H_∞/H_2 Model Reduction Through Dilated Linear Matrix Inequalities *

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Abstract: This paper presents sufficient dilated linear matrix inequalities (LMI) conditions to the H_∞ and H_2 model reduction problem. A special structure of the auxiliary (slack) variables allows the original model of order n to be reduced to an order $r = n/s$ where $n, r, s \in \mathbb{N}$. Arbitrary order of the reduced model can be enforced by including states in the original system with negligible input-to-output system norms. The use of dilated LMI conditions facilitates model reduction of parameter-dependent systems. When a reduced model determined by the sufficient LMI conditions does not satisfactorily approximates the original system, an iterative algorithm based on dilated LMIs is proposed to significantly improve the approximation bound. The effectiveness of the method is accessed by numerical experiments. The method is also applied to the H_2 order reduction of a flexible wind turbine model.

Keywords: Model Reduction; LMI Optimization; Parametric Uncertainties.

1. INTRODUCTION

The model reduction problem consists on the approximation of a given asymptotically stable system by a reduced order model according to a given minimum norm criteria on the approximation error. Several techniques and norm measures were investigated, giving rise to numerically reliable algorithms. A comparison of some of the algorithms for model reduction can be found in [Gugercin and Antoulas (2000)]. This problem can be formulated as an optimization problem with rank constraints [Skelton and de Oliveira (2001)] or posed as a set of nonlinear matrix equations [Haddad and Bernstein (1989)]. Due to the inherent non-convexity of these problems, they are very difficult to solve.

More recently, model reduction has been investigated under the linear matrix inequalities (LMI) framework, facilitating the use of classical norm criteria for the reduction error like H_∞ [Ebihara and Hagiwara (2004b)] and H_2. This framework is particularly suitable to address multichannel / mixed problems as well as uncertain models [Beck et al. (1996); Wu (1996); Trofino and Coutinho (2004)]. Unfortunately, the difficulties of non-convexity remains when formulating the model reduction problem as an LMI, typically involving an additional rank constraint [Wu and Jaramillo (2002)] or resulting in bilinear matrix inequalities [Grigoriadis (1997); Assunçao and Peres (1999)]. In order to circumvent the non-convexity of the problem, some authors reformulate the non-convex constraint by a linear constraint presenting a matrix variable that is fixed a priori [Trofino and Coutinho (2004); Geromel et al. (2005)]. The choice of the fixed variable influences the degree of suboptimality.

In this paper, we explore the usage of dilated (or extended) LMIs to the model reduction problem. See [Pipeleers et al. (2009)] for a survey on the history and different characterizations proposed in the literature. Dilated LMIs are composed of instrumental (slack) variables which facilitates a linear dependence of the LMI in the Lyapunov variables. This added flexibility is valuable for reducing conservatism in robust and multi-objective control. A sufficient LMI condition with a special structure of the slack variables is here proposed, allowing an original model of order n to be reduced to an order $r = n/s$ where $n, r, s \in \mathbb{N}$. Arbitrary order of the reduced model can be enforced by including states in the original system with negligible input-to-output system norms. This slack variable structure is trivially extended to cope with robust and parameter-dependent model reduction. When a reduced model determined by the sufficient LMI conditions does not satisfactorily approximates the original system, an iterative algorithm based on dilated LMIs is proposed to significantly improve the approximation bound with the expense of higher computational cost. The effectiveness of the method is accessed by numerical experiments. The method is successfully applied to the H_2 order reduction of a flexible wind turbine.

2. MODEL REDUCTION THROUGH DILATED LMI

2.1 Linear Time-Invariant Systems

We initially consider a stable MIMO LTI dynamical system of order n in state-space form

$$S : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$, $D \in \mathbb{R}^{n_y \times n_u}$. We seek a model of order $r < n$ denoted S_r

$$S_r : \begin{cases} \dot{x}_r = A_r x_r(t) + B_r u(t) \\ y(t) = C_r x_r(t) + D_r u(t) \end{cases} \qquad (2)$$

where $A_r \in \mathbb{R}^{r \times r}$, $B_r \in \mathbb{R}^{r \times n_u}$, $C_r \in \mathbb{R}^{n_y \times r}$, $D_r \in \mathbb{R}^{n_y \times n_u}$ such that the input-output difference between the original system S and the reduced system S_r is small in an H_∞ or H_2-norm sense. That is

$$\|S - S_r\|_{\infty \text{ or } 2} \leq \gamma \qquad (3)$$

where γ represents the upper bound on H_∞ or H_2, depending on the context. The input-output difference of S and S_r can be represented by the following state-space description denoted ΔS

$$\Delta S : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + \begin{bmatrix} B \\ B_r \end{bmatrix} u(t) \\ y_\Delta(t) = \begin{bmatrix} C & -C_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + (D - D_r) u(t) \end{cases} \qquad (4)$$

Hereafter, the system matrices of ΔS are denoted A_Δ, B_Δ, C_Δ, D_Δ. Our results benefit from the dilated LMI conditions for an upper bound on H_∞ [Xie (2008)] or H_2 [Ebihara and Hagiwara (2004a)]. Please consult [Pipeleers et al. (2009)] and references therein for a throughout exposure of dilated LMIs; we state them already in the context of our problem.

Lemma 1. $\|S - S_r\|_\infty \leq \gamma$ holds if, and only if, there exist a general auxiliary matrix Q, symmetric matrix X and a scalar $\mu > 0$ such that

$$\begin{bmatrix} A_\Delta Q + Q^T A_\Delta^T & \star & \star & \star \\ \mu Q^T A^T - Q + X & -\mu(Q + Q^T) & \star & \star \\ C_\Delta Q & \mu\, C_\Delta Q & -\gamma I & \star \\ B_\Delta^T & 0 & D_\Delta^T & -\gamma I \end{bmatrix} < 0, \qquad (5)$$

is satisfied.

The multiplication between the scalar μ and matrix variables in (5) makes a line search in μ necessary.

Lemma 2. $\|S - S_r\|_2 \leq \gamma$ holds if, and only if, there exist a general auxiliary matrix Q and symmetric matrices X, Z such that

$$\begin{bmatrix} A_\Delta Q + Q^T A_\Delta^T & \star & \star \\ -Q^T A_\Delta^T + Q - X & -(Q + Q^T) & \star \\ C_\Delta Q & -C_\Delta Q & -\gamma I \end{bmatrix} < 0,$$
$$\begin{bmatrix} Z & \star \\ B_\Delta & X \end{bmatrix} > 0, \quad \text{trace}(Z) < \gamma \qquad (6)$$

is satisfied.

The previous two lemmas state conditions for analysis of ΔS. In order to derive conditions to synthesize the reduced order matrices $A_r, \ldots, D_r$, let the general auxiliary matrix Q be partitioned as

$$Q := \begin{bmatrix} Q_1 & Q_2 & \ldots & Q_{s+1} \\ H & H & \ldots & H \end{bmatrix} \qquad (7)$$

where $Q_k, \in \mathbb{R}^{n \times r}$, $k = 1, \ldots, s+1$, $H \in \mathbb{R}^{r \times r}$, and $r = n/s$, r, n, $s \in \mathbb{N}$. Also define new matrix variables $\hat{A}_r$ and $\hat{C}_r$ resulting from the nonlinear change of variables

$$\hat{A}_r := A_r H, \quad \hat{C}_r := C_r H. \qquad (8)$$

With these definitions at hand, the LMI conditions for synthesis can be stated as follows.

Theorem 3. $\|S - S_r\|_\infty \leq \gamma$ holds if there exist general auxiliary matrices Q_k, $k = 1, \ldots, s+1$ and H, symmetric matrix X, general matrices $\hat{A}_r$, B_r, $\hat{C}_r$, D_r and a scalar $\mu > 0$ such that

$$\begin{bmatrix} \hat{A}_\Delta + \hat{A}_\Delta^T & \star & \star & \star \\ \mu\, \hat{A}_\Delta^T - Q + X & -\mu\left(Q + Q^T\right) & \star & \star \\ \hat{C}_\Delta & \mu\, \hat{C}_\Delta & -\gamma I & \star \\ B_\Delta^T & 0 & D_\Delta^T & -\gamma I \end{bmatrix} < 0, \qquad (9a)$$

$$\hat{A}_\Delta := \begin{bmatrix} AQ_1 & AQ_2 & \ldots & AQ_{s+1} \\ \hat{A}_r & \hat{A}_r & \ldots & \hat{A}_r \end{bmatrix},$$
$$B_\Delta := \begin{bmatrix} B \\ B_r \end{bmatrix}, \quad D_\Delta := D - D_r \qquad (9b)$$
$$\hat{C}_\Delta := \begin{bmatrix} CQ_1 - \hat{C}_r & CQ_2 - \hat{C}_r & \ldots & CQ_{s+1} - \hat{C}_r \end{bmatrix},$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to

$$A_r = \hat{A}_r H^{-1}, \quad C_r = \hat{C}_r H^{-1}. \qquad (10)$$

Proof. The LMIs (9) are obtained by trivial manipulations of (5), (7) and resorting to the nonlinear change of variables (8). To show that $\hat{A}_r$ and $\hat{C}_r$ can always be reconstructed according to (10), H should be invertible thus nonsingular. The fact that $-\mu\left(Q + Q^T\right) < 0$ with $\mu > 0$ implies nonsingularity of Q. Notice that H is the lower-right block of Q (see (7)), thus nonsingularity of H is also guaranteed. $\square$

The same rationale can be applied to turn Lemma 2 into synthesis conditions.

Theorem 4. $\|S - S_r\|_2 \leq \gamma$ holds if there exist general auxiliary matrices Q_k, $k = 1, \ldots, s+1$ and H, symmetric matrices X, Z and general matrices $\hat{A}_r$, B_r, $\hat{C}_r$, D_r such that

$$\begin{bmatrix} \hat{A}_\Delta + \hat{A}_\Delta^T & \star & \star \\ -\hat{A}_\Delta^T + Q - X & -(Q + Q^T) & \star \\ \hat{C}_\Delta & -\hat{C}_\Delta & -\gamma I \end{bmatrix} < 0,$$
$$\begin{bmatrix} Z & \star \\ B_\Delta & X \end{bmatrix} > 0, \quad \text{trace}(Z) < \gamma, \quad \text{and Eq. (9b),} \qquad (11)$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to (10).

The chosen structure (7) of the auxiliary variable Q restrains the dimension of A_r to be $r = n/s$, or in words,

the order of the reduced model is an integer fraction of the order of the original model. Therefore, the order of the reduced system can be chosen smaller by redefining the partitioning of Q. For example, in the case of $r = n/3$

$$Q := \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ H & H & H & H \end{bmatrix} \qquad (12)$$

with $\hat{A}_\Delta$ and $\hat{C}_\Delta$ changing accordingly

$$\hat{A}_\Delta := \begin{bmatrix} AQ_1 & AQ_2 & AQ_3 & AQ_4 \\ \hat{A}_r & \hat{A}_r & \hat{A}_r & \hat{A}_r \end{bmatrix},$$
$$\hat{C}_\Delta := \begin{bmatrix} CQ_1 - \hat{C}_r & CQ_2 - \hat{C}_r & CQ_3 - \hat{C}_r & CQ_4 - \hat{C}_r \end{bmatrix}. \qquad (13)$$

Being n a multiple of r limits the choice of the order of the reduced model. This fact can be circumvented by adding states on the original system S with negligible input-to-output norms. A convenient way to do so is by augmenting the system with modes appearing in the diagonal of A

$$A \rightarrow \begin{bmatrix} A & 0 \\ 0 & A_a \end{bmatrix}, \quad A_a = \mathrm{diag}(A_{a,i}), \quad B \rightarrow \begin{bmatrix} B \\ B_a \end{bmatrix},$$
$$B_a = \begin{bmatrix} B_{a,1} \\ B_{a,2} \\ \vdots \\ B_{a,i} \end{bmatrix}, \quad C \rightarrow [C \ C_a],$$
$$C_a = [C_{a,1} \ C_{a,2} \ \ldots \ C_{a,i}], \quad i = 1, \ldots, n_a.$$

where $\|C(sI - A)^{-1}B + D\| \ \gg \ \|C_a(sI - A_a)^{-1}B_a\|$. An arbitrary order r can be chosen by combining both strategies.

2.2 Parameter Dependent Systems

The linear time invariant conditions just presented are trivially extended to cope with model reduction of parameter-dependent (PD) systems. Consider the linear PD system of order n

$$S(\alpha) : \begin{cases} \dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t) \\ y(t) = C(\alpha)x(t) + D(\alpha)u(t). \end{cases} \qquad (14)$$

System matrices are polytopic with respect to the parameter α

$$A(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i A_i, \qquad B(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i B_i,$$
$$C(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i C_i, \qquad D(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i D_i, \qquad (15)$$
$$\Lambda := \left\{ \alpha : \sum_{i=1}^{N_\alpha} \alpha_i = 1, \ \alpha_i \geq 0 \right\}$$

as well as the symmetric matrices

$$X(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i X_i, \qquad Z(\alpha) = \sum_{i=1}^{N_\alpha} \alpha_i Z_i, \qquad (16)$$

where $\alpha \in \Lambda$. The aim is to find a reduced system $S_r(\alpha)$ with order $r < n$ and structure analogous to (14), (15) such that $\|S(\alpha) - S_r(\alpha)\| \leq \gamma$ for all $\alpha \in \Lambda$. The auxiliary matrices are considered parameter independent defined according to (7). New matrix variables $\hat{A}_{r,i}$ and $\hat{C}_{r,i}$ result from the nonlinear change of variables involving the reduced order matrices $A_{r,i}$ and $C_{r,i}$

$$\hat{A}_r(\alpha) := \sum_{i=1}^{N_\alpha} \alpha_i \hat{A}_{r,i}, \quad \hat{A}_{r,i} = A_{r,i}H,$$
$$\hat{C}_r(\alpha) := \sum_{i=1}^{N_\alpha} \alpha_i \hat{C}_{r,i}, \quad \hat{C}_{r,i} = C_{r,i}H, \qquad (17)$$
$$i = 1, \ldots, N_\alpha.$$

Theorem 5. $\|S(\alpha) - S_r(\alpha)\|_\infty \leq \gamma$ holds if there exist general auxiliary matrices Q_k, $k = 1, \ldots, s+1$ and H, symmetric matrices X_i, general matrices $\hat{A}_{r,i}$, $B_{r,i}$, $\hat{C}_{r,i}$, $D_{r,i}$ and a scalar $\mu > 0$ such that

$$\begin{bmatrix} \hat{A}_{\Delta i} + \hat{A}_{\Delta i}^T & \star & \star & \star \\ \mu \hat{A}_{\Delta i}^T - Q + X_i & -\mu (Q + Q^T) & \star & \star \\ \hat{C}_{\Delta i} & \mu \hat{C}_{\Delta,i} & -\gamma I & \star \\ B_{\Delta i}^T & 0 & D_{\Delta i}^T & -\gamma I \end{bmatrix} < 0, \quad (18a)$$

$$\hat{A}_{\Delta i} := \begin{bmatrix} A_i Q_1 & A_i Q_2 & \ldots & A_i Q_{s+1} \\ \hat{A}_{r,i} & \hat{A}_{r,i} & \ldots & \hat{A}_{r,i} \end{bmatrix},$$
$$B_{\Delta i} := \begin{bmatrix} B_i \\ B_{r,i} \end{bmatrix}, \quad D_{\Delta i} := D_i - D_{r,i}$$
$$\hat{C}_{\Delta i} := [C_i Q_1 - \hat{C}_{r,i} \ C_i Q_2 - \hat{C}_{r,i} \ \ldots \ C_i Q_{s+1} - \hat{C}_{r,i}],$$
$$i = 1, \ldots, N_\alpha, \qquad (18b)$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to

$$A_{r,i} = \hat{A}_{r,i}H^{-1}, \quad C_{r,i} = \hat{C}_{r,i}H^{-1}. \qquad (19)$$

Theorem 6. $\|S - S_r\|_2 \leq \gamma$ holds if there exist general auxiliary matrices Q_k, $k = 1, \ldots, s+1$ and H, symmetric matrices X_i, Z_i and general matrices $\hat{A}_{r,i}$, $B_{r,i}$, $\hat{C}_{r,i}$, D_r such that

$$\begin{bmatrix} \hat{A}_{\Delta i} + \hat{A}_{\Delta i}^T & \star & \star \\ -\hat{A}_{\Delta i}^T + Q - X_i & -(Q + Q^T) & \star \\ \hat{C}_{\Delta i} & -\hat{C}_{\Delta i} & -\gamma I \end{bmatrix} < 0,$$
$$\begin{bmatrix} Z_i & \star \\ B_{\Delta i} & X_i \end{bmatrix} > 0, \quad \mathrm{trace}\,(Z_i) < \gamma, \quad \text{and Eq. (18b)}, \qquad (20)$$

is satisfied. Once a solution is found, the reduced order system matrices can always be reconstructed according to (19).

2.3 Iterative Algorithm

If the reduced system does not satisfactorily approximate the dynamics of the original system, one can resort to an iterative LMI (ILMI) algorithm based on dilated LMIs to

find a better result. The reduced model resulted from the sufficient conditions just presented can be used to initialize the ILMI algorithm. The auxiliary (slack) variable Q is now considered a general matrix without any specific partitioning. The following matrices are also redefined under the ILMI context.

$$\hat{A}_{\Delta i} := \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} Q + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{r,i} & C_{r,i} \\ B_{r,i} & A_{r,i} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q$$

$$B_{\Delta i} := \begin{bmatrix} B_i \\ B_{r,i} \end{bmatrix}, \quad D_{\Delta i} := D_i - D_{r,i} \quad (21)$$

$$\hat{C}_{\Delta i} := [C_i \ 0] \, Q + [-I \ 0] \begin{bmatrix} D_{r,i} & C_{r,i} \\ B_{r,i} & A_{r,i} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q$$

For a clear exposure, only the ILMI algorithm for the H_∞ model reduction is described here. The H_2 case can be treated similarly. The notation $(\cdot)^{\{j\}}$ stands for the iteration index. The algorithm solves LMI problems by successively fixing the reduced order matrices $A_{r,i}, \ldots, D_{r,i}$ at one step and the slack variable Q at another step.

Algorithm 1. Consider $A_{r,i}^{\{1\}}$, $B_{r,i}^{\{1\}}$, $C_{r,i}^{\{1\}}$, $D_{r,i}^{\{1\}}$ as the solution of Theorem 5. Set a tolerance ϵ, $j = 1$ and start to iterate:

(1) Find $Q^{\{j\}}$ and $\gamma^{\{j\}}$ that solves the LMI problem:
Minimize $\gamma^{\{j\}}$ subject to (18a) and (21) with fixed $A_{r,i}^{\{j\}}, ..., D_{r,i}^{\{j\}}, i = 1, \ldots, N_\alpha$.
(2) Find $A_{r,i}^{\{j\}}, ..., D_{r,i}^{\{j\}}, i = 1, \ldots, N_\alpha$ and $\gamma^{\{j\}}$ that solves the LMI problem:
Minimize $\gamma^{\{j\}}$ subject to (18a) and (21) with fixed $Q^{\{j\}}, i = 1, \ldots, N_\alpha$.
(3) If $|\gamma^{\{j\}} - \gamma^{\{j-1\}}| < \epsilon$, stop. Else, set $j = j + 1$ and go to step 1.

The Lyapunov matrices X_i act as variables during the whole optimization, a benefit of using dilated LMI conditions in an iterative scheme.

3. NUMERICAL EXAMPLES

To solve the LMI problems, we have used the interface YALMIP [Löfberg (2004)] with semidefinite programming solver SeDuMi. Because the interest lies in finding reduced order models with minimal H_∞ / H_2 norm bounds, the optimization objective *Minimize* γ are included in the LMI conditions just presented.

3.1 Comparison With Other Results

In this subsection, some results obtained by the proposed conditions are compared with [Trofino and Coutinho (2004)] and references therein.

Example 1   Consider an LTI system with state-space matrices (22) [Assunçao and Peres (1999)] from which a first order model should be approximated in an H_2-norm sense. This example gives us a glimpse of the conservativeness of the proposed condition in face of severe order reduction (1/6 of the original system) that may occur due to partitioning the slack variable (7).

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.007 & -0.114 & -0.850 & -2.800 & -4.450 & -3.400 \end{bmatrix}$$

$$B = [0 \ 0 \ 0 \ 0 \ 0 \ 1]^T$$

$$C = [0.007 \ 0.014 \ 0 \ 0 \ 0 \ 0]$$

$$(22)$$

We obtain $\gamma^2 = 0.0283$ by applying Theorem 4 which is considerably close to Trofino and Coutinho (2004) ($\gamma^2 = 0.0205$) and better than the results in [Assunçao and Peres (1999)] ($0.0557 \le \gamma^2 \le 0.0616$). Note that, in contrast to [Trofino and Coutinho (2004)], no matrix involved on the formulation should be chosen a priori.

Example 2   A second-order reduced model of the uncertain system with state-space representation [Wu (1996)]

$$A(\alpha) = \begin{bmatrix} -2 & 3 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & a(\alpha) & 12 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B(\alpha) = \begin{bmatrix} -2.5 & b(\alpha) & -1.2 \\ 1.3 & -1 & 1 \\ 1.6 & 2 & 0 \\ -3.4 & 0.1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -2.5 & 1.3 & 1.6 & -3.4 \\ 0 & -1 & 2 & 0.1 \\ -1.2 & 1 & 0 & 2 \end{bmatrix},$$

$$a(\alpha) = -3.5\alpha_1 - 2.5\alpha_2, \quad b(\alpha) = -0.5\alpha_1 + 0.5\alpha_2$$

should be obtained in an H_∞ sense. Firstly, a parameter-independent reduced system is found by applying Theorem 5 with $A_{r,i} = A_r, \ldots, D_{r,i} = D_r$, $i = 1, \ldots, N_\alpha$. The minimum upper bound $\gamma = 6.2139$ at $\mu = 0.22$ is more conservative than [Trofino and Coutinho (2004)] ($\gamma = 5.54$). The resulting reduced-order system is

$$A_r = \begin{bmatrix} 1.8061 & 0.2829 \\ -2.5823 & -4.7980 \end{bmatrix}$$

$$B_r = \begin{bmatrix} 4.1664 & -0.1741 & -4.2018 \\ -2.2297 & 0.3139 & 2.9823 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 0.4039 & 1.0428 \\ 5.6700 & 8.7890 \\ -5.7262 & -5.3496 \end{bmatrix} \quad (23)$$

$$D_r = \begin{bmatrix} 1.7012 & 1.3074 & 0.9660 \\ -2.0854 & 2.3911 & 0.4805 \\ 0.7488 & 1.1724 & -2.0965 \end{bmatrix}$$

The ILMI Algorithm 1 is initialized with system (23) in an attempt to find a reduced system that better approximates the original one. Convergence tolerance is set to 1e-3. Figure 1(a) shows the convergence of $\gamma^{\{j\}}$ for three different values of $\mu = \{0.1, 0.22, 0.3\}$. For $\mu = 0.22$, the algorithm converges after 18 iteration to $\gamma = 3.995$. This upper bound is considerably better than [Trofino and Coutinho (2004)]. For $\mu = 0.1$, the proposed algorithm finds a parameter-independent reduced model with approximation error $\gamma = 3.578$, less conservative than [Wu (1996)] where a parameter-dependent reduced system is determined by an alternating projection method.

A reduced model with the same parameter dependence as the original one is now desired. Therefore, $A_r(\alpha)$ and $B_r(\alpha)$ depends on the parameter α, while C_r, D_r are parameter independent. The sufficient LMI condition results in an upper bound $\gamma = 6.1080$ (for $\mu = 0.22$), slightly better than the parameter-independent reduced system case. When resorting to the ILMI algorithm to find a parameter-dependent reduced system, an H_∞ upper bound of $\gamma = 3.506$ is reached for $\mu = 0.1$, expectedly less conservative than the parameter-independent reduced system. The convergence of the algorithm is depicted in Fig. 1(b).
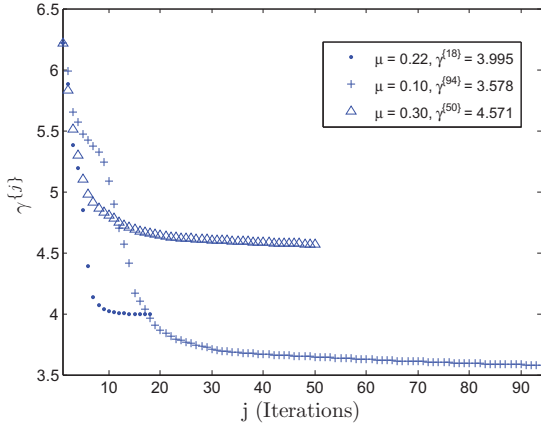

(a) Parameter-Independent
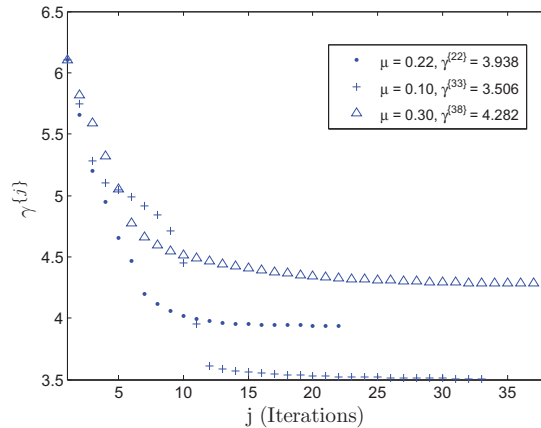
(b) Parameter-Dependent

Fig. 1. Convergence of ILMI model reduction.

3.2 Flexible Wind Turbine

In the wind energy industry, wind turbine models are often derived from high fidelity aeroelastic numerical tools. These large-scale models are not suitable for control analysis and synthesis. The order reduction conditions here presented will be applied to an industrial wind turbine reference model generated by the recently developed aeroelastic code HAWCstab2 [Hansen (2004)]. The LTI model contains several hundreds of flexible modes and aerodynamic delays, out of reach for the current capabilities of LMI-based model order reduction techniques. Therefore, the model was initially reduced from 880 to 20 states using a H_2 modal truncation method in which modes with natural frequencies higher than 4 Hz were also discarded.

The objective here is to reduce from 20 states to 10 states without compromising the quality of the model in an H_2 sense. Thus, $n = 20, s = 2, r = 10$. Magnitude plots in frequency domain of the original and reduced models are depicted in Fig. 2(a).
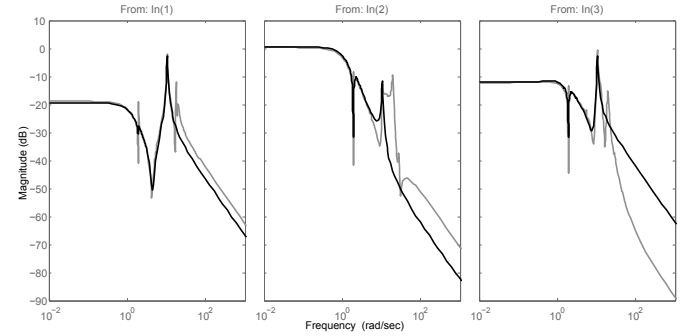
(a) H_2-norm
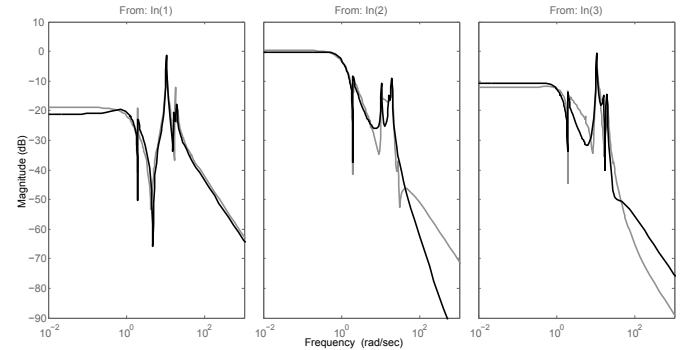
(b) Balanced Truncation

Fig. 2. Magnitude diagrams of a flexible wind turbine model with 20 states (gray) and reduced 10 states (dark).

Inputs 1 and 2 are controllable signals of generator torque and pitch angle, respectively, while input 3 is the wind speed disturbance. The output channel is the wind turbine rotational speed. Another 10 state model were derived based on the well known balanced truncation model reduction scheme using the MATLAB command *balred*. The comparison with the original model, in this case, is depicted in Fig. (2(b)). When compared to balanced truncation, the H_2 measure seems to be more appropriate in approximating the low frequency range of the model.

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