

Control of systems with costs related to switching: applications to air-condition systems

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Abstract—The objective of this paper is to investigate a low complexity method for controlling systems with binary inputs that have costs related to switching. The control objective for this type of systems is often a trade off between the deviation from the reference and the number of switches (weariness energy efficiency etc.). For such systems a steady state solution might never be attained, but rather the optimal behavior might be constituted by a limit cycle. In this paper we consider the problem of finding and controlling the system towards an optimal limit cycle. A low complexity approach giving a suboptimal solution, and avoiding the above problems will be proposed.

I. INTRODUCTION

In this paper we aim at developing a low complexity method for optimal control of system with binary inputs that have costs related to switching. The method is exemplified by a motivating example based on developing energy optimal control for Air-condition (AC) system.

The basic objective of an AC system is to remove heat from a room to keep the room temperature close to the reference. The AC system is composed of a compressor, an evaporator, a condenser, and an expansion valve, where the compressor determines the cooling capacity. In a typical AC system, the room temperature will vary around the reference level, due to the fact that the compressor only can be switched on/off. Hence determining the optimal switch frequency and duty cycle is a trade-off between comfort, energy consumption (which is measured by COP, coefficient of performance) and compressor weariness due to switching. It should be noted that the compressor weariness here refers to the compressor component weariness caused by a poor lubrication at a low speed. The reason the weariness is important in the AC system in our case study is that the compressor components wear in every startup due to the poor lubrication condition.

From the comfort point of view, a small temperature variation is preferred, hence the switch frequency should be very fast. With such a fast switching frequency, the compressor has problem with lubrication, but a high efficiency (COP) is achieved[1]. The reason for the high COP is that the evaporator will not get empty during the short off period, therefore the effect of low COP from filling the refrigerant into the evaporator is avoided. However the conclusion in [1] is based only on the criterion of energy

consumption. Therefore in this paper, we will focus on the switch frequency range which is much slower than above.

With a slower switching frequency, the weariness of compressor due to the poor lubrication condition in the startup phase is obvious and it gets worse with the increase of switch frequency. The temperature variation with the slower switching frequency is much bigger. COP varies with the switch frequency too. Therefore the 3 parameters of temperature variation, compressor weariness, and energy efficiency should all be used for evaluating the system performance. For simplicity, in this paper, we will use comfort and compressor weariness for the performance measure.

Several methods exist that can handle systems with binary inputs that have costs related to switching. They can be divided into two categories: low complexity and advanced methods. In the low complexity category, pulse width modulation (PWM) and hysteresis controllers are the two popular controller types. In the advanced methods category, methods related to model predictive control (MPC) is one example.

PWM has to have very fast switching frequency, which has been proven not suitable in the AC system control case.

Hysteresis controller are widely used today. The problem with hysteresis controller is that the system always work on the same constraint (usually temperature). The hysteresis controller are usually tuned under a certain working condition, which performs not so good when working condition changes, therefore the optimality is destroyed.

One of the standard methods for hybrid switched systems is MPC using the mixed logical dynamical framework (MLD)[2]. An experiment with simulation model has been done to investigate how the prediction horizon affects the resulted switch period (Fig.1). The cost function of the MPC composes of quadratic error (deviation from reference) and switches numbers and both terms are penalized. The results are presented in Fig.1). It has shown the problem of finite horizon prediction - the controller postpones the switches because of the penalty on switching, until at last a short switch period has to be used in order not to violate the constraint. One way to improve it is to use longer prediction horizon. It seems that with increasing prediction horizon, the resulting switch period converges to 16 minutes. It can be concluded from the results that an extremely long prediction horizon is needed in order to get a good result. However MPC with very long prediction horizon requires a high computational power, which is not feasible in an AC application.

A different way of using MPC is proposed in [3], where the first step is to find the optimal limit cycle and then use

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MPC to bring the system to an optimal limit cycle. The author has shown that this method can avoid the problem of finite horizon method for optimal limit cycles and this method provides good results for the particular application. However the method has to be modified for the AC system and the computational power is not affordable for the AC application.

Other MPC methods based on solving finite horizon optimization problems also exists such as [4] and [5]. Common for these methods based on finite horizon prediction is that they typically produce rather complex controllers requiring large computational power which is not feasible for mass production application and special care is not taken using finite horizon methods on systems with discrete inputs, it can lead to rather poor performance.

Our objective in this paper is to develop a method which has low complexity as PWM and hysteresis, but at the same time achieves close to the optimal solution for controlling systems with binary inputs.

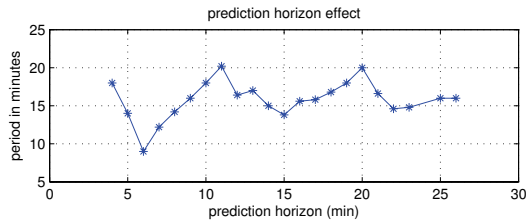


Fig. 1. Prediction horizon and periods. With increase of the prediction horizon, the compressor switch period moves towards 16 minutes.

The paper is organized in the following way. Modeling of the system is briefly presented in Section II. An analysis of performance function nature for discrete input system are carried out in section III. A new method is presented in Section IV. The conclusion is given in Section V.

II. SYSTEM DESCRIPTION AND MODELLING

The AC system is a unit which composes of compressor, evaporator, condenser, and evaporation valve. The refrigeration cycle works in this way: The compressor compresses the refrigerant gas into high pressure at a high temperature, and then gas is sent to condenser, where high pressure low temperature liquid comes out. Then expansion valve releases the liquid into low pressure liquid (or gas liquid mix), and the liquid flows into evaporator and evaporates there. During the evaporation, the refrigerant absorbs heat from outside of the evaporator, which produces coldness for the room. The switch on/off is referred to the compressor switches. The detailed explanation can be found in [6]

Whenever the compressor is switched on (assume that it is switched off long enough that the evaporator is empty), the evaporator has to be filled in first. This leads to low COP during the start up. Another reason for low COP during startup is also that the compressor is lubricated well enough, which then results in low energy efficiency.

For simplification, the air condition system is considered as a unit, which removes energy from the room. In order

to illustrate the method, for simplicity, the air-conditioned room is modelled as a first order system, which is shown in Fig.2. In this paper the AC system is not modeled in detail, but presented as an energy input to the room model.

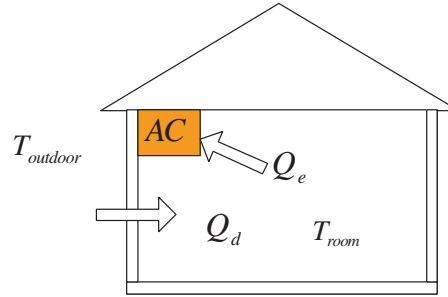


Fig. 2. First order room model

$$\frac{dT_{room}}{dt} = \frac{\dot{Q}_d - \dot{Q}_e}{C_p m} \quad (1)$$

$$\dot{Q}_d = \frac{(T_{outdoor} - T_{room})}{R} \quad (2)$$

Insert (2) into (1), the model can be formulated as (3),

$$\frac{dT_{room}}{dt} = \frac{-1}{C_p m R} T_{room} + \frac{-1}{C_p m} \dot{Q}_e + \frac{1}{C_p m R} T_{outdoor} \quad (3)$$

where T_{room} is the temperature of room air.

\dot{Q}_d is the heat flow from ambient to the room.

\dot{Q}_e is the energy removed by the AC system when it is switched on.

R is the thermal resistance from ambient to the room.

m is the mass of room air.

C_p , is the specific heat capacities of room air.

Equation (3) can be expressed by state space equation (5), where T_{room} is the state, \dot{Q}_e is the input u , and the last term in (3) is disturbance E . In this paper, the following parameters are used for simulation.

$$A = -2.00123 \times 10^{-4};$$

$$B = -4.4028 \times 10^{-6};$$

$$E = 0.0057;$$

$$u = 0 \text{ or } 300w$$

III. COST FUNCTION ANALYSIS

For the type of system described above, the existing methods have proved to be very unreasonably computationally demanding and some have fundamental problems with finite horizon prediction, therefor a new method has to be developed. We know that the optimal solution for the AC system with discrete input will be a stable limit cycle, where the compressor switches on and off with a certain frequency and duty cycle. The optimal solution will achieve the smallest deviation from the reference with the least number of the switches. Inspired by MPC, a cost function can be formulated in the similar way where both the comfort and cost of getting the comfort (switches in this problem) appear. Since the

system is running periodically, the system performance can be measured by only one period, which leads to (4). The widely accepted definition for comfort is Predicted Mean Vote (PMV) and Predicted Percentage Dissatisfied (PPD), (details see [7]), where temperature, humidity, draught, occupant activity level, cloth level etc. are all taken into account. For our case study, the PMV and PPD are both too complicated as a start point. Therefore a quadratic error is used. It might not represent the best comfort, but it gives an indication of the AC system performance. On the numerator of (4), the squared error of the output is accumulated through the whole period, and two switches in one period are used. On the denominator, period T_s is used to express that the cost J here is the cost per time unit. This makes it possible to calculate the performance for a system by observing only one period. The corresponding figure is presented in Fig.3.

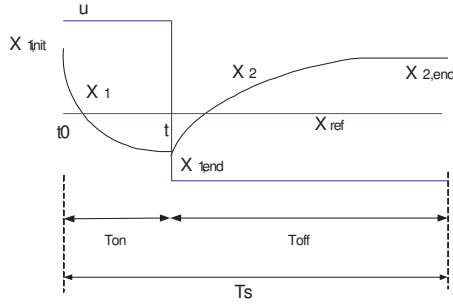


Fig. 3. one period

Before we start developing new method which can be measured by this cost function, a thorough study on the cost function (4) is necessary.

$$J(T_s) = \frac{Q \cdot \int_{t_0}^{t_0+T_s} (y - y_{ref})^2 dt + R \cdot J_{sw}}{T_s} \quad (4)$$

T_s is the total period. Q is the penalty on temperature deviation from the reference and R is the penalty on 2 switches in one period.

Since a period includes on and off parts, the cost J actually is a function of T_{on} and T_{off} . This function is not convex and it is not easy to determine when its gradient is zero. Our interest here is to find out if the cost J has only one optimum in T_{on} and T_{off} and if so, whether we can find the optimum explicitly. This type of generic cost function is relevant for a huge number of systems with discrete inputs.

Since in the discrete system, T_{on} and T_{off} have the equal meaning for the system as period T_s and duty cycle α . A question here might come up: 'is the optimal duty cycle the same as the load?'. (in this context: the load is defined as a percentage and it means that a system running with this percentage of the high limit power (300W in this example) will be controlled at the reference. In this example, 50% load means that running with 150W (50% \times 300) will make the system stay at the reference.)

The solution to the state space model (5) is (6) and (7), where (6) is for the on period and (7) is for the off period.

$x_{1,init}$ refers to the state when the system switches from off to on, and $x_{1,end}$ refers to the state when the system switches from on to off. When the input u is 0, which means that AC is switched off, the system becomes (7), where $x_{1,end}$ is the state when the system switches from on to off.

$$\dot{x} = Ax + Bu + E \quad (5)$$

$$x_1 = \frac{Bu}{-A} + (x_{1,init} + \frac{Bu}{A}) \cdot e^{At} \quad (6)$$

$$x_2 = x_{1,end} \cdot e^{A(t-T_{on})} \quad (7)$$

The cost J depending on T_{on} and T_{off} should be evaluated under the steady state conditions, otherwise it does not make sense to compare. Under steady state conditions, the system is periodical, which means that the states at the start of a cycle is same as the end of the cycle, which is described in equation (8). $x_{2,end}$ denotes the state when the system switches from off to on. From equation (6), (7) and (8), $x_{1,init}$ can be expressed as a function of T_{on} and T_{off} . Therefor $x_{1,init}$ and $x_{1,end}$ can be eliminated form the x_1 and x_2 expression, and be reformulated as (9) and (10)

$$x_{1,init} = x_{2,end} \quad (8)$$

$$x_1 = \frac{Bu}{-A} + \left[\frac{Bu}{A} \cdot \frac{e^{A \cdot (T_{on} + T_{off})} - e^{A \cdot T_{off}}}{1 - e^{A \cdot (T_{on} + T_{off})}} + \frac{Bu}{A} \right] \cdot e^{A \cdot t} \quad (9)$$

$$x_2 = \left\{ \frac{Bu}{-A} + \left[\frac{Bu}{A} \cdot \frac{e^{A \cdot (T_{on} + T_{off})} - e^{A \cdot T_{off}}}{1 - e^{A \cdot (T_{on} + T_{off})}} + \frac{Bu}{A} \right] \cdot e^{A \cdot T_{on}} \right\} \cdot e^{A \cdot (t - T_{on})} \quad (10)$$

The output of the system is

$$y_1 = c \cdot x_1$$

$$y_2 = c \cdot x_2$$

The cost function now becomes

$$J(T_{on}, T_{off}) = \frac{Q \cdot \int_0^{T_{on}} (x_1 - x_{ref})^2 dt}{T_{on} + T_{off}} + \frac{Q \cdot \int_{T_{on}}^{T_{on} + T_{off}} (x_2 - x_{ref})^2 dt + R \cdot J_{sw}}{T_{on} + T_{off}} \quad (11)$$

Inserting 9 and 10 into 11, the corresponding plot (cost J) can be seen in Fig. 4, where T_1 is the on period, and T_2 is the off period. From the figure it can be seen that there is only one optimum. Using Newton's method with (11), the optimal period can be found easily. However, the initial guess for the Newton search is actually critical for this problem, which is due to the irregularity of cost surface J .

An adjustment of (11) has been done to check if it could help to get surface for which is easier to find the optimal solution. What we did is to parameterize the cost J with period T_s and duty cycle α . This can be simply done by replacing T_{on} by $T_s \cdot \alpha$, and T_{off} by $T_s \cdot (1 - \alpha)$. The result is plotted in Fig.5. The surface J is much more regular, which means that the initial guess for the Newton search is not

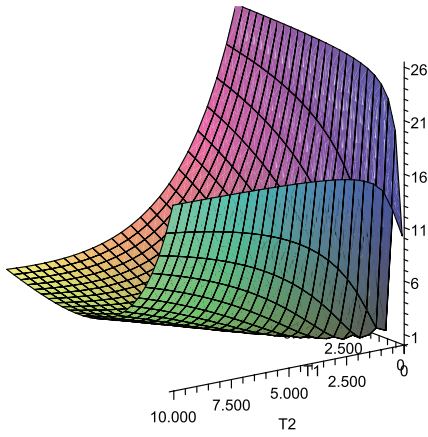


Fig. 4. Cost J in T_1 (T_{on}) and T_2 (T_{off})

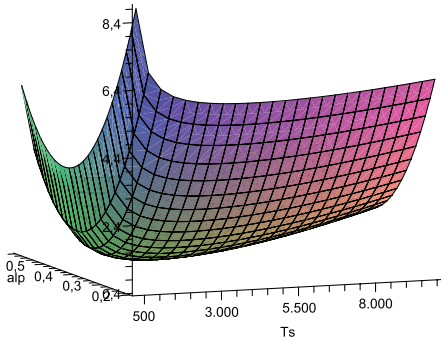


Fig. 5. Cost J in period T_s and duty cycle α

that strict anymore. Form several tests, it seems like that the initial guess is much less strict than parameterizing the system over T_{on} and T_{off} .

Applying the above idea on the system with 90% load, we found that the optimal period is 6447s, and the duty cycle is 91%. The optimal solution (duty cycle 91%) is different from the load (90%), which answers the question in beginning of this section.

A further investigation was carried out to learn if this is a special case or holds in general, what factors affect the difference between the load and optimal duty cycle and what are the reasons that that cause the difference between the load and optimal duty cycle.

To answer these questions, two experiments were designed. One is to run with different load to check if the load has influence on the difference between the load and optimal duty cycle. The other is to change penalty ratio Q and R . As an example the results from 90% and 10% load with different penalty ratio is presented in Tab. I.

The results show that for duty cycle α (or $1 - \alpha$) close to 1, there is a difference between optimal duty cycle and load. When the load is close to 1, the optimal duty cycle is

weight factor Q/R	load			
	90%		10%	
	period (S)	duty cycle	period (S)	duty cycle
2/500	6447	90.91%	6447	9.09%
2/2500	13947	93.12%	13947	6.88%

TABLE I

EXPERIMENTAL RESULT - THE DIFFERENCE BETWEEN LOAD AND DUTY CYCLE.

always bigger than the load, but when the load is close to 0, the optimal duty cycle is always smaller than the load. The reason is that, the error accumulation of the longer half period (either on or off period) has smaller gradient in the longer period than the short half period error accumulation in the short half period, which is shown in Fig.3. Therefore the optimal will always prolong the longer half period which results in a duty cycle different than load. The value of the difference depends on the ratio of Q and R . With the increase of switch penalty R , the difference will become bigger. For 50% load, both on and off period has the same gradient over the corresponding period, therefore, the optimal duty cycle is the same as the load.

The computational load of finding the optimum grows fast with increase of the system size.

IV. A SIMPLE METHOD

In previous section, we have demonstrated that there is a unique optimum for the cost function J at least for this example. A method of finding the optimum has been presented, and following that the system will be controlled to the optimal period and duty cycle. This is a way of doing it, but it is very computationally demanding, especially for higher order system. What we are really looking for here is a simple method which requires little computation power and at the same time overcomes the fundamental problem of the finite prediction horizon. An idea for such a simple method is presented below.

We have known that cost function J has only one optimum, and it should make the derivative of (4) over T_s give 0 which is (12), but it is not easy to find the analytical solution. (12) can be rewritten as (13) and it has 3 terms, the cost of current output, the integration of error in a period, and the switch. If we rearrange the three terms as in (13), it can be understood in this way: the left side involves only the current output, while the right side involves the accumulated comfort error and a fixed switch cost. The two sides can only become equal when the period is optimal, since the cost function J has no local optimum. Hence, it can be used to find the optimum. For simplicity, we rename both sides of (13) as in (14) and (15). The controller always compare Acu (which is the error accumulation from the beginning of a period) and Cur with the value from last sample at each time sample to check if the switch should happen at the current time sample. We have the knowledge that if $T_s < T_{optimal}$, $\frac{\partial J}{\partial T_s} < 0$ and that if $T_s > T_{optimal}$, $\frac{\partial J}{\partial T_s} > 0$. Therefore we can

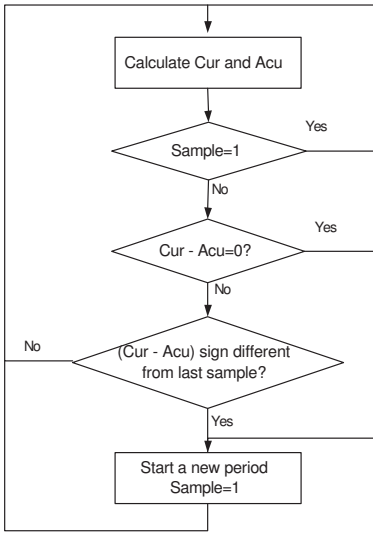


Fig. 6. Flow chart of the proposed method. *Cur* is the current cost and *Acu* is the accumulated error and switch cost.

use the switch criterion of $Acu-Cur=0$ (in reality if the sign of $Acu-Cur$ is different from last sample, then a switch should happen, although care should be taken that measurement noise do not cause undesirable switches) for switch. The system should switch when this condition fulfills. Otherwise, the system should just wait for next sample. The flow chart of the algorithm is shown in Fig.6, from which it can be seen that the algorithm is very simple and requires very little computation power.

$$\frac{\partial J}{\partial T_s} = 0 \quad (12)$$

$$Q \cdot (y_{t_0+T_s} - y_{ref})^2 T_s = Q \cdot \int_{t_0}^{t_0+T_s} (y - y_{ref})^2 dt + R \cdot J_{sw} \quad (13)$$

$$Cur = Q \cdot (y_{t_0+T_s} - y_{ref})^2 T_s \quad (14)$$

$$Acu = Q \cdot \int_{t_0}^{t_0+T_s} (y - y_{ref})^2 dt + R \cdot J_{sw} \quad (15)$$

The basic idea of the method has been explained, but some adjustments have to be made in order to apply it in applications. The reason is that with this method, a decision of switch can only be made for the current sample, but in one period, there are two switches. The following two adjustments have been made without introducing prediction for the next half period, and they are named Method 1 and Method 2 and explained in details below.

A. Method 1

For this method, the idea is to use the past period decided just before the current period. This past period and the current period together composes a whole period with both on and off. By doing this, prediction for the next half period

is avoided. So the system will only use the information from the past. Equation (13) is modified to (16) according to the adjustments.

The switch criteria will be

$$Q \cdot (y_{t_0+T_s+T_{last}} - y_{ref})^2 (T_s + T_{last}) = Q \cdot \int_{t_0}^{t_0+T_s+T_{last}} (y - y_{ref})^2 dt + R \cdot J_{sw} \quad (16)$$

It should be noted that when we calculate for the first switch, the error accumulated from earlier period is 0, therefore the switch cost should be only half of the later calculation.

Two experiments using this method have been carried out with the on/off AC-system example. One experiment is to apply this controller for the same system with different load and the result is shown in Fig.II, in the 'on/off together' block. Compared with the optimal solution from cost analysis result, the results from this method are close to the optimal (in period, duty cycle and cost).

The other experiment is to use the same load, but apply different initial conditions. The results indicate that different initial conditions lead to different results. For example, with the same load, initial conditions 22.8°C and 24°C lead to different results. Some more experiments have been carried with different initial conditions. The conclusions is that, for the first order system, if the initial guess is bigger than the reference, then the result will not reach the optimal, but when the initial is smaller than the reference, the system reaches optimal.

Since it is difficult to visualize the result from first order system, a second order system example is chosen here. The experiment is to run the system with 50% load from different initial conditions and the resulting state trajectories are plotted in Figure 7. The initial conditions located in the lower part of the right half plane out outside of the larger limit circle converges to the big limit circle, but others converge to the small one. The experimental results lead to the same conclusion that Method 1 is sensitive to the initial conditions. The reason is that the controller carries the influence of the initial condition because it always takes the last half period into account.

B. Method 2

Method 2 aims at changing (16) to look at the two half periods (T_{on}, T_{off}) separately.

The same experiments with the conditions as in subsection IV-A have been conducted using the method 2.

The experimental results with different loads for the same first order system are shown in Fig.II, on/off separately block. Compared with the solution from Method 1, the solutions from Method 2 are further away from the optimal (in period, duty cycle and cost).

The experimental results with different initial conditions are also conducted, and the results show that Method 2 is insensitive to the initial condition. For visualization, the results for 50% load from second order system with two different initial conditions are plotted in Fig.7. It is obvious

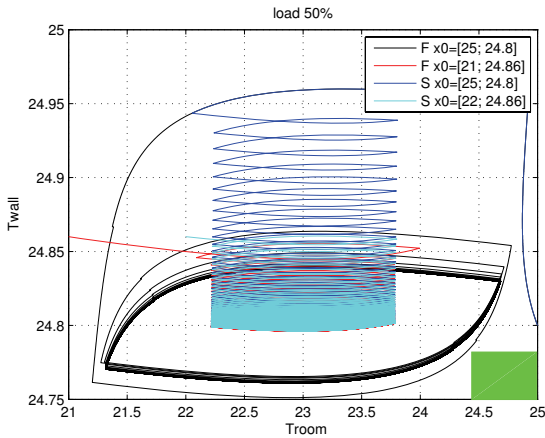


Fig. 7. Phase trajectory 2^{nd} order system. F means Method 1, and S means Method 2. x_0 is initial temperature

load (%)	optimal from cost analysis			on/off period separately (initial: 22.8)			on/off period together (initial: 22.8)		
	period (S)	cost	duty cycle (%)	period	cost	duty cycle (%)	period	cost	duty cycle (%)
90	6447	0.242	90.91	7796	0.290	89.20	6460	0.242	90.90
45	3059	0.492	44.92	3072	0.493	45.18	3062	0.492	44.94
85	4919	0.311	85.69	5630	0.347	84.05	4930	0.311	85.68
15	4919	0.311	14.31	5630	0.347	15.95	4930	0.311	14.32
50	3037	0.495	50.00	3044	0.495	50.00	3044	0.495	50.00

TABLE II

COMPARISON OF THE RESULTS FROM THE 2 METHODS AND THE OPTIMAL SOLUTION FROM THE COST ANALYSIS.

that taking the on and off periods separately, the system converges to the same limit cycle here.

More initial conditions in different part of the plane have been tested, and they all converge to the same limit cycle. Trajectories for the two initial conditions, that can be used for comparing with the results from method 1, are illustrated in Fig.7.

An idea is to combine Method 1 and Method 2, where Method 2 will be used first to take the states to the right track, and then Method 1 can be used to get the solution closer to the optimal. Due to space limitations, these will not be presented here.

C. stability problem.

The focus of this paper is to demonstrate the feasibility of a computationally simple approach to optimization. Analyzing and proving stability is omitted due to space limitations.

V. CONCLUSION

A low complexity method of controlling discrete input systems has been developed. With this method, problem of the inherited finite prediction horizon and heavy computation demand has been avoided. It was shown that for a class of systems it is capable of achieving close to optimal solutions. The computational power required for the developed method is very low which is only several memories. This can be applied for the mass produced controllers. The methods proposed are empirical, and it is not claimed that they hold in

all generality. The authors, however, are convinced that the methods can be applied to a wide class of systems, including many arising from practical problems.

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