Structured Control of LPV Systems with Application to Wind Turbines

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Abstract— **This paper deals with structured control of linear parameter varying systems (LPV) with application to wind turbines. Instead of attempting to reduce the problem to linear matrix inequalities (LMI), we propose to design the controllers via an LMI-based iterative algorithm. The proposed algorithm can synthesize structured controllers like decentralized, static output and reduced order output feedback for discrete-time LPV systems. Based on a coordinate decent, it relies on a sufficient matrix inequality condition extended with slack variables to an upper bound on the induced $\mathcal{L}_2$-norm of the closed-loop system. Algorithms for the computation of feasible as well as optimal controllers are presented. The general case where no restrictions are imposed on the parameter dependence is treated here due to its suitability for modeling wind turbines. A comprehensive numerical example of a gain-scheduled LPV controller design with prescribed pattern for wind turbines illustrate the utilization of the proposed algorithm.**

I. Introduction

Practical considerations often dictate structural constraints on the controller. Control practitioners face the challenge of designing low-order, decentralized, observed-based, PID control structures, among others. These control problems are naturally formulated as Bilinear Matrix Inequalities (BMI), and to which equivalent convex reformulations based on Linear Matrix Inequalities (LMI) are not known to exist. Add to that some systems inherently exhibit time-varying nonlinear dynamics along their nominal operating trajectory, motivating the use of advanced control techniques such as gain-scheduling, to counteract performance degradation or even instability problems by continuously adapting to the dynamics of the plant. A systematic way of designing controllers for systems with linearized dynamics that vary significantly with the operating point is within the framework of linear parameter-varying (LPV) control. Wind turbines are naturally inserted in this context. Firstly, because gain-scheduling is an usual approach to deal with varying dynamics dependent on the operating point [11]. Secondly, the structure of wind turbine industrial controllers often have a prescribed pattern [14].

In this context, our interest lies in the synthesis of LPV controllers with structural constraints, more specifically, the $\mathcal{L}_2$-norm minimization problem,

$$\begin{gathered} minimize \, \|T_{z\to w}(\theta, K(\theta))\|_2 \\ K(\theta) \in \mathcal{K} \end{gathered}$$

where $T_{z\to w}(\cdot)$ is an input-output system operator, $K(\theta)$ is a linear parameter varying controller dependent on a vector of time-varying parameters θ, and $\mathcal{K}$ represents a structural constraint in the controller matrices.

The diversification of LPV controller structures is not extensively addressed in the literature. The static state feedback and full-order dynamic output feedback are by far the most investigated structures. There are some proposals on the design of static output feedback controllers [1], [2]. A few works can be found on other controller structures like decentralized [3], fixed-order dynamic output for single-input single-output polynomial systems [4]. Synthesis conditions based on Linear Matrix Inequalities (LMI) is a common feature to all these papers. Recently, static output [6] and full-order dynamic output [7] synthesis procedures relies on extended LMI conditions with slack variables [5].

Instead of an attempt to reduce the problem to linear matrix inequalities (LMI), this paper investigates the design of structured LPV controllers via an LMI-based iterative algorithm. Iterative LMI algorithms with slack matrices were investigated in the context of robust [8] and affine LPV [9] control. Decentralized of any order, fixed-order output, static output and simultaneous plant-control design are among the possible control structures. Based on a coordinate decent, it relies on extended LMI conditions to an upper bound on the induced $\mathcal{L}_2$-norm of the closed-loop system. We propose a relaxation on the LMI condition useful for computing feasible controllers. After a feasible controller is found, the objective is cost minimization until the solution converges to a stationary point. The general case where no restrictions are imposed on the parameter dependence is treated here due to its suitability for modeling wind turbines.

Realizing advanced gain-scheduled controllers can be difficult in practice and may lead to numerical challenges [11], [10]. Usually, several plant and controller matrices must be stored on the controller memory. Moreover, matrix factorizations and inversions are among the operations that must be done online by the controller at each sampling time [12]. The proposed synthesis methodology can be of practical relevance because the resulting controllers have simple implementation.

This paper is organized as follows. Section II describes the system, controller and some possible controller structures. Section III presents known extended matrix inequalities conditions for the induced $\mathcal{L}_2$ norm and the proposed relaxation. Section IV describes the iterative LMI algorithm along with convergence and computational considerations. Section V revisits the design of wind turbine industry-standard controllers under the LPV framework.

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II. System and Controller Description

An open-loop, discrete-time augmented LPV system with state-space realization of the form,

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B_w(\theta)w(k) + B_u(\theta)u(k) \\ z(k) &= C_z(\theta)x(k) + D_{zw}(\theta)w(k) + D_{zu}(\theta)u(k) \\ y(k) &= C_y(\theta)x(k) + D_{yw}(\theta)w(k), \end{aligned} \quad (1)$$

is considered for the purpose of synthesis, where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ is the vector of disturbance, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $z(k) \in \mathbb{R}^{n_z}$ is the controlled output, and $y(k) \in \mathbb{R}^{n_y}$ is the measured output. $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$are continuous functions of some time-varying parameter vector $\theta = [\theta_1, \ldots, \theta_{n_\theta}]$. Assume θ ranges over a hyperrectangle denoted Θ,

$$\Theta = \left\{\theta : \ \underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i, \ i = 1, \ldots, n_\theta \right\}.$$

The rate of variation $\Delta\theta = \theta(k+1) - \theta(k)$ belongs to a hypercube denoted $\mathcal{V}$,

$$\mathcal{V} = \left\{\Delta\theta : \ |\Delta\theta_i| \leq v_i, \ i = 1, \ldots, n_\theta \right\}.$$

The LPV controller has the form,

$$\begin{aligned} x_c(k+1) &= A_c(\theta)x_c(k) + B_c(\theta)y(k) \\ u(k) &= C_c(\theta)x_c(k) + D_c(\theta)y(k), \end{aligned} \quad (2)$$

where $x_c(k) \in \mathbb{R}^{n_c}$ and the controller matrices are continuous functions of θ. Note that depending on the controller structure, some of the matrices may be zero. The controller matrices can be represented in a compact way,

$$K(\theta) := \begin{bmatrix} D_c(\theta) & C_c(\theta) \\ B_c(\theta) & A_c(\theta) \end{bmatrix}. \quad (3)$$

The interconnection of system (1) and controller (2) leads to the following closed-loop LPV system denoted S_{cl},

$$\begin{aligned} S_{cl}: \quad x(k+1) &= \mathcal{A}(\theta, K(\theta))x_{cl}(k) + \mathcal{B}(\theta, K(\theta))w(k) \\ z(k) &= \mathcal{C}(\theta, K(\theta))x_{cl}(k) + \mathcal{D}(\theta, K(\theta))w(k). \end{aligned} \quad (4)$$

This general system structure can be particularized to some usual control topologies. In the case $K(\theta)$ is an *unconstrained* matrix, if $n_c = 0$, the problem becomes a static output feedback. The static state feedback is a particular case of static output, when the system output is a full rank linear transformation of the state vector $\forall\theta$. If $n = n_c$, the full-order dynamic output feedback arises. In a structured control context, more elaborate control systems can be designed by constraining $K(\theta)$. A fixed-order dynamic output feedback has $n_c < n$. For decentralized controllers of arbitrary order, the structure of $K(\theta)$ is constrained to be,

$$K(\theta) := \begin{bmatrix} \text{diag}(D_c(\theta)) & \text{diag}(C_c(\theta)) \\ \text{diag}(B_c(\theta)) & \text{diag}(A_c(\theta)) \end{bmatrix}$$

where $\text{diag}(\cdot)$ stands that $(\cdot)$ has a block-diagonal structure.

In the general parameter dependence case, the open-loop system matrices are dependent on arbitrary functions of the varying parameters,

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}(\theta) = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_0 + \sum_i \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}_i \rho_i(\theta), \quad i = 1, \ldots, n_\rho \quad (5)$$

where $\rho_i(\theta)$ are scalar functions known as basis functions that encapsulate possible system's nonlinearities and n_ρ is the number of basis functions. The controller matrices are continuous functions of θ with similar type of dependence,

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}(\theta) = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}_0 + \sum_{i=1}^{n_\theta} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}_i \rho_i(\theta) \quad (6)$$
$$i = 1, \ldots, n_\theta$$

III. Induced L_2-norm Performance

The design of a closed-loop system usually considers performance specifications that can be characterized in different ways. Define $T_{zw}(\theta)$ as the input-output operator that represents the forced response of (4) to an input signal $w(k) \in \mathcal{L}_2$ for zero initial conditions. The induced $\mathcal{L}_2$-norm of a given input-output operator,

$$\|T_{zw}\|_{\mathcal{L}_2} := \sup_{\theta \in \Theta \times \mathcal{V}} \ \sup_{\|w\|_{\mathcal{L}_2} \neq 0} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}$$

is commonly utilized as a measure of performance of LPV systems and allows formulating the control specification as in $\mathcal{H}_\infty$ control theory. The LPV system (4) is said to have performance level γ when it is exponentially stable and $\|T_{zw}\|_{\mathcal{L}_2} < \gamma$ holds. An extension of the bounded real lemma (BRL) for parameter dependent systems is a sufficient condition for checking the $\mathcal{L}_2$ performance level of system S_{cl}.

Lemma 1 (Extended $\mathcal{L}_2$ Performance): [5], [7] For a given controller $K(\theta)$, if there exist $\mathcal{P}(\theta) = \mathcal{P}(\theta)^T$ and $\mathcal{Q}(\theta)$ satisfying (7) with $r = 1$ for all $(\theta, \Delta\theta) \in \Theta \times \mathcal{V}$, then the system S_{cl} is exponentially stabilizable by the controller $K(\theta)$ and $\|T_{zw}(\theta)\|_{\mathcal{L}_2} < \gamma$.

The term r^2 multiplying the Lyapunov matrix at the (1,1) entry of (7) is, in the present paper, artificially inserted into the formulation. For frozen θ (LTI system) r represents the z-plane circle radius, thus $r = 1$ in the Schur stability criteria. By imposing $r > 1$ the z-plane circle would be enlarged, meaning that even unstable closed-loop systems (eigenvalues lying out of the unit circle) would satisfy Lemma 1. For parameter varying systems, this notion of enlargement still exists when $r > 1$, defined by the following lemma.

Lemma 2 (Enlarged $\mathcal{L}_2$ Performance): For a given controller $K(\theta)$, if there exist $\mathcal{P}(\theta) = \mathcal{P}(\theta)^T$ and $\mathcal{Q}(\theta)$ satisfying (7) with $r = r_e > 1$ for all $(\theta, \Delta\theta) \in \Theta \times \mathcal{V}$, then the system S_{cl} satisfies the enlarged $\mathcal{L}_2$ performance with

$$\begin{bmatrix} r^2\mathcal{P}(\theta^+) & \mathcal{A}(\theta,K(\theta))\mathcal{Q}(\theta) & \mathcal{B}(\theta,K(\theta)) & 0 \\ \star & -\mathcal{P}(\theta)+\mathcal{Q}(\theta)^T+\mathcal{Q}(\theta) & 0 & \mathcal{Q}(\theta)^T\mathcal{C}(\theta,K(\theta))^T \\ \star & \star & \gamma I & \mathcal{D}(\theta,K(\theta))^T \\ \star & \star & \star & \gamma I \end{bmatrix} > 0 \tag{7}$$

$\|T_{zw}(\theta)\|_{\mathcal{L}_2, r=r_e} < \gamma$.

Even systems that are not exponentially stabilizable may satisfy the enlarged $\mathcal{L}_2$-norm condition. This fact will be utilized in the proposed algorithms for finding a feasible controller. The shifted-$\mathcal{H}_\infty$-norm is a similar concept for continuous-time LTI systems [13].

The Lyapunov and slack variables mimic the general parameter dependence of the plant and controller,

$$\mathcal{P}(\theta) = P_0 + \sum_{i=1}^{n_\theta} \rho_i(\theta) P_i \tag{8a}$$

$$\mathcal{Q}(\theta) = Q_0 + \sum_{i=1}^{n_\theta} \rho_i(\theta) Q_i \tag{8b}$$

The Lyapunov function at $\theta^+ := \theta + \Delta\theta$ can be described as,

$$\mathcal{P}(\theta^+) = P_0 + \rho_i(\theta^+) P_i \tag{9}$$

Conveniently, the basis functions at θ^+ are approximated by a linear function of $\rho(\theta)$ and $\Delta\theta$,

$$\rho_i(\theta^+) := \rho_i(\theta) + \frac{\partial \rho_i(\theta)}{\partial \theta}\Delta\theta, \tag{10}$$

thereby turning inequality (7) affine dependent on the rate of variation $\Delta\theta$. This approximation makes sufficient to verify (7) with (9)-(10) only at Vert $\mathcal{V}$.

IV. Optimization Algorithm

The optimization algorithm iterates between LMI problems by fixing the controller variables and the slack variable alternatively. In this way, the parameter dependent Lyapunov matrix remains as a variable during the whole optimization process. In the general parameter dependence case, the controller is designed in a gridded parameter space. A gridding procedure consists of defining a gridded parameter subset denoted $\Theta_g \subset \Theta$, designing a controller that satisfies the matrix inequalities constraints $\forall \theta \in \Theta_g$, and checking the inequalities constraints in a denser grid. If the last step fails, the process is repeated with a finer grid.

In order to save text during the exposure of the algorithms, denote the inequality constrains by

$$\Pi_Q(x) := (7),\ \forall (\Theta, \Delta\theta) \in \Theta_g \times \text{Vert } \mathcal{V}$$

The algorithm for computing a feasible structured LPV controller is described next. The aim is to create a sequence of r convergent to 1, that is, for a certain tolerance ϵ, $r^{(j)} \geq 1 - \epsilon$, $r^{(j)} \to 1 \pm \epsilon$, as $j \to \infty$.

Algorithm 1: (Feasibility) Given initial slack matrix $\mathcal{Q}^{(1)}(\theta) = I$, $\forall \theta \in \Theta_g$, an initial radius $r^{(1)} > 1$, a target radius $r_{tg} \leq 1$ and a convergence tolerance ϵ_1. Set $j = 1$ and start to iterate:

1) Find $\mathcal{P}(\theta)$, $K(\theta)$, and γ that solves the LMI problem,
   Minimize γ subject to $\Pi_Q(x)$ with $r = r^{(j)}$, and frozen $\mathcal{Q}(\theta) = \mathcal{Q}^{(j)}(\theta)$ $\forall \theta \in \Theta_g$.
2) If Step 1 is feasible, $K^{(j)}(\theta) = K(\theta)$. Else, $K^{(j)}(\theta) = K^{(j-1)}(\theta)$.
3) Find $\mathcal{P}(\theta)$, $\mathcal{Q}(\theta)$, and γ that solves the LMI problem,
   Minimize γ subject to $\Pi_Q(x)$ with $r = r^{(j)}$, and frozen $K(\theta) = K^{(j)}(\theta)$, $\forall \theta \in \Theta_g$
4) If Step 3 is feasible, $\mathcal{Q}(\theta)^{(j+1)} = \mathcal{Q}(\theta)$. Else, $\mathcal{Q}(\theta)^{(j+1)} = \mathcal{Q}^{(j)}(\theta)$
5) If Step 1 and step 3 are feasible, $r^{(j+1)} = 0.5\,(r^{(j)} + r_{tg})$ and $\Delta r^{(j+1)} = |r^{(j+1)} - r^{(j)}|$ (Reduced radius).
   Elseif Step 3 is feasible, $r^{(j+1)} = r^{(j)}$ and $\Delta r^{(j+1)} = \Delta r^{(j)}$ (Same radius).
   Else, $r^{(j+1)} = r^{(j)} + 0.5|r^{(j-1)} - r^{(j)}|$ and $\Delta r^{(j+1)} = |r^{(j+1)} - r^{(j)}|$ (Increased radius).
6) If $|r^{(j+1)} - r^{(j)}| < \epsilon_1$, stop. Else, $j = j + 1$ and go to step 1.

The initial radius $r^{(1)}$ should be made large enough to make the first iteration feasible. Our experience shows that $r^{(1)} = 2$ suffices for most situations. The target radius r_{tg} can be made slightly smaller than 1. Once the radius reaches the target radius within a certain tolerance ϵ_1, the objective is only to minimize the performance level γ.

Algorithm 2: (Performance Level) Given initial controller $K^{(j)}(\theta)$, $\forall \theta \in \Theta_g$, and a convergence tolerance ϵ_2. Set $j = 1$ and start to iterate:

1) Find $\mathcal{P}(\theta)$, $K^{(j)}(\theta)$, and γ that solves the LMI problem,
   Minimize γ subject to $\Pi_Q(x)$ with $r = 1$, and frozen $\mathcal{Q}(\theta) = \mathcal{Q}^{(j)}(\theta)$ $\forall \theta \in \Theta_g$.
2) Find $\mathcal{P}(\theta)$, $\mathcal{Q}(\theta)$, and $\gamma^{(j)}$ that solves the LMI problem,
   Minimize γ subject to $\Pi_Q(x)$ with $r = 1$, and frozen $K(\theta) = K^{(j)}(\theta)$, $\forall \theta \in \Theta_g$.
3) If $|\gamma^{(j)} - \gamma^{(j-1)}| < \epsilon_2$, stop. Else, $j = j + 1$ and go to step 1.

Algorithm 2 generates a convergent sequence of solutions such that the cost is non-increasing, that is, $\gamma^{(1)} \geq \gamma^{(j)} \geq \gamma^{(*)}$. To realize this, notice that taking the slack variable equal to the Lyapunov variable implies sufficiency of Lemma 1 [5]. Therefore, $\mathcal{P}(\theta)$ computed at step 1 is a solution for $\mathcal{Q}(\theta)$ at step 2, implying feasibility of step 2 with at least the same value of γ of step 1. The controller $K(\theta)$ at the iteration

j is also a solution for the step 1 at iteration $j+1$, implying feasibility of step 1 with at least the same performance level as iteration j.

Algorithms 1 and 2 are in fact very similar and can be unified in a single algorithm.

Computational Load

Depending on the system/controller order and number of basis functions, the procedure may be computationally expensive. Slight modifications on the algorithms alleviate computational load at the expense of some conservatism.

- The step at which the slack matrix is computed can be replaced by an update rule of the form
$$\mathcal{Q}^{(j+1)}(\theta) = \mathcal{P}^{(j)}(\theta),\ \forall\theta \in \Theta_g.$$
Indeed, taking the slack variable equal to the Lyapunov variable implies sufficiency of Lemma 1 [5].
- The slack matrix can be made parameter independent, e.g. $\mathcal{Q}(\theta) = \mathcal{Q}$.
- Parameter dependent matrix variables may also depend on a fewer number of basis functions/varying parameters than the plant, thus reducing the number of optimization variables. Some basis functions are more representative of system's nonlinearities than others. For example, the LPV controller can be made dependent of some basis functions while being designed robust to the reminiscent ones by including them in the Lyapunov variable.

Controller Implementation

Due to the fact that no linearizing change of variables is involved in the formulation, the resulting controller can be easily implemented in practice. The iterative LMI optimization algorithm provides the controller matrices $A_{\text{c},i}$, $B_{\text{c},i}$, $C_{\text{c},i}$, $D_{\text{c},i}$, for $i = 0, 1, \ldots, n_\rho$. These matrices, the basis functions, and the value of the scheduling variables are the only required information to determine the control signal $u(k)$. At each sample time k, the scheduling variable $\theta(k)$ is measured (or estimated) and a control signal is obtained as follows.

1) Compute the value of the basis functions $\rho_i(\theta(k))$, for $i = 0, 1, \ldots, n_\rho$. The basis functions may be stored in a lookup table that takes $\theta(k)$ as an input and outputs an interpolated value of $\rho(\theta(k))$.
2) With the value of the basis functions in hand, determine the controller matrices $A_\text{c}(\theta(k))$, $B_\text{c}(\theta(k))$, $C_\text{c}(\theta(k))$, $D_\text{c}(\theta(k))$ according to (6).
3) Once the controller matrices have been found, the control signal $u(k)$ can be obtained by the dynamic equation (2) of the LPV controller, only involving multiplications and sums.

V. Wind Turbine LPV Control

At high wind speeds, the power generated by a wind turbine should be maintained at rated value. A common control strategy is to regulate the generator speed (Ω_g) by varying the blade pitch angles (β) while maintaining a constant generator torque (Q_g). The wind energy industry relies on the proportional and integral (PI) controller to accomplish such task. The PI speed control using pitch angle as controlled input strongly couples with the tower dynamics, denoting a multivariable problem, and should be properly designed. The adopted control structure depicted in Fig. 1 includes the most common control loops of a industry standard Region III controller [14].

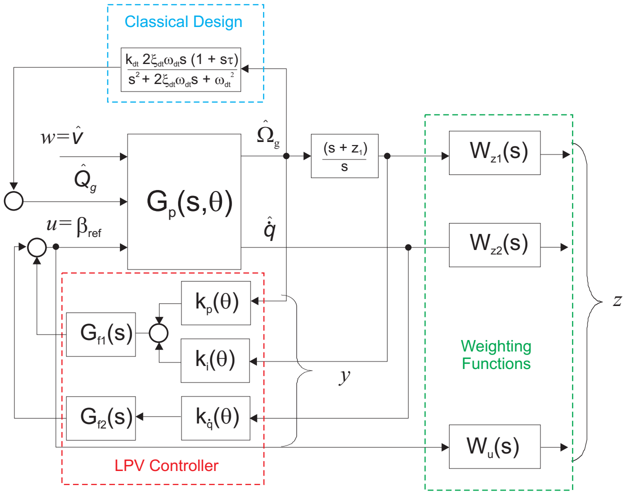

Fig. 1: Control loops of generator speed and tower damping.

The generator speed is regulated by a PI controller of the form,
$$G_\text{PI} := \ k_\text{p}(\theta) + k_\text{i}(\theta) G_\text{I}(s)$$
where s denotes the Laplace operator. Instead of a pure integrator, the PI controller is composed by an integrator filter,
$$G_\text{I}(s) := \frac{s + z_\text{I}}{s},$$
where the filter zero z_I is a design parameter. The PI controller is connected in series with a parameter independent filter $G_{\text{f}1}(s)$. It is possible to provide an extra signal by using an accelerometer mounted in the nacelle, allowing the controller to better recognize between the effect of wind speed disturbances and tower motion on the measured power or generator speed. With this extra feedback signal, tower bending moment loads can be reduced without significantly affecting speed or power regulation. Therefore, it is assumed that tower velocity $\dot{q}$ is available for measurement, by integrating tower acceleration $\ddot{q}$, and is multiplied by a parameter-dependent constant $k_{\dot{q}}(\theta)$ for feedback. A parameter independent filter $G_{\text{f}2}(s)$ completes the tower feedback loop. The order of the filters $G_{\text{f}1}(s)$ and $G_{\text{f}2}(s)$ can be arbitrarily chosen. The choice trades-off closed-loop performance and number of controller states. High order filters leads to better performance with the expense of higher controller complexity.

The drive train of a wind turbine presents a poorly damped torsional mode when a constant torque control strategy is adopted. To counteract this, active drive train damping is deployed by adding a signal to the generator torque (Q_{g}) to compensate for the oscillations in the drive train. For a didactic and clear exposure, the compensation of the drive train damper is considered ideal. Therefore, for synthesis purposes, the drive train torsional mode is neglected and the rotor speed is proportional to the generator speed. The LPV controller can now be designed to trade off the tracking of generator speed and tower oscillations with control effort (wear on pitch actuator). Wind turbine aerodynamics is the main source of nonlinearities. A linearization-based LPV model depends on partial derivatives of aerodynamic torque (Q) and thrust (T) forces with respect to rotor speed (Ω_r), wind speed (V) and pitch angle. These partial derivatives, also known as aerodynamic gains, vary with the operating point. Thus, they are natural candidates for the basis functions [16],

$$\begin{aligned}\rho_1 &:= \frac{1}{J_{\mathrm{e}}}\frac{\partial Q}{\partial \Omega}\bigg|_{\theta},\ \rho_2 := \frac{1}{J_{\mathrm{e}}}\frac{\partial Q}{\partial V}\bigg|_{\theta},\ \rho_3 := \frac{1}{J_{\mathrm{e}}}\frac{\partial Q}{\partial \beta}\bigg|_{\theta},\\ \rho_4 &:= \frac{1}{M_{\mathrm{t}}}\frac{\partial T}{\partial \Omega}\bigg|_{\theta},\ \rho_5 := \frac{1}{M_{\mathrm{t}}}\frac{\partial T}{\partial V}\bigg|_{\theta},\ \rho_6 := \frac{1}{M_{\mathrm{t}}}\frac{\partial T}{\partial \beta}\bigg|_{\theta}.\end{aligned} \tag{11}$$

In the above expressions, J_r and J_g is the rotor and generator inertia, which combined with the gearbox ratio N_g results in the equivalent rotational inertia in the rotor side $J_{\mathrm{e}} := J_{\mathrm{r}} + J_{\mathrm{g}} N_{\mathrm{g}}^2$. M_t is the equivalent modal mass of the first bending moment of the tower. Basis functions with equivalent inertia and tower mass were chosen to improve numerical conditioning. The operating point of a wind turbine varies according to the effective wind speed $\theta(t) = \bar{V}(t)$ driving the rotor. The dynamic model of the variable-speed wind turbine can then be expressed as an LPV model of the form,

$$G : \begin{cases} \dot{x} = A(\theta)\, x + B_w(\theta)\, \hat{V} + B_u(\theta)\, \beta_{\mathrm{ref}} \\ y = C_y\, x \end{cases}$$

where states, controllable input and measurements are,

$$\begin{aligned} x &= \begin{bmatrix} \Omega_{\mathrm{r}} & \dot{q} & q & \dot{\beta} & \beta & x_{\Omega,i} \end{bmatrix}^T, \\ u &= \beta_{\mathrm{ref}}, \quad y = \begin{bmatrix} \Omega_{\mathrm{g}} & y_{\Omega,i} & \dot{q} \end{bmatrix}^T. \end{aligned}$$

with open-loop system matrices,

$$A(\theta) =$$

$$\begin{bmatrix} \rho_1(\theta) & -\rho_2(\theta) & 0 & 0 & \rho_3(\theta) & 0 \\ \rho_4(\theta) & -\dfrac{1}{M_{\mathrm{t}}}B_{\mathrm{t}} - \rho_5(\theta) & -\dfrac{K_{\mathrm{t}}}{M_{\mathrm{t}}} & 0 & \rho_6(\theta) & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}2\zeta\omega_{\mathrm{n}} & -\omega_{\mathrm{n}}^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ N_g & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_w(\theta) = \begin{bmatrix} \rho_2(\theta) & \rho_5(\theta) & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$B_u = \begin{bmatrix} 0 & 0 & 0 & \omega_{\mathrm{n}}^2 & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$C_y = \begin{bmatrix} N_{\mathrm{g}} & 0 & 0 & 0 & 0 \\ z_{\mathrm{I}} & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Notice the PI controller integrator filter G_{I} conveniently augmented into the state-space of G, represented by the state $x_{\Omega,i}$ and the output $y_{\Omega,i}$. The plant G_{p} is defined as the wind turbine model solely (plant G without the augmentation of G_{I}). A state-space realization of the control structure depicted in Fig. 1 is given by,

$$\begin{aligned} A_{\mathrm{c}} &:= \begin{bmatrix} A_{\mathrm{f1}} & 0 \\ 0 & A_{\mathrm{f2}} \end{bmatrix}, \quad B_{\mathrm{c}}(\theta) := \begin{bmatrix} B_{k_p}(\theta) & B_{k_i}(\theta) & 0 \\ 0 & 0 & B_{k_{\dot{q}}}(\theta) \end{bmatrix}, \\ C_{\mathrm{c}} &:= \begin{bmatrix} C_{\mathrm{f1}} & C_{\mathrm{f2}} \end{bmatrix}, \quad D_{\mathrm{c}} := \begin{bmatrix} D_{\mathrm{f1}} & D_{\mathrm{f2}} \end{bmatrix}. \end{aligned}$$

where the size of sub-matrices depends on the chosen orders of the filters. The parameter-dependent controller matrix has the general dependence form,

$$\begin{bmatrix} B_{k_p}(\theta) & B_{k_i}(\theta) & 0 \\ 0 & 0 & B_{k_{\dot{q}}}(\theta) \end{bmatrix} :=$$

$$\begin{bmatrix} B_{k_p} & B_{k_i} & 0 \\ 0 & 0 & B_{k_{\dot{q}}} \end{bmatrix}_0 + \sum_{m=1}^{n_\rho} \begin{bmatrix} B_{k_p} & B_{k_i} & 0 \\ 0 & 0 & B_{k_{\dot{q}}} \end{bmatrix} \rho_{\mathrm{m}}(\theta).$$

The Lyapunov matrix is chosen dependent on all basis functions (11), and the slack matrix is chosen parameter independent.

Weight W_{z1} and W_u governs the tradeoff between rotational speed regulation and pitch wear. In this example, W_{z1} is chosen as a scalar k_1, turning the first performance channel similar to an integral square error measure ($z_1 = W_{\mathrm{z1}} G_{\mathrm{I}} \hat{\Omega}_{\mathrm{r}}$). W_u is taken as a first order high-pass filter that penalizes high-frequency content on the pitch angle. Due to the resonance characteristics of the transfer function from $\hat{V}$ to $\dot{q}$, the weighting function W_{z2} is chosen as a scalar k_2, that tradeoffs the desired tower damping. Considering the plant and weighting functions just mentioned, the augmented plant has 7 states. $G(s)_{\mathrm{f1}}$ and $G(s)_{\mathrm{f1}}$ are chosen as first order and second order filters, respectively, therefore the controller is comprised of 3 states.

Remember that the iterative LMI algorithm is a synthesis procedure in discrete time. Therefore, the augmented LPV plant in continuous time is discretized using a bilinear (Tustin) approximation [15] with sampling time $T_{\mathrm{s}} = 0.02$ s, at each point $\Theta_g \times$ Vert $\mathcal{V}$. The effective wind speed ranges $\theta = \bar{V} \in$ [12 m/s, 25 m/s] and its rate of variation ranges $\Delta\theta(t) = \Delta\bar{V}(t) \in [-2$ m/s^2, 2 m/s^2]. The grid is comprised of seven equidistant points. The rate of variation of the scheduling variables in continuous-time must as well be converted to discrete-time by the relation $\Delta\theta(k) = T_s \Delta\theta(t)$.

The numerical example is based on data from a typical 2MW utility scale wind turbine. The evolution of radius $r^{(j)}$ and performance level $\gamma^{(j)}$ during the course of the optimization is illustrated on Fig. 2. During the feasibility phase, as the radius gradually converges to 1, the performance level value increases. The algorithm switches to optimization phase by maintaining $r = 1$ during the subsequent iterations, being the cost monotonically decreasing to a stationary point.

Wind disturbance step responses under different operating points (frozen θ) are depicted in Fig. 3. The rotor speed is well regulated around the origin. A similar response irrespective of the operating point is noticeable, meaning that

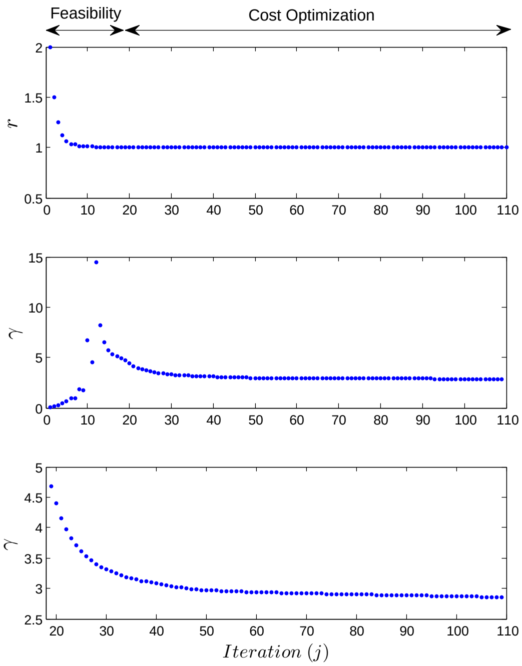

Fig. 2: Evolution of variables during the iterative LMI optimization.

the controller is gain-scheduling to adapt to the nonlinearities of the plant. This is corroborated by the magnitude plots of transfer functions from wind disturbance to rotor speed and tower velocity, for the open-loop and closed-loop systems. The increased damping of the tower fore-aft motion is noticeable in Fig.4d where the magnitude of the open-loop system (dashed line) is plotted for comparison.

VI. ACKNOWLEDGMENTS

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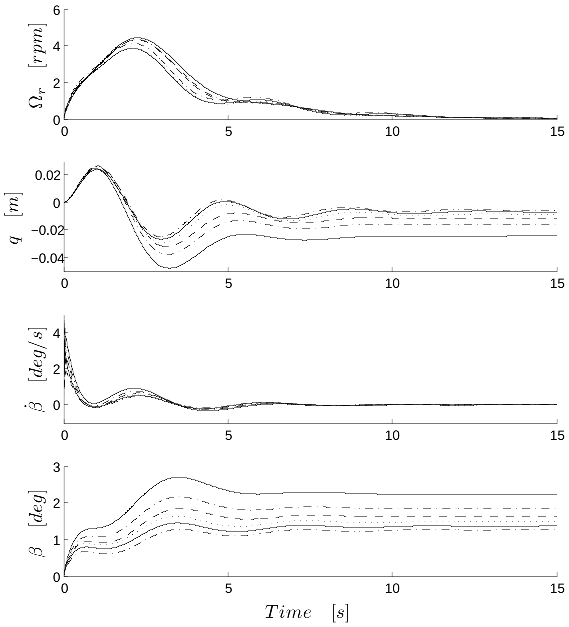
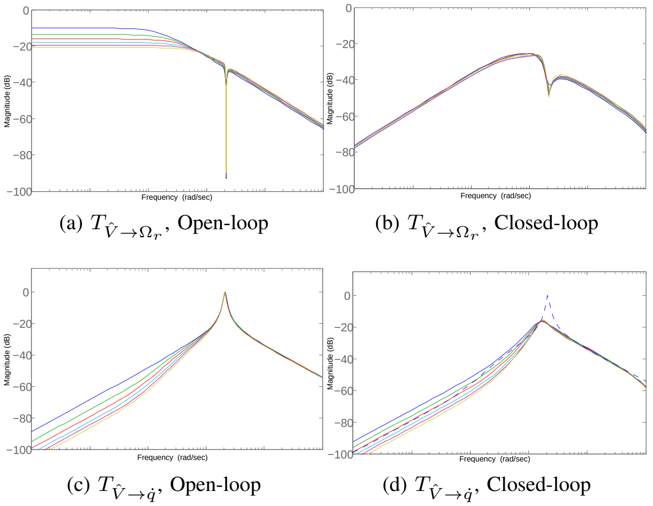


Fig. 3: Step responses under different operating points.



(a) $T_{\hat{V}\to\Omega_r}$, Open-loop

(b) $T_{\hat{V}\to\Omega_r}$, Closed-loop

(c) $T_{\hat{V}\to\dot{q}}$, Open-loop

(d) $T_{\hat{V}\to\dot{q}}$, Closed-loop

Fig. 4: Magnitude of transfer functions from wind disturbance to rotor speed and tower velocity.

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