}\mathrm{tr}\sum_{\mathrm{t}=0}^{\mathrm{N}-1}\mathbf{B}_2(\mathrm{t})^{\mathrm{T}}\tilde{\mathbf{Q}}(\mathrm{t})\mathbf{B}_2(\mathrm{t})\right)^{\frac{1}{2}}. \qquad (28)$$

3 LMI

The design of the optimal periodic H_2 control algorithm addressed in this section will be observer based. The argument for using this paradigm is that the separation principle is valid for periodic systems [2].

Consider the system s_1 with full state space information, i.e. $\mathbf{C}_2 = \mathbf{I}$, and $\mathbf{D}_{21} = \mathbf{0}$, and periodic state feedback $\mathbf{u}(t) = \mathbf{K}(t)\mathbf{x}(t)$, $\mathbf{K}(t+N) = \mathbf{K}(t)$. The objective of the control design is to compute a gain $\mathbf{K}(t)$ for which the transfer function

$$
\begin{aligned}
&s_c : \mathbf{w} \mapsto \mathbf{z},\\
&\mathbf{x}(t+1) \;=\; \mathbf{A}_c(t)\mathbf{x}(t) + \mathbf{B}_1(t)\mathbf{w}(t)\\
&\mathbf{z}(t) \;=\; \mathbf{C}_c(t)\mathbf{x}(t),
\end{aligned} \qquad (29)
$$

where $\mathbf{A}_c(t) = \mathbf{A}(t) + \mathbf{B}_2(t)\mathbf{K}(t)$, $\mathbf{C}_c(t) = \mathbf{C}_1(t) + \mathbf{D}_{12}(t)\mathbf{K}(t)$ satisfies

$$||s_c||_2 < \gamma. \qquad (30)$$

The main results are summarized in the following theorem.

Theorem 1 *Consider a periodic discrete time system s_c, $(\mathbf{A}(t), \mathbf{B}_2(t))$ stabilizable. The suboptimal H_2 problem Eq. (30) is solvable if and only if there exists a symmetric periodic matrix $\mathbf{Q}(t)$ and a periodic $\mathbf{Z}(t)$ such that*

$$
\begin{aligned}
&\left(\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{A}(t) + \mathbf{W}_2(t)^{\mathrm{T}}\mathbf{C}_1(t)\right)\mathbf{Q}(t-1)\\
\times\ &\left(\mathbf{A}(t)^{\mathrm{T}}\mathbf{W}_1(t) + \mathbf{C}_1(t)^{\mathrm{T}}\mathbf{W}_2(t)\right)\\
-\ &\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{Q}(t)\mathbf{W}_1(t) - \mathbf{W}_2(t)^{\mathrm{T}}\mathbf{W}_2(t) < \mathbf{0},
\end{aligned} \qquad (31)
$$

$$\begin{bmatrix}\mathbf{Q}(t) & \mathbf{B}_1(t)\\ \mathbf{B}_1(t)^{\mathrm{T}} & \mathbf{Z}(t)\end{bmatrix} > \mathbf{0}, \qquad (32)$$

$$\mathrm{tr}\Big(\sum_{\mathrm{t}=0}^{\mathrm{N}-1}\mathbf{Z}(\mathrm{t})\Big) < \mathrm{N}\gamma^2, \qquad (33)$$

where $\mathrm{im}\begin{bmatrix}\mathbf{W}_1(t)\\ \mathbf{W}_2(t)\end{bmatrix} = \ker\left[\mathbf{B}_2(t)^{\mathrm{T}} \quad \mathbf{D}_{12}(t)^{\mathrm{T}}\right].$

We shall use Projection Lemma [12] in the proof of Theorem 1. For consistency of presentation it is provided below.

Lemma 2 (Projection Lemma) *For arbitrary matrices $\mathbf{\Psi}_a$ and $\mathbf{\Psi}_b$ and a symmetric $\mathbf{P}$, the LMI*

$$\mathbf{\Psi}_a^{\mathrm{T}}\mathbf{X}\mathbf{\Psi}_b + \mathbf{\Psi}_b^{\mathrm{T}}\mathbf{X}\mathbf{\Psi}_a + \mathbf{P} < \mathbf{0}, \qquad (34)$$

is solvable if and only if

$$\mathbf{W}_a^{\mathrm{T}}\mathbf{P}\mathbf{W}_a < 0 \ \textit{and}\ \mathbf{W}_b^{\mathrm{T}}\mathbf{P}\mathbf{W}_b < 0, \qquad (35)$$

where $\mathbf{W}_a, \mathbf{W}_b$ are any matrices with columns forming bases for the null spaces of $\mathbf{\Psi}_a$ and $\mathbf{\Psi}_b$.

Proof of Theorem 1 *Using the findings of Section 2.3, Eq. (28) the generalized H_2 norm $||s_c||_2 < \gamma$ is equivalent to*

$$\mathrm{tr}\sum_{\mathrm{t}=0}^{\mathrm{N}-1}\mathbf{B}_2(\mathrm{t})^{\mathrm{T}}\mathbf{Q}^{-1}(\mathrm{t})\mathbf{B}_2(\mathrm{t}) < \mathrm{N}\gamma^2, \qquad (36)$$

where $\tilde{\mathbf{Q}}$ is N-periodic and satisfies the inequality

$$\mathbf{Q}^{-1}(t-1) - \mathbf{A}_c(t)^{\mathrm{T}}\mathbf{Q}^{-1}(t)\mathbf{A}_c(t) - \mathbf{C}_c(t)^{\mathrm{T}}\mathbf{C}_c(t) > \mathbf{0}, \qquad (37)$$

but Eq. (36) is equivalent to

$$\mathrm{tr}\Big(\sum_{\mathrm{t}=0}^{\mathrm{N}-1}\mathbf{Z}(\mathrm{t})\Big) < \mathrm{N}\gamma^2, \qquad (38)$$

where $\mathbf{Z}(t)$ is a solution of the following LMI

$$\mathbf{B}_1(t)^{\mathrm{T}}\mathbf{Q}^{-1}(t)\mathbf{B}_1(t) < \mathbf{Z}. \qquad (39)$$

The result of applying the Schur complement [12] on Eq. (39) is the LMI (32).

The next step is to use the Schur complement twice on Eq. (37) which gives two equivalent forms

$$\begin{bmatrix} -\mathbf{Q}^{-1}(t-1)+\mathbf{A}_c(t)^{\mathrm{T}}\mathbf{Q}^{-1}(t)\mathbf{A}_c(t) & \mathbf{C}_c(t)^{\mathrm{T}} \\ \mathbf{C}_c(t) & -\mathbf{I} \end{bmatrix} < \mathbf{0} \tag{40}$$

$$\Updownarrow$$

$$\begin{bmatrix} -\mathbf{Q}(t) & \mathbf{A}_c(t) & \mathbf{0} \\ \mathbf{A}_c(t)^{\mathrm{T}} & -\mathbf{Q}^{-1}(t-1) & \mathbf{C}_c(t)^{\mathrm{T}} \\ \mathbf{0} & \mathbf{C}_c(t) & -\mathbf{I} \end{bmatrix} < \mathbf{0}. \tag{41}$$

For the purpose of the control synthesis Eq. (41) is grouped into $\mathbf{K}(t)$ *and* $\mathbf{Q}(t)$ *dependent terms*

$$\begin{aligned} &\begin{bmatrix} -\mathbf{Q}(t) & \mathbf{A}(t) & \mathbf{0} \\ \mathbf{A}(t)^{\mathrm{T}} & -\mathbf{Q}^{-1}(t-1) & \mathbf{C}_1(t)^{\mathrm{T}} \\ \mathbf{0} & \mathbf{C}_1(t) & -\mathbf{I} \end{bmatrix} \\ &+ \begin{bmatrix}\mathbf{B}_2(t)^{\mathrm{T}} & \mathbf{0} & \mathbf{D}_{12}(t)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\mathbf{K}(t)\begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix} \\ &+ \begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix}^{\mathrm{T}}\mathbf{K}(t)^{\mathrm{T}}\begin{bmatrix}\mathbf{B}_2(t)^{\mathrm{T}} & \mathbf{0} & \mathbf{D}_{12}(t)^{\mathrm{T}}\end{bmatrix} < \mathbf{0}, \end{aligned} \tag{42}$$

but the structure of Eq. (42) corresponds to Eq. (34), thus the LMI (42) is solvable if and only if

$$\mathbf{W}_a(t)^{\mathrm{T}}\begin{bmatrix} -\mathbf{Q}(t) & \mathbf{A}(t) & \mathbf{0} \\ \mathbf{A}(t)^{\mathrm{T}} & -\mathbf{Q}^{-1}(t-1) & \mathbf{C}_1(t)^{\mathrm{T}} \\ \mathbf{0} & \mathbf{C}_1(t) & -\mathbf{I} \end{bmatrix}\mathbf{W}_a(t) < 0, \tag{43}$$

$$\mathbf{W}_b(t)^{\mathrm{T}}\begin{bmatrix} -\mathbf{Q}(t) & \mathbf{A}(t) & \mathbf{0} \\ \mathbf{A}(t)^{\mathrm{T}} & -\mathbf{Q}^{-1}(t-1) & \mathbf{C}_1(t)^{\mathrm{T}} \\ \mathbf{0} & \mathbf{C}_1(t) & -\mathbf{I} \end{bmatrix}\mathbf{W}_b(t) < 0, \tag{44}$$

where

$$\mathbf{W}_a(t) = \begin{bmatrix} \mathbf{W}_1(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{W}_2(t) & \mathbf{0} \end{bmatrix} \text{ and } \mathbf{W}_b(t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{45}$$

The LMI (44) is always fulfilled, whereas (43) is equivalent to

$$\begin{bmatrix} \boldsymbol{\Omega}_{11}(t) & \boldsymbol{\Omega}_{12}(t) \\ \boldsymbol{\Omega}_{21}(t) & \boldsymbol{\Omega}_{22}(t) \end{bmatrix} < \mathbf{0}, \text{ where} \tag{46}$$

$$\begin{aligned} \boldsymbol{\Omega}_{11}(t) &= -\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{Q}(t)\mathbf{W}_1(t) - \mathbf{W}_2(t)^{\mathrm{T}}\mathbf{W}_2(t), \\ \boldsymbol{\Omega}_{12}(t) &= \mathbf{W}_1(t)^{\mathrm{T}}\mathbf{A} + \mathbf{W}_2(t)^{\mathrm{T}}\mathbf{C}_1(t), \\ \boldsymbol{\Omega}_{21}(t) &= \mathbf{A}(t)^{\mathrm{T}}\mathbf{W}_1(t) + \mathbf{C}_1(t)^{\mathrm{T}}\mathbf{W}_2(t), \\ \boldsymbol{\Omega}_{22}(t) &= -\mathbf{Q}^{-1}(t-1). \end{aligned} \tag{47}$$

Applying the Schur complement in Eq. (46) the LMI (31) results. □

The control design scheme consists of first finding a periodic matrix $\mathbf{Q}(t)$ solving the LMIs (31) to (33), and then subsequently a controller $\mathbf{K}(t)$ fulfilling the LMI (42) has to be computed.

Algorithm 1

1. *Find using the LMI technique a symmetric matrix* $\mathbf{Q}(t)$ *and a matrix* $\mathbf{Z}(t)$ *for* $t = 0 \ldots N-1$ *satisfying*

$$\begin{aligned} &\left(\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{A}(t) + \mathbf{W}_2(t)^{\mathrm{T}}\mathbf{C}_1(t)\right)\mathbf{Q}(t-1) \\ \times\ &\left(\mathbf{A}(t)^{\mathrm{T}}\mathbf{W}_1(t) + \mathbf{C}_1(t)^{\mathrm{T}}\mathbf{W}_2(t)\right) \\ -\ &\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{Q}(t)\mathbf{W}_1(t) - \mathbf{W}_2(t)^{\mathrm{T}}\mathbf{W}_2(t) < \mathbf{0} \end{aligned}$$

$$\begin{aligned} \operatorname{im}\begin{bmatrix}\mathbf{W}_1(t) \\ \mathbf{W}_2(t)\end{bmatrix} &= \ker\begin{bmatrix}\mathbf{B}_2(t)^{\mathrm{T}} & \mathbf{D}_{12}(t)^{\mathrm{T}}\end{bmatrix} \\ \begin{bmatrix}\mathbf{Q}(t) & \mathbf{B}_1(t) \\ \mathbf{B}_1(t)^{\mathrm{T}} & \mathbf{Z}(t)\end{bmatrix} &> \mathbf{0} \\ \operatorname{tr}\left(\sum_{t=0}^{N-1}\mathbf{Z}(t)\right) &< N\gamma^2 \\ \mathbf{Q}(0) &= \mathbf{Q}(N). \end{aligned}$$

2. *For each* $t = 0 \ldots N-1$ *find using the LMI technique a matrix* $\mathbf{K}(t)$*, which satisfies*

$$\begin{aligned} &\begin{bmatrix} -\mathbf{Q}(t) & \mathbf{A}(t) & \mathbf{0} \\ \mathbf{A}(t)^{\mathrm{T}} & -\mathbf{Q}^{-1}(t-1) & \mathbf{C}_1(t)^{\mathrm{T}} \\ \mathbf{0} & \mathbf{C}_1(t) & -\mathbf{I} \end{bmatrix} \\ &+ \begin{bmatrix}\mathbf{B}_2(t)^{\mathrm{T}} & \mathbf{0} & \mathbf{D}_{12}(t)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\mathbf{K}(t)\begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix} \\ &+ \begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix}^{\mathrm{T}}\mathbf{K}(t)^{\mathrm{T}}\begin{bmatrix}\mathbf{B}_2(t)^{\mathrm{T}} & \mathbf{0} & \mathbf{D}_{12}(t)^{\mathrm{T}}\end{bmatrix} \\ &< \mathbf{0}. \end{aligned}$$

Remark 1 *The* H_2 *suboptimal controller suggested in this paper fulfills Eq. (37), hence the system is stable according to Corollary 1.*

Remark 2 *The observer synthesis reduces to an application of the duality argument. The following system is considered*

$$\begin{aligned} &s_o : \mathbf{w} \mapsto \mathbf{z}, \\ &\mathbf{x}(t+1) = \mathbf{A}_o(t)\mathbf{x}(t) + \mathbf{B}_o(t)\mathbf{w}(t) \\ &\mathbf{z}(t) = \mathbf{C}_1(t)\mathbf{x}(t), \end{aligned} \tag{48}$$

where $\mathbf{A}_o(t) = \mathbf{A}(t) + \mathbf{L}(t)\mathbf{C}_2(t)$, $\mathbf{B}_o(t) = \mathbf{B}_1(t) + \mathbf{L}(t)\mathbf{D}_{21}(t)$. *The observer synthesis is such that the gain* $\mathbf{L}(t)$ *fulfills*

$$\|s_o\|_2 < \gamma. \tag{49}$$

The problem Eq. (49) is solvable if and only if there exists a symmetric periodic matrix $\mathbf{Q}(t)$ *and a periodic* $\mathbf{Z}(t)$ *such that*

$$\begin{aligned} & \left(\mathbf{W}_1(t)^{\mathrm{T}}\mathbf{A}(t)^{\mathrm{T}} + \mathbf{W}_2(t)^{\mathrm{T}}\mathbf{B}_1(t)^{\mathrm{T}}\right)\mathbf{Q}(t-1) \\ \times & \;\; (\mathbf{A}(t)\mathbf{W}_1(t) + \mathbf{B}_1(t)\mathbf{W}_2(t)) \\ - & \;\; \mathbf{W}_1(t)^{\mathrm{T}}\mathbf{Q}(t)\mathbf{W}_1(t) - \mathbf{W}_2(t)^{\mathrm{T}}\mathbf{W}_2(t) < \mathbf{0}, \end{aligned} \quad (50)$$

$$\begin{bmatrix} \mathbf{Q}(t) & \mathbf{C}_2^{\mathrm{T}}(t) \\ \mathbf{C}_2(t) & \mathbf{Z}(t) \end{bmatrix} > \mathbf{0}, \quad (51)$$

$$\mathrm{tr}\Big(\sum_{\mathrm{t}=0}^{N-1} \mathbf{Z}(\mathrm{t})\Big) < \mathrm{N}\gamma^2, \quad (52)$$

where $\mathrm{im}\begin{bmatrix}\mathbf{W}_1(t) \\ \mathbf{W}_2(t)\end{bmatrix} = \ker\begin{bmatrix}\mathbf{C}_2(t) & \mathbf{D}_{21}(t)\end{bmatrix}$. *The observer gain is a solution of the following LMI*

$$\begin{aligned} & \begin{bmatrix} -\mathbf{Q}(t) & \mathbf{A}(t)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{A}(t) & -\mathbf{Q}^{-1}(t-1) & \mathbf{B}_1(t) \\ \mathbf{0} & \mathbf{B}_1(t)^{\mathrm{T}} & -\mathbf{I} \end{bmatrix} \\ + & \;\; \begin{bmatrix}\mathbf{C}_2(t) & \mathbf{0} & \mathbf{D}_{21}(t)\end{bmatrix}^{\mathrm{T}} \mathbf{L}(t)^{\mathrm{T}} \begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix} \\ + & \;\; \begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix}^{\mathrm{T}} \mathbf{L}(t) \begin{bmatrix}\mathbf{C}_2(t) & \mathbf{0} & \mathbf{D}_{21}(t)\end{bmatrix} < \mathbf{0}. \end{aligned}$$

4 Conclusions

This paper focused on the generalized H_2 suboptimal control synthesis for discrete time periodic systems. The necessary and sufficient conditions for solvability of this problem were formulated. The algorithm for state feedback control synthesis was suggested.

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