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Switching between multivariable controllers

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SUMMARY

A concept for implementation of multivariable controllers is presented in this paper. The concept is based on the Youla–Jabr–Bongiorno–Kucera (YJBK) parameterization of all stabilizing controllers. Using this scheme for implementation of multivariable controllers, it is shown how it is possible to smoothly switch between multivariable controllers with guaranteed closed-loop stability. This includes also the case where one or more controllers are unstable.

The concept for smooth on-line changes of multivariable controllers based on the YJBK architecture can also handle the start-up and shut down of multivariable systems. Furthermore, the start-up of unstable multivariable controllers can be handled as well. Finally, implementation of (unstable) controllers as a stable Q parameter in a Q-parameterized controller can also be achieved. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: multivariable controllers; parameterization; switching; controller implementation; stabilizing controllers

1. MOTIVATION—AN EXAMPLE

Some aspects of stability in connection with implementation of controllers for multivariable systems are considered in this paper. This includes both implementation of unstable controllers as well as on-line change between a number of controllers.

Even for stable systems, most (post-) modern control techniques based on various optimization techniques, such as $\mathscr{H}_2$, $\mathscr{H}_\infty$, $\mathscr{L}_1/\ell_1$ norm based or μ optimization-based designs tend to provide unstable controllers.

The industrial use of unstable controllers has been limited. This is unfortunate, considering that for some plants, no stable controller will achieve optimality (in a mixed sensitivity sense). Moreover, for some plants, no stable controller will robustly stabilize the system. Finally, for

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some unstable plants—violating the interlacing property—no stable controller will stabilize even the nominal system.

Another aspect is the change of controllers, e.g. in the case where a simple controller is applied in the start-up of the process, but which is later replaced by a more advanced controller. This is normally performed by using a linear interpolation between the two controllers. As the following small example show, there is in general no guarantee that a linear combination of two stabilizing controllers will also stabilize the system.

Consider the following state space description of a generalized nominal 2×2 system,

$$\Sigma : \begin{cases} \dot{x} = Ax + B_w w + B_u u \\ z = C_z x + D_{zw} w + D_{zu} u \\ y = C_y x + D_{yw} w + D_{yu} u \end{cases} \tag{1}$$

where $x \in \mathscr{R}^n$ is the state vector, $w \in \mathscr{R}^r$ is a disturbance input vector, $u \in \mathscr{R}^m$ the control input signal vector, $z \in \mathscr{R}^q$ is the external output signal vector to be controlled, and $y \in \mathscr{R}^p$ is the measurement vector. Let the nine matrices in the general system in (1) be given by

$$\left(\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right) = \left(\begin{array}{ccc|cc} 7.0 & 0 & 0 & 0.1 & 1.0 \\ 1.0 & -7.0 & -2.4495 & 0.0 & 0.0 \\ 0.0 & 2.4495 & 0 & 0.1 & 0.0 \\ \hline \multicolumn{3}{c|}{(1.0 \;\; 0.0 \;\; 1.0)} & 0 & 1 \\ \multicolumn{3}{c|}{(1.0 \;\; -5.0 \;\; 253.1139)} & 1 & 0 \end{array}\right)$$

i.e. $n = 3,\ r = 1,\ m = 1,\ q = 1$ and $p = 1$.

The system is unstable, but can be stabilized by a P controller given by

$$u = -D_P y \tag{2}$$

The system is closed-loop stable for D_P given by

$$D_P \in [334, \infty)$$

Let us use $D_P = 1000$ as the gain for the P controller. This controller results in the following stable closed-loop poles:

$$\text{poles}_{\text{cl},P} = \begin{pmatrix} -998.67 & \\ -0.6660 & +25.027i \\ -0.6660 & -25.027i \end{pmatrix}$$

Let the other controller be a dynamic controller given by

$$u = K_1(s) y \tag{3}$$

where the controller $K_1(s)$ has the following state-space realization:

$$\Sigma_C : \begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u = C_c x_c + D_c y \end{cases} \tag{4}$$

where $x_c \in \mathscr{R}^{n_c}$ is the controller state vector. The controller has been designed as an observer-based feedback controller, using an $\mathscr{H}_2$ design method on the full 2×2 system described above. The resulting controller K_1 is given by

$$K_1 : \left(\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right) = \left(\begin{array}{c|c} \begin{pmatrix} -15.070 & 45.992 & -2309.7 \\ 0.35357 & -3.7679 & -166.07 \\ -0.13121 & 3.1056 & -33.212 \end{pmatrix} & \begin{pmatrix} 9.1283 \\ 0.64643 \\ 0.13121 \end{pmatrix} \\ \hline (-12.941 \;\; 0.35054 \;\; 0.85619) & 0 \end{array}\right)$$

Note that the state feedback gain F and the observer gain L for the controller is given as C_c and $-B_c$, respectively.

The poles for the closed-loop system using K_1 are given by

$$\text{poles}_{\text{cl},K_1} = \begin{pmatrix} -25.1218 \\ -0.9022 \\ -7.7082 + 1.1005i \\ -7.7082 - 1.1005i \\ -5.3047 + 1.1643i \\ -5.3047 - 1.1643i \end{pmatrix}$$

Let a controller K_α be given as a linear combination of the P and K_1 controllers, i.e.

$$K_\alpha = (1 - \alpha)D_P + \alpha K_1, \quad \alpha \in [0, 1] \tag{5}$$

This controller is used for on-line change between the two controllers. However, it turns out that the controller K_α given by (5) is not stable for all $\alpha \in [0, 1]$. The closed-loop system is not stable for

$$\alpha \in [0.66768, 0.99995)$$

This example clearly show that using a direct linear change between two controllers can result in a stability problem. This problem will become even more distinct in the case where we want to change between more that two controllers. A direct jump from one stabilizing controller to another stabilizing controller is not in general a useful method. This will in many cases result in spikes in the outputs, which is not acceptable. It is therefore necessary to provide a systematic way to obtain on-line controller change without getting stability or transient problems.

In the following, a concept based on the Youla–Jabr–Bongiorno–Kucera (YJBK) parameterization will be introduced for handling on-line controller changes without resulting

in any closed-loop stability problems. As a result of this, the implementation of unstable controllers by using stable transfer functions only is also considered in the following. At last, the connection with gain scheduling control will be considered.

The rest of this paper is organized as follows. The YJBK parameterization is briefly introduced in Section 2. The main results are given in Section 3. The connection between gain scheduling control and the presented results in this is considered in Section 4. The example from this section is considered again in Section 5 followed by a conclusion in Section 6.

2. THE YJBK PARAMETERIZATION

The YJBK parameterization is briefly introduced in this section. Let the state space system given in (1) be described by transfer functions as follows:

$$\Sigma : \begin{cases} z = G_{zw}w + G_{zu}u \\ y = G_{yw}w + G_{yu}u \end{cases} \tag{6}$$

Moreover, let a co-prime factorization of the system $G_{yu}(s)$ from (1) and a stabilizing controller $K(s)$ from (4) be given by

$$G_{yu} = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \quad N, M, \tilde{N}, \tilde{M} \in \mathscr{RH}_\infty$$

$$K = UV^{-1} = \tilde{V}^{-1}\tilde{U}, \quad U, V, \tilde{U}, \tilde{V} \in \mathscr{RH}_\infty \tag{7}$$

where the eight matrices in (7) must satisfy the double Bezout equation given by, see Reference [1]

$$\begin{aligned} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &= \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} \\ &= \begin{pmatrix} M & U \\ N & V \end{pmatrix} \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \end{aligned} \tag{8}$$

Assume that the controller $K(s)$ is an observer-based feedback controller given by

$$K(s) = \left(\begin{array}{c|c} A + B_uF + LC_y + LD_{yu}F & -L \\ \hline F & 0 \end{array} \right) \tag{9}$$

where F is a stabilizing state feedback gain such that $A + B_uF$ is stable and L is a stabilizing observer gain such that $A + LC_y$ is stable. One possible way to construct the eight stable co-prime matrices in (7) is then

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} = \left(\begin{array}{c|cc} A + B_uF & B_u & -L \\ \hline F & I & 0 \\ C_y + D_{yu}F & D_{yu} & I \end{array} \right)$$

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} = \left(\begin{array}{c|cc} A + LC_y & -(B_u + LD_{yu}) & L \\ \hline F & I & 0 \\ C_y & -D_{yu} & I \end{array}\right) \tag{10}$$

Based on the above co-prime factorization of the system $G_{yu}(s)$ and the controller $K(s)$, we can give a parameterization of all controllers that stabilize the system in terms of a stable parameter $Q(s)$, i.e. all stabilizing controllers are given by [2]:

$$K(Q) = U(Q)V(Q)^{-1} \tag{11}$$

where

$$U(Q) = U + MQ, \quad V(Q) = V + NQ, \quad Q \in \mathscr{RH}_\infty$$

or by using a left factored form

$$K(Q) = \tilde{V}(Q)^{-1}\tilde{U}(Q) \tag{12}$$

where

$$\tilde{U}(Q) = \tilde{U} + Q\tilde{M}, \quad \tilde{V}(Q) = \tilde{V} + Q\tilde{N}, \quad Q \in \mathscr{RH}_\infty$$

Using the Bezout equation, the controller given either by (11) or by (12) can be realized as an LFT in the parameter Q,

$$K(Q) = \mathscr{F}_l(J_K, Q) \tag{13}$$

where J_K is given by

$$J_K = \begin{pmatrix} UV^{-1} & \tilde{V}^{-1} \\ V^{-1} & -V^{-1}N \end{pmatrix} = \begin{pmatrix} \tilde{V}^{-1}\tilde{U} & \tilde{V}^{-1} \\ V^{-1} & -V^{-1}N \end{pmatrix} \tag{14}$$

3. CONTROLLER IMPLEMENTATION

First, let us consider a controller change between two stabilizing controllers by using the YJBK parameterization. The following theorem shows that it is possible to switch from a stabilizing controller to another stabilizing controller while maintaining stability.

Theorem 3.1
Let the system be given by (1) and let a number of stabilizing controllers for the system be given by K_i. Then K_i, $i = 1, \ldots, p$, can be implemented as $K_0(Q_i) := \mathscr{F}_l(J_{K_0}, Q)$ where J_{K_0} is formed in analogy with (14) and where the stable Q_i parameter is given by

$$Q_i = X_i(\tilde{U}_i V_0 - \tilde{V}_i U_0) = X_i \bar{Q}_i, \quad i = 1, \ldots, p$$

or

$$Q_i = X_i(\tilde{V}_i(K_i - K_0)V_0), \quad i = 1, \ldots, p$$

with

$$X_i = M_0^{-1} M_i, \quad i = 1, \ldots, p$$

The proof of Theorem 3.1 is given in Appendix A. The proof can also be found in Reference [3].

Note that in the above theorem, it is not assumed that the co-prime factorization of G_{yu} is the same, i.e. that N and M are the same for both controllers K_0 and K_i as assumed in Reference [4], for example.

A state space realization of the Q_i parameter from Theorem 3.1 is given in the following lemma.

Lemma 3.2
Assume that the stabilizing controllers K_i are given as observer-based feedback controllers. Possible state space realizations of J_K, $\bar{Q}_i$ and X_i are then given by

$$J_K = \left(\begin{array}{c|cc} A + B_u F + LC_y + LD_{yu}F & -L & B_u + LD_{yu} \\ \hline F & 0 & I \\ -(C_y + D_{yu}F) & I & -D_{yu} \end{array}\right)$$

$$\bar{Q}_i = \left(\begin{array}{c|cc} A + L_i C_y & B_u + L_i D_{yu} & -L_i \\ \hline F_i & -I & 0 \end{array}\right) \times \left(\begin{array}{c|c} A + B_u F_0 & -L_0 \\ \hline F_0 & 0 \\ C_y + D_{yu} F_0 & I \end{array}\right)$$

$$X_i = \left(\begin{array}{c|c} A + B_u F_i & B_u \\ \hline F_i - F_0 & I \end{array}\right)$$

Proof
Lemma 3.2 follows directly from Theorem 3.1 by using the state-space description of the co-prime factors given by (10). □

In general, the most typical case is when we want to change from a P controller to a more advanced controller, such as an observer-based controller. Let us consider the case where the nominal controller is a P controller and the second controller is an observer-based controller. The state space realization of a P controller is given by

$$K_p(s) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_P \end{array}\right) \tag{15}$$

From the general state space description of the co-prime factors given in Reference [2], we can derive state space realizations of the factors when a P controller is applied. The co-prime factors are given by

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} = \left(\begin{array}{c|cc} A + B_u F_P & B_u & 0 \\ \hline F_P & I & D_P \\ C_y & 0 & I \end{array}\right)$$

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} = \left(\begin{array}{c|cc} A + B_u D_P C_y & -B_u & B_u D_P \\ \hline F_P - D_P C_y & I & D_P \\ C_y & 0 & I \end{array}\right) \tag{16}$$

where F_P is a fictitious state feedback gain that stabilizes $A + B_u F_P$. The derivation of (16) is based on the assumption that the system does not include a direct term.

To simplify the Q term, we can select F_P to be the same as the state feedback gain F_1 in the observer-based controller K_1. This will make $X_1 = I$, see Lemma 3.2. Note that for more than one observer-based controller, X_i, $i = 2, \ldots, p$ will, in general, not be equal to the identity matrix. Based on the co-prime factorization given in (16), we have the following result:

Lemma 3.3
Assume that the nominal stabilizing controller K_0 is a P controller and K_1 is an observer-based feedback controller. State space realizations of J_K, $\bar{Q}_1$ and X_1 are then given by

$$J_K = \left(\begin{array}{c|cc} A + B_u F_1 & 0 & B_u \\ \hline (F_1 - D_P C_y) & D_P & I \\ -C_y & I & 0 \end{array}\right)$$

$$\bar{Q}_1 = \left(\begin{array}{c|c} A + L_1 C_y & B_u D_p - L_1 \\ \hline F_1 & -D_p \end{array}\right)$$

$$X_1 = I$$

In connection with the above lemma, it is important to point out that the implementation involves a separate implementation of the state feedback dynamics and the observer dynamics, respectively.

These results show how it is possible to implement a controller as a stable Q parameter based on another stabilizing controller. The result also shows that it is possible to change the controller online without any jumps, just by scaling the Q parameter from zero to full value continuously. The closed-loop system is guaranteed to be stable for all values of Q_i. This is very useful in connection with implementation of unstable controllers.

Moreover, the above result can also be applied in connection with implementation of unstable controllers for a stable system, where no other stabilizing controller is implemented. From Theorem 3.1, we have the following result.

Lemma 3.4
Let $K_{\text{unstable}} = K_1 = U_1 V_1^{-1} = \tilde{V}_1^{-1}\tilde{U}_1,\ U_1, V_1, \tilde{U}_1, \tilde{V}_1 \in RH_\infty$ be an unstable controller for a stable system $G_{yu}(s) = N = \tilde{N},\ N, \tilde{N} \in RH_\infty$. The unstable controller can then be implemented as

$$K_1 = K_0(Q_1) = Q_1(I + NQ_1)^{-1}$$

where

$$Q_1 = M_1\tilde{U}_1$$

where M_1, $\tilde{U}_1$ and $\tilde{V}_1$ satisfies the Bezout equations:

$$\tilde{V}_1 M_1 - \tilde{U}_1 N_1 = I$$

Proof
The proof of Lemma 3.4 follows directly from Theorem 3.1 by using that $K_0(Q)$ is given by (12) with G_{yu} stable and $K_0 = 0$. □

To implement an unstable controller K_1 as described in Lemma 3.4, it is also possible to describe the controller as

$$K_1 = M_1\tilde{U}_1(I + N_1\tilde{U}_1)^{-1} \tag{17}$$

It is easy to show that the implementation of an unstable controller given in Reference [5] is equivalent with the above implementation based on the YJBK parameterization, [6]. Further, note that K_1 given in (17) can also be obtained directly from Lemma 3.3 by using $D_P = 0$.

If the system is unstable, the above results cannot be applied directly. Instead, Theorem 3.1 can be used, provided the system is strongly stabilizable. In this case, there will exist stable controllers that will stabilize the unstable system, [7]. The unstable controller can then be implemented using a stable stabilizing preliminary controller and the controller implementation of Theorem 3.1 to implement the unstable controller by using stable transfer functions (co-prime factors) only.

The result in Theorem 3.1 gives an implementation of a multivariable controller as a specific stable Q parameter in a parameterization of all stabilizing controllers. Theorem 3.1 provides one way to change the applied controller from K_0 to K_i online in closed-loop and also in a way such that the closed-loop system is stable for all applied controllers. Further, we do not necessarily need to be limited to the use of two controllers given by K_0 and K_i. It is not only possible to change the controller from K_0 to one of the p controllers given by K_i, it is also possible to change the controller K_i to K_j, $i,j = 1,\ldots,p,\ i \neq j$. In the case where we want to apply controllers that are a combination of all p (or a subset) stabilizing controllers, we get the following result.

Theorem 3.5
Let the system $G_{yu}(s)$ be given by (1) and let p stabilizing controllers for the system be given by $K_i,\ i = 1,\ldots,p$. Further, let the controllers be implemented as

$$K_i = K_0(Q) = K_0 + \tilde{V}_0^{-1}Q_i(I + V_0^{-1}N_0Q_i)^{-1}V_0^{-1}, \quad Q_i \in RH_\infty, \quad i = 1,\ldots,p$$

with Q_i given by

$$Q_i = X_i(\tilde{U}_i V_0 - \tilde{V}_i U_0), \quad i = 1, \ldots, p$$

Moreover, let a linear combination of the Q_i parameters be given by

$$Q = \sum_{i=1}^{p} \alpha_i Q_i$$

with $\sum_{i=1}^{p} \alpha_i = 1$. Then the resulting controller K is independent of K_0 and is given by

$$K(Q) = \left(\sum_{i=1}^{p} \alpha_i M_i \tilde{V}_i \right)^{-1} \sum_{i=1}^{p} \alpha_i M_i \tilde{U}_i$$

The proof of Theorem 3.5 is given in Appendix B.

Remark 3.1
It is stated in the theorem that the final controller is independent of K_0. The reason is that it is assumed that the scaling parameters α_i satisfy $\sum_{i=1}^{p} \alpha_i = 1$. However, from a stability point of view, there is no reason to require that the scaling parameters α_i need to satisfy that the sum is equal to 1. If they do indeed not satisfy this condition, the final controller will also be a function of K_0. It should also be pointed out that the scaling parameters does not even need to be positive. Negative values can be allowed without any closed-loop stability problems.

Using the complete description of the controller $K(s)$ given in Theorem 3.5 as a feedback controller, it is interesting to give an explicit equation for the closed-loop system. Such an explicit description of the closed-loop system can be applied in connection with the tuning of the controller, i.e. the selection of the α vector, such that the closed-loop system is optimized with respect to the operating point.

Let the complete open-loop system be described by (1). The closed-loop system from w to z, $T_{zw}(s)$, is then given by

$$T_{zw}(s) = \mathscr{F}_l(\Sigma, K) = G_{zw} + G_{zu} K (I - G_{yu} K)^{-1} G_{yw} \tag{18}$$

We can now give an explicit description of the closed-loop system T_{zw} when the controller $K(Q)$ given in Theorem 3.5 is applied.

Theorem 3.6
Let the closed-loop transfer function be given by (18). Further, let the stabilizing controller $K(Q)$ be given by

$$K(Q) = \left(\sum_{i=1}^{p} \alpha_i M_i \tilde{V}_i \right)^{-1} \sum_{i=1}^{p} \alpha_i M_i \tilde{U}_i$$

with $\sum_{i=1}^{p} \alpha_i = 1$. Then the closed-loop transfer function T_{zw} is given by

$$T_{zw}(s) = G_{zw} + G_{zu}\left(\sum_{i=1}^{p} \alpha_i M_i \tilde{U}_i\right) G_{yw}$$

The proof of Theorem 3.6 is given in Appendix C.

Again, it should be pointed out that $\sum_{i=1}^{p} \alpha_i = 1$ is not really required. If it is not satisfied, however, the nominal controller will also be part of the closed-loop transfer function T_{zw}.

The controller can now be designed based on an optimization of the closed-loop transfer function T_{zw} given in Theorem 3.6. If M_i and $\tilde{U}_i$ are stable and satisfy the Bezout equation in (8), then the closed-loop transfer function T_{zw} is stable. The design of the controller can then be done in open loop, which make e.g. multiobjective controller design more easy. This concept has been used in Reference [8] in connection with a multi-objective design method based on optimization of sensitivity functions.

4. GAIN SCHEDULING CONTROL

The relation between the proposed switching concept of multivariable controllers and gain scheduling control is considered in this section. For a description of gain scheduling control, see e.g. References [9–11].

The main idea in gain scheduling control is to switch between a number of pre-designed controllers with respect to the variation in the system. The variation can be a result of parameter variations and non-linearities in the system. In the switching method presented in this paper, the switching is derived with respect to changes of performance conditions.

This difference has also a major impact on the stability conditions for the closed-loop systems. For the method presented in this paper, the closed-loop stability is obtained by requiring that the nominal closed-loop system is stable and that the YJBK parameter Q is stable. Stability of systems including gain scheduling controllers are much more involved. Here, it is required that the applied controller will stabilize the non-linear system at the actual working point. The stability condition will not be changed much if the gain scheduling controllers are implemented by using the switching approach from this paper. In this case, stability of the nominal feedback loop together with stability of the YJBK parameter will not guarantee closed-loop stability. The reason is that the closed-loop system T_{zw} given in Theorem 3.6 will not be an affine function of the YJBK parameter Q. Q will also appear in the feedback loop of the closed-loop system, [2, 12]. This means that Q needs to be considered in connection with a feedback system.

However, it is possible to apply the switching method from this paper in connection with gain scheduling control with advantages. Using this approach, it is possible to separate the gain scheduling controller into two parts, a nominal controller related with the nominal performance of the system and a controller part related with the robustness of the feedback system. The last part is implemented by the YJBK parameter Q.

Let a gain scheduling controller be given by $K(\theta)$, where θ is the scheduling parameter. The controller for the nominal system is given by $K(0)$. Based on this controller, the YJBK

parameterization is derived resulting in the following feedback controller:

$$u = K(\theta)y = \mathscr{F}_l(J_K, Q(\theta))y \tag{19}$$

or

$$\begin{pmatrix} u \\ s \end{pmatrix} = J_K \begin{pmatrix} y \\ r \end{pmatrix}$$

$$r = Q(\theta)s \tag{20}$$

It can be shown, see References [2, 12], that the open-loop transfer function from r to s depends directly of the parameter variations and non-linearities in the system. The transfer function will be zero in the nominal case. This means that there will be an explicit decoupling of the YJBK parameter $Q(\theta)$ in the feedback controller. It is then clear that the $Q(\theta)$ part of the controller is closely related with the robustness of the closed-loop feedback system.

From the above description of the application of the YJBK parameterization in connection with gain scheduling control, it is also clear that it cannot be guaranteed that $Q(\theta)$ will always be a stable system. Further, this gives also a very direct connection between the variations in the system and the associated feedback part of the controller, given by $Q(\theta)$. It is possible to apply this connection in order to establish a performance validation of the closed-loop system.

5. EXAMPLE

Now, let us again consider the motivation example from Section 1. Based on the results given in the above section, we are now able to give a correct set-up for the controller K_α described in (5). Using the result from Lemma 3.3, we get directly that K_α needs to be given as

$$K_\alpha = K(Q(\alpha)) = \mathscr{F}_l(J_K, Q(\alpha))$$

where J_K and $Q(\alpha)$ are given by

$$J_K = \left(\begin{array}{ccc|cc} \begin{pmatrix} -5.9413 & 0.35054 & 0.85619 \\ 1.00 & -7.00 & -2.4495 \\ 0.0 & 2.4495 & 0.0 \end{pmatrix} & & & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1.0 \\ 0.0 \\ 0.0 \end{pmatrix} \\ \hline (987.06 & -4999.6 & 253110) & -1000 & 1.0 \\ (-1.0 & 5.0 & -253.1139) & 1.0 & 0 \end{array}\right)$$

$$Q(\alpha) = \alpha \times \left(\begin{array}{ccc|c} \begin{pmatrix} -2.1283 & 45.642 & -2310.5 \\ 0.35357 & -3.7679 & -166.07 \\ -0.13121 & 3.1056 & -3.3212 \end{pmatrix} & & & \begin{pmatrix} -990.87 \\ 0.64643 \\ 0.13121 \end{pmatrix} \\ \hline (-12.941 & 0.35054 & 0.85619) & 1000 \end{array}\right)$$

with this structure we have

$$K_{\alpha=0} = D_P$$

and

$$K_{\alpha=1} = K_1$$

Note that using the above controller K_α, the α parameter is not restricted to be in the interval [0, 1].

Further, note that the closed-loop poles of the system is invariant of the α parameter. The closed-loop system has the following stable closed-loop poles:

$$\text{poles}_{\text{cl},Q} = \begin{pmatrix} -998.67 \\ -25.1218 \\ -0.9022 \\ -7.7082 + 1.1005i \\ -7.7082 - 1.1005i \\ -5.3047 + 1.1643i \\ -5.3047 - 1.1643i \\ -0.6660 + 25.027i \\ -0.6660 - 25.027i \end{pmatrix}$$

The closed-loop poles given above are the combination of the closed-loop poles when the P controller and when the $\mathscr{H}_2$ controller is applied, respectively.

6. CONCLUSIONS

This paper demonstrates a number of successful applications of the YJBK-parameterization to problems of implementing multivariable controllers.

First of all, by using the YJBK-parameterization, it is possible to switch between controllers in a stable way. If the switch is established by a simple linear interpolation of the transfer functions of two stabilizing controllers, stability is not guaranteed during the transition. This lack of closed-loop stability is removed by using a parameterization in connection with the controller implementation.

Furthermore, it is also possible to optimize a controller given as a combination of a number of pre-designed controllers. This optimization can be done on-line, thereby facilitating adaptive optimization of the controller.

Another important issue is implementation of unstable controllers. Again, by using the YJBK-parameterization, it has been shown how unstable controllers can be implemented by using stable transfer functions only. This is especially important in connection with starting up unstable controllers.

APPENDIX A: PROOF OF THEOREM 3.1

Proof

Clearly, $\bar{Q}_i$ is a stable transfer matrix. We just need to show that $K_0(Q_i) = K_i$ when the above Q is applied and X_i is stable.

$$
\begin{aligned}
K_0(Q_i) &= K_0 + \tilde{V}_0^{-1} Q_i (I + V_0^{-1} N_0 Q_i)^{-1} V_0^{-1} \\
&= K_0 + \tilde{V}_0^{-1} X_i \tilde{V}_i (K_i - K_0) V_0 (I + V_0^{-1} N_0 X_i \tilde{V}_i (K_i - K_0) V_0)^{-1} V_0^{-1} \\
&= K_0 + \tilde{V}_0^{-1} X_i \tilde{V}_i (K_i - K_0)(I + N_0 X_i \tilde{V}_i (K_i - K_0))^{-1} \\
&= K_0 + \tilde{V}_0^{-1} X_i \tilde{V}_i (I + (K_i - K_0) N_0 X_i \tilde{V}_i)^{-1} (K_i - K_0) \\
&= K_0 + \tilde{V}_0^{-1} X_i (\tilde{V}_i^{-1} + (K_i - K_0) N_0 X_i)^{-1} (K_i - K_0) \\
&= K_0 + \tilde{V}_0^{-1} X_i (\tilde{V}_i^{-1} + \tilde{V}_i^{-1} \tilde{U}_i N_0 X_i - \tilde{V}_0^{-1} \tilde{U}_0 N_0 X_i)^{-1} (K_i - K_0) \\
&= K_0 + \tilde{V}_0^{-1} X_i (\tilde{V}_i^{-1} (I + \tilde{U}_i N_i) - \tilde{V}_0^{-1} \tilde{U}_0 N_i)^{-1} (K_i - K_0) \\
&= K_0 + \tilde{V}_0^{-1} X_i (\tilde{V}_0 M_i - \tilde{U}_0 N_i)^{-1} \tilde{V}_0 (K_i - K_0) \\
&= K_0 + \tilde{V}_0^{-1} (\tilde{V}_0 M_0 - \tilde{U}_0 N_0)^{-1} \tilde{V}_0 (K_i - K_0) \\
&= K_0 + (K_i - K_0) \\
&= K_i, \quad i = 1, \ldots, p
\end{aligned}
$$

From Lemma 3.2, we directly have that X_i is stable. □

APPENDIX B: PROOF OF THEOREM 3.5

Proof

The proof of Theorem 3.5 is derived for the case when $p = 2$. This will simplify the proof and it is without loss of generality.

Let the Q parameter be given by

$$
\begin{aligned}
Q &= xQ_1 + yQ_2 \\
&= xM_1(\tilde{U}_1 V_0 - \tilde{V}_1 U_0) + yM_2(\tilde{U}_2 V_0 - \tilde{V}_2 U_0) \\
&= xM_1 \tilde{V}_1 (K_1 - K_0) V_0 + yM_2 \tilde{V}_2 (K_2 - K_0) V_0
\end{aligned}
$$

with $x + y = 1$.

The resulting controller $K(Q)$ is then given by

$$
\begin{aligned}
K(Q) &= K_0 + \tilde{V}_0^{-1} Q (I + V_0^{-1} N_0 Q)^{-1} V_0^{-1} \\
&= K_0 + \tilde{V}_0^{-1} (x X_1 \tilde{V}_1 (K_1 - K_0) + y X_2 \tilde{V}_2 (K_2 - K_0)) \\
&\quad \times (I + N_0 (x X_1 \tilde{V}_1 (K_1 - K_0) + y X_2 \tilde{V}_2 (K_2 - K_0)))^{-1} \\
&= K_0 + \tilde{V}_0^{-1} (I + (x X_1 \tilde{V}_1 (K_1 - K_0) + y X_2 \tilde{V}_2 (K_2 - K_0)) N_0)^{-1} \\
&\quad \times (x X_1 \tilde{V}_1 (K_1 - K_0) + y X_2 \tilde{V}_2 (K_2 - K_0)) \\
&= K_0 + \tilde{V}_0^{-1} (M_0 + (x M_1 (\tilde{U}_1 - \tilde{V}_1 K_0) \\
&\quad + y M_2 (\tilde{U}_2 - \tilde{V}_2 K_0)) N_0)^{-1} \\
&\quad \times (x M_1 \tilde{V}_1 (K_1 - K_0) + y M_2 \tilde{V}_2 (K_2 - K_0)) \\
&= K_0 + (M_0 \tilde{V}_0 + (x M_1 (\tilde{U}_1 - \tilde{V}_1 K_0) \\
&\quad + y M_2 (\tilde{U}_2 - \tilde{V}_2 K_0)) N_0 \tilde{V}_0)^{-1} \\
&\quad \times (x M_1 \tilde{V}_1 (K_1 - K_0) + y M_2 \tilde{V}_2 (K_2 - K_0)) \\
&= K_0 + (M_0 \tilde{V}_0 + x M_1 (\tilde{U}_1 N_0 \tilde{V}_0 - \tilde{V}_1 U_0 \tilde{N}_0) \\
&\quad + y M_2 (\tilde{U}_2 N_0 \tilde{V}_0 - \tilde{V}_2 U_0 \tilde{N}_0))^{-1} \\
&\quad \times (x M_1 \tilde{V}_1 (K_1 - K_0) + y M_2 \tilde{V}_2 (K_2 - K_0)) \\
&= K_0 + (x M_1 \tilde{V}_1 + y M_2 \tilde{V}_2 \\
&\quad + (I - x M_1 \tilde{V}_1 - y M_2 \tilde{V}_2) M_0 \tilde{V}_0 \\
&\quad + (x M_1 \tilde{U}_1 + y M_2 \tilde{U}_2) N_0 \tilde{V}_0)^{-1} \\
&\quad \times (x M_1 \tilde{V}_1 (K_1 - K_0) + y M_2 \tilde{V}_2 (K_2 - K_0))
\end{aligned}
$$

$$
\begin{aligned}
&= K_0 + (xM_1\tilde{V}_1 + yM_2\tilde{V}_2 + x\tilde{U}_1(\tilde{M}_1N_0 - \tilde{N}_1M_0)\tilde{V}_0 \\
&\quad + y\tilde{U}_2(\tilde{M}_2N_0 - \tilde{N}_2M_0)\tilde{V}_0)^{-1} \\
&\quad \times (xM_1\tilde{V}_1(K_1 - K_0) + yM_2\tilde{V}_2(K_2 - K_0)) \\
&= K_0 + (xM_1\tilde{V}_1 + yM_2\tilde{V}_2)^{-1} \\
&\quad \times (xM_1\tilde{V}_1(K_1 - K_0) + yM_2\tilde{V}_2(K_2 - K_0)) \\
&= (xM_1\tilde{V}_1 + yM_2\tilde{V}_2)^{-1}(xM_1\tilde{U}_1 + yM_2\tilde{U}_2) \qquad \square
\end{aligned}
$$

APPENDIX C: PROOF OF THEOREM 3.6

Proof

Using a YJBK parameterization, it can be shown that the closed-loop system is given by [2]

$$T_{zw}(s) = G_{zw} + G_{zw}M_0\tilde{U}_0G_{ew} + G_{zu}M_0Q\tilde{M}_0G_{yw}$$

Using the Q given by

$$Q = \sum_{i=1}^{p} \alpha_i Q_i$$

where

$$Q_i = X_i(\tilde{U}_iV_0 - \tilde{V}_iU_0)$$

with $X_i = M_0^{-1}M_i$. Without loss of generality, let $p = 2$. The closed-loop transfer function T_{zw} is then given by

$$
\begin{aligned}
T_{zw} &= G_{zw} + G_{zu}(M_0\tilde{U}_0 + xM_1(\tilde{U}_1V_0 - \tilde{V}_1U_0)\tilde{M}_0 + yM_2(\tilde{U}_2V_0 - \tilde{V}_2U_0)\tilde{M}_0)G_{yw} \\
&= G_{zw} + G_{zu}((1 - xM_1\tilde{V}_1 - yM_2\tilde{V}_2)M_0\tilde{U}_0 + xM_1(\tilde{U}_1 + yM_2(\tilde{U}_2)V_0\tilde{M}_0))G_{yw} \\
&= G_{zw} + G_{zu}((xM_1\tilde{U}_1 + yM_2\tilde{U}_2) + (xM_1\tilde{U}_1 + yM_2\tilde{U}_2)N_0\tilde{U}_0 - (xU_1\tilde{N}_1 + yU_2\tilde{N}_2)M_0\tilde{U}_0)G_{yw} \\
&= G_{zw} + G_{zu}((xM_1\tilde{U}_1 + yM_2\tilde{U}_2) + xU_1(\tilde{M}_1N_0 - \tilde{N}_1M_0)\tilde{U}_0 + yU_2(\tilde{M}_2N_0 - \tilde{N}_2M_0)\tilde{U}_0)G_{yw} \\
&= G_{zw} + G_{zu}(xM_1\tilde{U}_1 + yM_2\tilde{U}_2)G_{yw} \qquad \square
\end{aligned}
$$

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