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A Parameterization of Observer-Based Controllers: Bumpless Transfer by Covariance Interpolation

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Abstract— **This paper presents an algorithm to interpolate between two observer-based controllers for a linear multivariable system such that the closed loop system remains stable throughout the interpolation. The method interpolates between the inverse Lyapunov functions for the two original state feedbacks and between the Lyapunov functions for the two original observer gains to determine an intermediate observer-based controller.**

I. INTRODUCTION

Observer-based controllers play a dominating role in modern control theory due to their wide generality and in industry due to their appealing qualities of monitoring process variables that are not directly accessible by measurements, and by allowing these estimated variables to be used for feedback. Even controllers that are not directly formulated as observer based controllers can usually be re-written into this form, see [12]. Several decades of research in observer based controllers have produced a vast number of design techniques for such controllers.

Ideally, one would like to design a controller that is both fast and has good measurement noise rejection properties. Clearly this is not possible, as increasing the bandwidth of the closed loop system will also make the system more sensitive to measurement noise [1]. Then the option is to design two distinct controllers: A controller K_1 which has a low closed loop bandwidth and is therefore not very sensitive to noise but exhibits a slow response and a controller K_2 which has a high bandwidth and is therefore fast but very sensitive to noise. Another reason to design two controllers for a certain plant can be associated with actuator saturation [8]. Also achieving some predefined output properties in the system performance can lead to follow a scheduled controller approach. Having designed the two controllers, the next issue which has to be addressed is how to switch between these two controllers. In many systems jumps in the input to the system are not desirable. Thus, finding a smooth way to switch between the two controllers comes up as a crucial problem. In [10], an approach is presented based on interpolation of Lyapunov functions. This approach, however, is based on continuity arguments, and is not guaranteed to cover the whole transition from one controller to another as opposed to the approached presented in the present paper.

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One important step in actual gain scheduling involves implementing the family of linear controllers such that the controller coefficient (gains) are varied (scheduled) according to the current value of the scheduling variable, also called scheduling signal that may be either exogenous signal or endogenous signal with respect to the plant [9]. Various issues arise here. An issue about the observer-based controllers here is that a simple gain interpolation technique which usually works well potentially can leave the closed loop system unstable for some intermediate points if applied to interpolate between two observer based controllers.

This paper presents an algorithm for interpolation between two observer-based controllers, designed to control a linear multivarible system, which renders the closed loop system stable for all values of the interpolation parameter. The family of observer-based controllers which will be introduced here can help the designer to achieve a safe bumpless transfer between two observer-based controllers to reach the control objectives. Finally, two numerical examples illustrate our claims.

II. PRELIMINARIES

The following notations are used in this paper. X^* indicates the transpose for X which is either a matrix or a vector. $X < 0$ ($X > 0$) means that X is symmetric and negative definite (positive definite). $Re(X)$ denotes the real part of a complex number. Finally, I stands for an identity matrix with appropriate dimension.

Consider the open loop system

$$\dot{x} = Ax + Bu\,, \quad y = Cx + Du$$

then:

- The system is asymptotically stable if all eigenvalues of A satisfy $Re(\lambda) < 0$ [4].
- A matrix A is Hurwitz if and only if for any given positive definite symmetric matrix Q there exists a positive definite matrix P that satisfies the Lyapunov equation [3]:

$$PA + A^*P = -Q\,, \quad Q = Q^*$$

or equivalently

$$PA + A^*P < 0$$

- The previous criterion can be written as

$$AP^{-1} + P^{-1}A^* < 0$$

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III. Main Results

Throughout this paper we will assume that (A,B) is controllable and (C,A) is observable. It should be noted that a slightly weaker results can also result even if (A,B) and (C,A) are only stabilizable and detectable.

J. Bertram in 1959 was perhaps the first to realize that if a given system realization was state controllable, then any desired characteristic polynomial could be obtained by state-variable feedback [5]. Since then state feedback and state space based output feedback controllers have been two of the most researched and written about issues in modern control theory. There is, of course, a long history of gain scheduling in applications too. However, bumpless transfer (soft switching) between two state feedbacks need some precise considerations because the gain interpolation of gain scheduled state feedbacks can leave the closed loop system unstable for the intermediate points. The following lemma presents an algorithm for interpolating between two state feedbacks while the closed loop system remains stable for all intermediate points.

Lemma 1: Consider the following control system:

$$\dot{x} = Ax + Bu$$

and assume that $u = F_0\ x$ and $u = F_1\ x$ are both stabilizing state feedback laws, with Lyapunov functions:

$$V_0(x) = x^*\ X_0\ x \quad \text{and} \quad V_1(x) = x^*\ X_1\ x$$

respectively, with $X_i > 0, i = 0,1$. Then, a family of state feedback gains $F(\alpha)$ which stabilizes the system for every α, $0 \le \alpha \le 1$ is given by:

$$F(\alpha) = \mathscr{F}_\ell(J_F, \alpha I) \tag{1}$$

where

$$J_F = \begin{pmatrix} F_0 & (F_1 - F_0)X \\ I & I - X \end{pmatrix}, \quad X = X_1^{-1} X_0$$

Furthermore, $F(\alpha)$ satisfies $F(0) = F_0$ and $F(1) = F_1$.

Proof: Defining $Y_0 = X_0^{-1}$ and $Y_1 = X_1^{-1}$, we can rewrite the Lyapunov inequalities corresponding to $V_0(x)$ and $V_1(x)$ as:

$$Q_0 := (A + BF_0)Y_0 + Y_0(A + BF_0)^* < 0$$

and

$$Q_1 := (A + BF_1)Y_1 + Y_1(A + BF_1)^* < 0$$

respectively. We will demonstrate, that the matrix valued function

$$Y(\alpha) = (1 - \alpha)Y_0 + \alpha Y_1$$

which is positive definite for $\alpha \in (0;1)$, satisfies

$$(A + BF(\alpha))Y(\alpha) + Y(\alpha)(A + BF(\alpha))^* < 0$$

for all $\alpha \in (0;1)$. To that end, we observe that:

$$\begin{aligned}
&(A + BF(\alpha))Y(\alpha) \\
&\quad = \left(A + B\left(F_0 + \alpha(F_1 - F_0)X(I - \alpha(I - X))^{-1}\right)\right) \\
&\qquad ((1-\alpha)Y_0 + \alpha Y_1) \\
&\quad = \left(A + BF_0 + \alpha B(F_1 - F_0)Y_1Y_0^{-1}\left(I - \alpha\left(I - Y_1Y_0^{-1}\right)\right)^{-1}\right) \\
&\qquad ((1-\alpha)Y_0 + \alpha Y_1) \\
&\quad = \left(A + BF_0 + \alpha B(F_1 - F_0)Y_1\left((1-\alpha)Y_0 + \alpha Y_1\right)^{-1}\right) \\
&\qquad ((1-\alpha)Y_0 + \alpha Y_1) \\
&\quad = (A + BF_0)\left((1-\alpha)Y_0 + \alpha Y_1\right) + \alpha B(F_1 - F_0)Y_1 \\
&\quad = (1-\alpha)\left(A + BF_0\right)Y_0 + \alpha\left(A + BF_1\right)Y_1
\end{aligned}$$

from which we conclude that:

$$\begin{aligned}
&(A + BF(\alpha))Y(\alpha) + Y(\alpha)(A + BF(\alpha))^* \\
&\quad = (1-\alpha)Q_0 + \alpha Q_1 < 0, \qquad\qquad \forall \alpha \in (0;1)
\end{aligned}$$

which establishes the proof.

From the last argument, note that in the special case $Q_0 = Q_1$, which is often obtainable, the proposed feedback will actually remain stable for *all* α, not just for $\alpha \in (0;1)$. ■

Note also that if there is a common Lyapunov function for the both state feedback controllers the Lemma 1 interpolation reduces to simple gain interpolation.

In most practical applications, the system states are not completely accessible and all the designer knows are the outputs and the inputs. Hence, the estimation of the system states is often necessary to realize some specific design objectives. The important issue in designing the observer gain (L) is to have $A + LC$ as a stable system. Thus, the critical point in bumpless transfer between two observers is the stability of $A + LC$. The subsequent lemma expresses an algorithm for interpolating between two observers while the stability of $A + LC$ is guaranteed.

Lemma 2: Let L_0 and L_1 be two different Luenberger observer gains for the following system:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

and suppose that

$$V_0(x) = x^*\ Z_0\ x \quad \text{and} \quad V_1(x) = x^*\ Z_1\ x$$

are the corresponding Lyapunov functions to $A + L_0C$ and $A + L_1C$, respectively, with $Z_i > 0, i = 0,1$. Then a family of observer gains $L(\beta)$, $0 \le \beta \le 1$ is given by:

$$L(\beta) = \mathscr{F}_\ell(J_L, \beta I) \tag{2}$$

where

$$J_L = \begin{pmatrix} L_0 & I \\ Z(L_1 - L_0) & I - Z \end{pmatrix}, \quad Z = Z_0^{-1}\ Z_1$$

Moreover, $L(\beta)$ satisfies $L(0) = L_0$ and $L(1) = L_1$.

Proof: The intermediate points admit the Lyapunov function given by

$$Z(\beta) = (1 - \beta)\ Z_0 + \beta\ Z_1$$

To verify the above claim, we have to show that

$$Z(\beta)(A + L(\beta)C) + (A + L(\beta)C)^* Z(\beta) < 0$$

The first term in left side of the Lyapunov inequality can be rewritten as:

$$\begin{aligned}
&Z(\beta)(A+L(\beta)C)\\
&\quad=((1-\beta)Z_0+\beta Z_1)\\
&\qquad\big(A+\big(L_0+\beta(I-\beta(I-Z))^{-1}Z(L_1-L_0)\big)\,C\big)\\
&\quad=((1-\beta)Z_0+\beta Z_1)\\
&\qquad\big(A+L_0C+\beta(I-\beta(I-Z_0^{-1}Z_1))^{-1}Z_0^{-1}Z_1(L_1-L_0)C\big)\\
&\quad=((1-\beta)Z_0+\beta Z_1)\\
&\qquad\big(A+L_0C+\beta((1-\beta)Z_0+\beta Z_1))^{-1}Z_1(L_1-L_0)C\big)\\
&\quad=(1-\beta)Z_0\,(A+L_0C)+\beta Z_1\,(A+L_1C)
\end{aligned}$$

So, we can conclude:

$$\begin{aligned}
&Z(\beta)(A+L(\beta)C)+(A+L(\beta)C)^*Z(\beta)\\
&\quad=(1-\beta)(Z_0(A+L_0C)+(A+L_0C)^*Z_0)+\\
&\qquad\beta(Z_1(A+L_1C)+(A+L_1C)^*Z_1)
\end{aligned}$$

According to the assumptions Z_0 and Z_1 are Lyapunov functions for $A+L_0C$ and $A+L_1C$, respectively. Thus, we have:

$$Z_0(A+L_0C)+(A+L_0C)^*Z_0<0$$

and

$$Z_1(A+L_1C)+(A+L_1C)^*Z_1<0$$

Then the proof is immediate. ■

According to the separation principle the problem of designing an observer-based controller can be broken into two separate parts: observer design and state feedback design. This approach facilitates the design procedure. Lemma 1 presented an algorithm for interpolation between two state feedbacks while satisfying the stability criterion. Similar algorithm was described in Lemma 2 for observers. Combining the results from the two previous lemmas leads to an algorithm for bumpless transfer between two observer-based controllers.

Theorem 1: Consider two observer-based controllers

$$K_0=\left(\begin{array}{c|c} A+BF_0+L_0C+L_0DF & -L_0\\ \hline F_0 & 0\end{array}\right)$$

and

$$K_1=\left(\begin{array}{c|c} A+BF_1+L_1C+L_1DF & -L_1\\ \hline F_1 & 0\end{array}\right)$$

for the minimal system

$$\dot{x}=Ax+Bu\,,\quad y=Cx+Du$$

.

which have been already designed [6].

Then a family of observer-based controllers for the mentioned system is denoted as

$$K(\gamma)=\mathscr{F}_\ell(J_K,\gamma I)\,,\quad 0\le\gamma\le 1 \tag{3}$$

where

$$J_K=\begin{pmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}$$

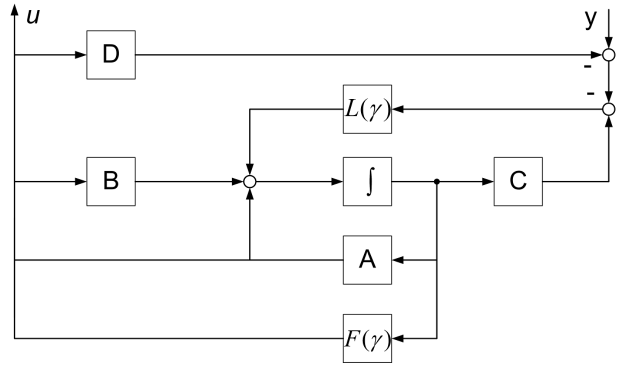

Fig. 1. The family of observer-based controllers introduced by Theorem 1

$$M_{11}=\left(\begin{array}{c|c} A+BF_0+L_0DF_0+L_0C & -L_0\\ \hline F_0 & 0\end{array}\right)$$

$$M_{12}=\begin{pmatrix} (L_0D+B)\ (F_1+F_0)\ X & I\\ (F_1-F_0)\ X & 0\end{pmatrix}$$

$$M_{21}=\begin{pmatrix} I & 0\\ Z\ (L_1-L_0)\ (C+DF_0) & -Z\ (L_1-L_0)\end{pmatrix}$$

$$M_{22}=\begin{pmatrix} I-X & 0\\ Z\ (L_1-L_0)\ D\ (F_1-F_0)\ X & I-Z\end{pmatrix}$$

X and Z are as those defined in (1) and (2).

Also, $K(\gamma)$ satisfies $K(0)=K_0$ and $K(1)=K_1$.

Proof: Fig. 1 shows the family of observer-based controllers presented by Theorem 1 (equation (3) is the LFT representation of the illustrated block diagram). Applying the principle of separation and then results in Lemmas 1 and 2, the proof is immediate. ■

It is interesting to see that if there is a common Lyapunov function for the closed loop system composed of the plant and the family of observer-based controllers the interpolation reduces to the simple gain interpolation. Furthermore, the closed loop system is stable for any γ (not only $0\le\gamma\le 1$) and any rate of switching [1]. Otherwise, in the general case which was addressed in Theorem 1 we assume that the scheduling variable is slow enough not to cause stability problems.

It should be emphasized that if the results above (as suggested) are applied to facilitate a transition from one controller to another, the stability arguments only hold during the transition if this is sufficiently slow (rate limited).

IV. Numerical Examples

Example 1: This example illustrates the fact that the gain interpolation between two stabilizing observer-based controller can cause instability for some intermediate points. However, it is shown that the algorithm proposed by Theorem 1 does not have this deficiency.

Consider the following third order system,

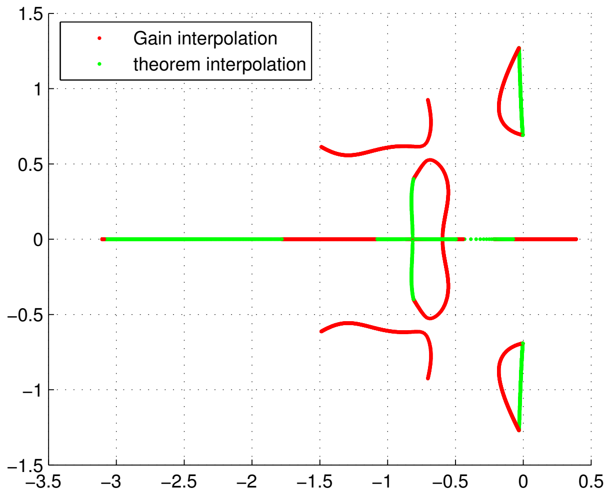

Fig. 2. Eigenvalue Plot of The Closed Loop System in Example 1 where Gain Interpolation (red curve) and Theorem Interpolation (green curve) of Observer-Based Controllers are applied.

$$A=\begin{pmatrix} -0.597 & -0.038 & 0.832 \\ 1.636 & -0.121 & 0.068 \\ -0.334 & -0.968 & -0.311 \end{pmatrix},\ B=\begin{pmatrix} -0.638 \\ 0.091 \\ 0.363 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 1 & 1 \end{pmatrix},\ D=0$$

This system is unstable with eigenvalues of -1.3195 and $0.1453 \pm 1.0314i$. Then two different observer-based controllers have been designed for stabilizing the system:

$$K_0=\left(\begin{array}{ccc|c} -1.863 & -1.995 & -0.393 & 1.353 \\ 0.762 & -0.896 & -0.811 & 0.861 \\ -1.75 & -1.992 & -1.751 & 1.367 \\ \hline -0.135 & 0.947 & -0.200 & 0 \end{array}\right)$$

and

$$K_1=\left(\begin{array}{ccc|c} -1.152 & -1.103 & -0.834 & 0.916 \\ 1.033 & -0.651 & -0.376 & 0.551 \\ -1.441 & -1.784 & -0.785 & 0.901 \\ \hline -0.567 & 0.233 & 1.175 & 0 \end{array}\right)$$

Fig. 2 illustrates the eigenvalue plot of the closed loop system where the gain interpolation and the interpolation proposed by the previous theorem for observer-based controllers are applied for bumpless transfer between the two designed controllers. The plot reveals that the naive gain interpolation of the controllers fails to maintain the stability of the closed loop system while the interpolation appeared in the previous theorem renders the closed loop system stable for all $0 \leq \gamma \leq 1$.

Example 2: In this example we will show the bumpless transfer between two state feedbacks designed to meet different objectives in a HVAC system applying the algorithm described by Theorem 1.

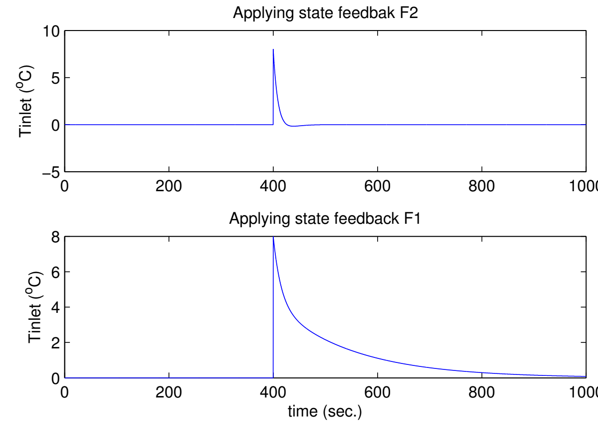

Fig. 3. Inlet temperature while applying state feedbacks F_1 and F_2

We consider here the control of the inlet air temperature of a ventilation system (a water-to-air heat exchanger). In accordance with the linearized model of a water-to-air heat exchanger described in [7] the linear model from primary (supply) water flow ($\dot{m}_{ws}$) to inlet air temperature ($Tinlet$) can be explained as following:

$$\begin{bmatrix} \dot{T}inlet \\ \dot{T}wout \end{bmatrix} = \begin{bmatrix} a_4 & a_3 \\ 0 & a_1 \end{bmatrix} \cdot \begin{bmatrix} Tinlet \\ Twout \end{bmatrix} + \begin{bmatrix} b_3 \\ b_1 \end{bmatrix} \cdot \dot{m}_{ws}$$

$$y=\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} Tinlet \\ Twout \end{bmatrix}$$

where

$$a1=-0.0352,\quad a3=0.0564,\quad a4=-0.5961$$

$$b1=17232,\quad b3=227635$$

and $Twout$ represents the temperature of the water that leaves the coil. Two state feedbacks $F_1 = \begin{bmatrix} -0.2464\cdot10^{-5} & 0.0155\cdot10^{-5} \end{bmatrix}$ and $F_2 = \begin{bmatrix} -0.2103\cdot10^{-5} & 0.0421\cdot10^{-5} \end{bmatrix}$ are designed for this system. Fig. 3 illustrates the system output ($Tinlet$) while state feedbacks F_1 and F_2 are applied to remove the step disturbance occurring at $400sec.$. The response resulted from applying of F_1 is slow but no overshoot happens. However, applying F_2 results in a faster response with overshoot. The fact is that the overshoot in the response is not desirable because in the real system it causes some oscillations which damps very slowly.

To overcome the problem of designing a fast controller with no overshoot we combine the two state feedbacks: When the output of the system ($Tinlet$) is more than (1^oC) away from the set-point F_2 will be the active controller but when the system output reaches the bound of ($\pm1^oC$) from the set-point a bumpless transfer, applying the algorithm described in Theorem 1, from F_2 to F_1 happens (γ is scheduled in accordance with the distance from the set-point). Fig. 4 shows the result of applying the proposed control strategy to remove a step disturbance occurring at

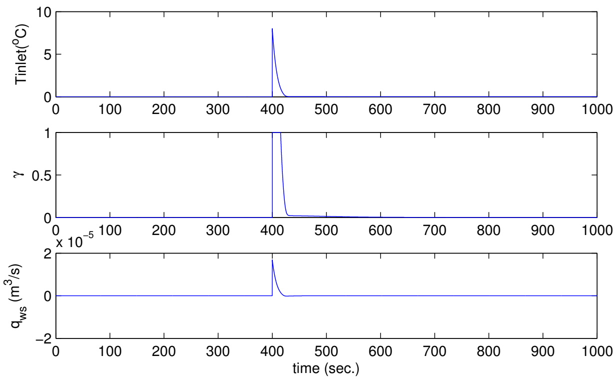

Fig. 4. Inlet temperature, scheduling parameter (γ), and the control input when a family of state feedbacks presented by Theorem 1 acting upon the HVAC system

400*sec*.. As can be seen, we have a fast response with no overshoot. Therefore, the proposed control strategy meets the control objectives.

It should be noted that as the bumpless transfer approach presented in this paper relies on stability considerations only, there is no guarantee as regards the magnitude of the transients, if the transition is performed quickly. We refer to [11] for an excellent treatment of the issue of bumpless transfer for systems with fast switching.

V. CONCLUSIONS

In this paper an algorithm to interpolate between two observer-based controllers was presented. The proposed algorithm guaranteed the stability of the closed loop system for the intermediate points. At the end, two numerical examples were presented. The first example showed that the naive gain interpolation failed to maintain the stability of the closed loop system while the algorithm in Theorem 1 worked perfectly to keep the closed loop system stable. The second example illustrated the application of the proposed interpolation algorithm to bumpless transfer between two observer-based controllers.

It should be noted that although the method has been described only for two state feedbacks and two observer gains for simplicity, it can easily be extended to larger numbers of either.

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