Stability of linear systems in second order form based on structure preserving similarity transformations

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Abstract

This paper deals with two stability aspects of linear systems of the form $I\ddot{x}+B\dot{x}+Cx=0$ given by the triple (I,B,C). A general transformation scheme is given for a structure and Jordan form preserving transformation of the triple. We investigate how a system can be transformed by suitable choices of the transformation parameters into a new system (I,B_1,C_1) with a symmetrizable matrix C_1. This procedure facilitates stability investigations. We also consider systems with a *Hamiltonian* spectrum which discloses marginal stability after a Jordan form preserving transformation.

1 Introduction

Systems of second order linear differential equations of the form

$$A\ddot{x}+B\dot{x}+Cx=0 \tag{1}$$

are characterized by the set (A,B,C), where A, B and C are real quadratic matrices. Such systems are frequently used as models in mechanics. The important issue here is the question of the stability of a given system. The stability is determined by the roots λ of the characteristic polynomial given by

$$\det(\lambda^2 A+\lambda B+C) = 0 \quad .$$

If all the eigenvalues have $Re(\lambda) < 0$, then system (1) is said to be *asymptotically stable*. Here, we call the system (1) *marginally stable* if all eigenvalues lie on the imaginary axis and are semi-simple. This case is in the literature often denoted by 'completely' (or totally) marginally stable. For simplicity we drop the word 'completely'. Thus, marginal stability basically means that all solutions are superpositions of harmonic oscillations. In the modelling of a physical system the matrices A, B and C represent the physical parameters. Therefore it can be interesting to get an estimation of the stability limits expressed by properties of the system matrices without carrying out a direct computation of the eigenvalues. In 2000 Adhikari [1] addressed simultaneous symmetrization of the damping matrix B and the stiffness matrix C by a similarity transformation [2, 3] and by an equivalence and congruence transformation. By using these methods one can simplify the investigation of stability. The subject of this note is to use a structure preserving similarity transformation, which deals with all systems with the same Jordan structure. The stability analysis will focus solely on the structure of the system matrices. Algebraic criteria like Routh-Hurwitz are not taken under consideration in this study.

2 Analysis

For simplicity we assume in what follows that the system given by (A,B,C) is *regular*, that means that the matrix A is non-singular or $\det(A) \neq 0$. Then without loss of generality we suppose that $A = I$, where I is the identity matrix. The system (1) can then be written

$$I\ddot{x}+B\dot{x}+Cx = 0 \quad . \tag{2}$$

Introducing a new variable Y

$$Y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad ,$$

Equation (2) can be rewritten as a first order system

$$\dot{Y} = LY \quad .$$

The $2n \times 2n$ *system matrix* L is given by

$$L = \begin{bmatrix} 0 & I \\ -C & -B \end{bmatrix} \quad . \tag{3}$$

We want to find all second order systems (I, B_1, C_1) which are *equivalent* (also called isospectral) to (I, B, C) in the sense that

$$L_1 = \begin{bmatrix} 0 & I \\ -C_1 & -B_1 \end{bmatrix} \tag{4}$$

has the same Jordan form as L. This means that L and L_1 have the same eigenvalues with the same partial multiplicities. Then L and L_1 must be similar which is expressed by $L_1 = T^{-1}LT$, where T is a $2n \times 2n$ non-singular transformation matrix. If we write T as

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \quad , \quad \det(T) \neq 0 \quad , \tag{5}$$

then the similarity between L and L_1 can be expressed by

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} 0 & I \\ -C & -B \end{bmatrix} = \begin{bmatrix} 0 & I \\ -C_1 & -B_1 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \quad . \tag{6}$$

The $n \times n$ matrices T_1, T_2, T_3 and T_4 must be chosen in such a way that T is non-singular. To guarantee the preservation of the structure the matrices must fulfill

$$\begin{aligned} T_3 &= -T_2 C \\ T_4 &= T_1 - T_2 B \\ -T_4 C &= -C_1 T_1 - B_1 T_3 \\ T_3 - T_4 B &= -C_1 T_2 - B_1 T_4 \quad . \end{aligned} \tag{7}$$

By eliminating T_3 and T_4 we get for the transformation matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ -T_2 C & T_1 - T_2 B \end{bmatrix} \quad . \tag{8}$$

The last two equations in (7) leads to the the following

$$[C_1\,, B_1] \begin{bmatrix} T_1 & T_2 \\ -T_2 C & T_1 - T_2 B \end{bmatrix} = [T_1 - T_2 B\,, T_2] \begin{bmatrix} C & B \\ 0 & C \end{bmatrix} \quad . \tag{9}$$

From this we can determine the new equivalent system (I, B_1, C_1). We then have

Theorem 1 *A given second order system (I,B,C) is equivalent to a new second order system (I,B_1,C_1), where the new matrices B_1 and C_1 are given by (9) and the matrices T_1 and T_2 must be chosen such that the matrix T given in (8) is non-singular.*

In general the new system matrices B_1 and C_1 need not to be similar to B and C, respectively. However, if we choose one of the transformation matrices T_1 or T_2 to be zero, e.g. $T_2 = 0$, then we find immediately the similarity relations

$$B_1 = T_1 B T_1^{-1} \quad ,$$

and

$$C_1 = T_1 C T_1^{-1} \quad .$$

If now C_1 is symmetric, this could give the possibility for a stability prediction, e.g. by the well-known theorem of Kelvin-Tait-Chetaev [9]. The idea for *simultaneously symmetrizing* the matrices B and C to symmetric forms B_1 and C_1 was proposed by Inman [8], but was shown to have no practical importance for non-conservative systems, when the matrix order exceeds $n = 2$, see [6, 7] .

To simplify Equation (9) further we introduce a new variable Z and new similar system matrices $\tilde{C}_1$ and $\tilde{B}_1$ by

$$\tilde{C}_1 = T_2^{-1} C_1 T_2 \quad , \quad \tilde{B}_1 = T_2^{-1} B_1 T_2 \quad , \quad Z = T_2^{-1} T_1 \quad , \tag{10}$$

where we have assumed that $\det(T_2) \neq 0$. Then (9) can be written

$$[\tilde{C}_1 \, , \tilde{B}_1] \begin{bmatrix} Z & I \\ -C & Z - B \end{bmatrix} = [Z - B \, , I] \begin{bmatrix} C & B \\ 0 & C \end{bmatrix} \quad . \tag{11}$$

The coefficient matrix of $[\tilde{C}_1, \tilde{B}_1]$ satisfy the relation

$$\begin{bmatrix} Z & I \\ -C & Z - B \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & Z \end{bmatrix} = \begin{bmatrix} Z & 0 \\ -C & M \end{bmatrix} \quad ,$$

where we have introduced the abbreviation

$$M = Z^2 - BZ + C \quad . \tag{12}$$

This shows that $\tilde{B}_1$ and $\tilde{C}_1$ are uniquely determined if Z and M are non-singular. Multiplying (11) from the right by

$$\begin{bmatrix} I & -I \\ 0 & Z \end{bmatrix} \begin{bmatrix} Z^{-1} & 0 \\ M^{-1} C Z^{-1} & M^{-1} \end{bmatrix} = \begin{bmatrix} Z^{-1} - M^{-1} C Z^{-1} & -M^{-1} \\ Z M^{-1} C Z^{-1} & Z M^{-1} \end{bmatrix} \quad ,$$

we find

$$[\tilde{C}_1\,,\tilde{B}_1] = \left[(Z-B)M-(Z-B)^2 Z\,,M-(Z-B)^2\right]\left[\begin{array}{cc} Z^{-1}-M^{-1}CZ^{-1} & -M^{-1} \\ ZM^{-1}CZ^{-1} & ZM^{-1} \end{array}\right] \quad ,$$

After evaluating the above matrix product we find the following general expressions for $\tilde{C}_1$ and $\tilde{B}_1$

$$C_1 = MZM^{-1}CZ^{-1} \quad , \tag{13}$$

$$B_1 = MZM^{-1} - (Z-B) \quad , \tag{14}$$

where we for simplicity have removed the ˜ from the symbols. This leads to the following theorem

Theorem 2 *Given a second order system (I,B,C) and a matrix $Z \in \mathbb{R}^{n\times n}$ chosen in such a way that both Z and the matrix $M = Z^2 - BZ + C$ are non-singular. Then the system is equivalent to a new second order system (I,B_1,C_1) given by the equations (13) and (14) .*

Using Theorem 2 we can construct all equivalent second order systems with the same eigenvalues and multiplicities. Particularly in stability theory we are interested to test if a given system (I,B,C) can be transformed into a new system (I,B_1,C_1) for which C_1 is symmetrizable with positive eigenvalues.

3 Transforming C into a symmetrizable matrix

In the framework of stability it is especially feasible if the stiffness matrix C is symmetric and positive definite. To this end we recall the concept of symmetrizability.

Lemma 1 *Consider a real square matrix C. Then the following four conditions are equivalent:*

1. *There exists a nonsingular matrix W such that $W^{-1}CW$ is symmetric.*
2. *C possesses only real eigenvalues and a full set of eigenvectors.*
3. *C is the product of two symmetric matrices, one of which is positive definite.*
4. *C becomes symmetric when multiplied by a suitable positive definite matrix.*

A real square matrix C satisfying any of these four conditions is called symmetrizable.

In mechanical systems of form (I,B,C) we can easily point out examples where the stiffness matrix C is not symmetric. For example mathematical models of articulated pipes or models of lateral vibrations of rotor systems are non-conservative systems where the stiffness matrix C includes a skew-symmetric part due to internal damping or to non-symmetrical steam flow in turbines. And this skew-symmetric part (also called circulatory) can cause instability of the system. We will show, that a suitable choice of the transformation (14) may bring a non-symmetric stiffness matrix C into a symmetrizable matrix C_1 which is similar to a symmetric form, see Item 1 in the above Lemma 1.

Example 1 Consider system (I,B,C) with

$$B = \begin{bmatrix} 5.8186 & 3.6667 \\ -3.6667 & 0.1814 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -0.5000 & 2.2500 \\ -2.2500 & -0.5000 \end{bmatrix} \quad .$$

This example was reported by Merkin [11] to show, that an inequality by Metelitsyn sufficient for stability, see e.g. [12], was not satisfied. Neither B nor C is symmetrizable and the structure of the matrices does not disclose whether the system is stable or not. If we choose the the generating matrix $Z = (B+B^T)^{-1}(C-C^T)$, then by Theorem 2 we find the new system matrices

$$B_1 = \begin{bmatrix} 5.8186 & -0.7264 \\ 9.9340 & 0.1814 \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} 5.1804 & -0.7268 \\ 6.9658 & 0.0483 \end{bmatrix} \quad ,$$

where C_1 is real symmetrizable, since it has two different real eigenvalues, see Item 2 in Lemma 1. Now the similarity transformation $B_2 = W^{-1}B_1W$ and $C_2 = W^{-1}C_1W$ where W is an eigenvector matrix of C_1 results in

$$B_2 = \begin{bmatrix} 3.8858 & -0.0367 \\ 1.5324 & 2.1142 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 3.8481 & 0 \\ 0 & 1.3805 \end{bmatrix} \quad .$$

Here $\frac{1}{2}(B_2 + B_2^T)$ and C_2 are both positive definite. In contrast to the original system the inequality of Metelitsyn [12] is now satisfied and shows asymptotic stability. But for this purpose the Kelvin-Tait-Chetaev theorem [9] could be used as well. ■

Example 2 Consider system (I,B,C) with

$$B = \begin{bmatrix} 5 & 3 & -2 \\ -6 & -3 & 6 \\ 1 & 0 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 10 & -1 & -4 \\ -3 & 5 & -5 \\ 1 & -1 & 7 \end{bmatrix} \quad .$$

None of these matrices are symmetrizable, since they possess imaginary eigenvalues. Taking $Z = (B+B^T)^{-1}(C+C^T)$ the transformation given in Theorem 2 leads to a new system with a symmetrizable stiffness matrix C_1 with both positive and negative eigenvalues. If we instead choose $Z = (I+C-C^T)^{-1}(I+B-B^T)$ we again get a system with a symmetrizable stiffness matrix C_1, but now with pure positive eigenvalues. ■

In Example 1 and in Example 2 we found the matrix C_1 to be real symmetrizable by using Item 2 in Lemma 1. But this is equivalent to calculating the eigenvalues of the original system. Therefore we will focus on Item 3 in Lemma 1 and on Theorem 2. This approach will be demonstrated in the following.

3.1 Determination of Z

The procedure of determining Z is as follows.

Theorem 3 *The transformed matrix $C_1 = MZM^{-1}CZ^{-1}$ is real symmetrizable if we have*

1. $Z = SC \quad , \quad S = S^T \quad , \quad S, C \in \mathbb{R}^{n\times n} \quad , \quad det(S) \neq 0 \quad .$

   Thus $CZ^{-1} = S^{-1} = (CZ^{-1})^T$ is symmetric.

2. $MZM^{-1} = (MZM^{-1})^T \quad , \quad M = Z^2 - BZ + C \in \mathbb{R}^{n\times n} \, .$

   $Z = SC$ is similar to a real symmetric matrix. This means that the symmetric matrix S must be chosen in such a way that Z has real eigenvalues.

3. *If the above Items 1. and 2. are satisfied we have*

$$C_1 = MZM^{-1}S^{-1} \quad , \tag{15}$$
$$B_1 = MZM^{-1} - SC + B \quad , \tag{16}$$

and the matrix C_1 is real symmetrizable if one of the symmetric matrices MZM^{-1} or S is positive definite.

Examples of the use of Theorem 3 are shown below.

Example 3 Given a system (I,B,C) by

$$B = \begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} \quad .$$

If we choose

$$S = \begin{bmatrix} 0.4818 & 1.0000 \\ 1.0000 & 2.5000 \end{bmatrix} > 0$$

and $Z = SC$, then

$$MZM^{-1} = \begin{bmatrix} 0.7475 & -5.9307 \\ -5.9307 & 58.2808 \end{bmatrix} > 0 \quad .$$

The new system matrices given by (15) and (16) are

$$C_1 = \begin{bmatrix} -8.1281 & -1.6645 \\ 131.0553 & 24.7468 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} -2.6431 & -0.6577 \\ 45.4129 & 9.6431 \end{bmatrix} \quad ,$$

where C_1 is real symmetrizable according to Lemma 1 , Item 3. ∎

Condition 2 in Theorem 3 is not easy to fulfill except for the case $n = 2$, as we shall see in the next example.

Example 4 This 2×2 system (I,B,C) is given by

$$C = \begin{bmatrix} p & q \\ -q & p \end{bmatrix} \quad , \quad B = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \quad ,$$

where $p,q,r > 0$. The example is taken from rotor dynamics, where q is representing a destabilizing circulatory force. According to Theorem 3 can Z be written $Z = SC$. For simplicity we here take

$$S = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad , \quad a > |b| > 0 \quad .$$

Then $CZ^{-1} = S^{-1} > 0$ and Z has *positive eigenvalues*. We have

$$Z = \begin{bmatrix} ap - bq & aq + bp \\ -aq + bp & ap + bq \end{bmatrix} \quad ,$$

and

$$CZ^{-1} = \frac{1}{a^2 - b^2} \begin{bmatrix} a & -b \\ -b & a \end{bmatrix} \quad .$$

At last we must determine Z such that $M^T M Z = Z^T M^T M$ to guarantee the symmetry of $H = MZM^{-1}$. This results in the four solutions given below for b as function of a

$$b = \pm\sqrt{\frac{k_1 \pm \sqrt{k_2}}{2a(p^2 + q^2)}} \quad ,$$

where

$$\begin{aligned} k_1 &= 2a^3 \left(p^2 + q^2\right) - 2r \left(a^2 p + 1\right) + a \left(r^2 + 2p\right) \quad , \\ k_2 &= \left[2r \left(a^2 p + 1\right) - a \left(r^2 + 2p\right)\right]^2 - 4a^2 \left(p^2 + q^2\right) \quad . \end{aligned}$$

As an example we now choose q as a parameter and take $r = 3$ and $p = 16$. To estimate the stability limit we set $a = 1$ and select the solution (17) for b given by the plus signs. If we write $S^{-1} = W^T W$ and make the similarity transformations $C_2 = W C_1 W^{-1}$ and $B_2 = W B_1 W^{-1}$ then $C_2 = C_2^T > 0$. The eigenvalues of C_2 and of the symmetric part of B_2 , which is $B_{2S} = \frac{1}{2}(B_2 + B_2^T)$, are shown as functions of q in Figure 1 on the next page. If $q < 9.416$ then $B_{2S} > 0$ and the system (I, B_2, C_2) is stable according to Kelvin-Tait-Chetaev [9]. A better estimate of the stability limit can be obtained by using Theorem 2, Condition a) given in [5]. From this theorem we obtain stability for $q < 11.8018$. These estimates should be compared with the exact stability limit $q_0 = r\sqrt{p} = 12$.

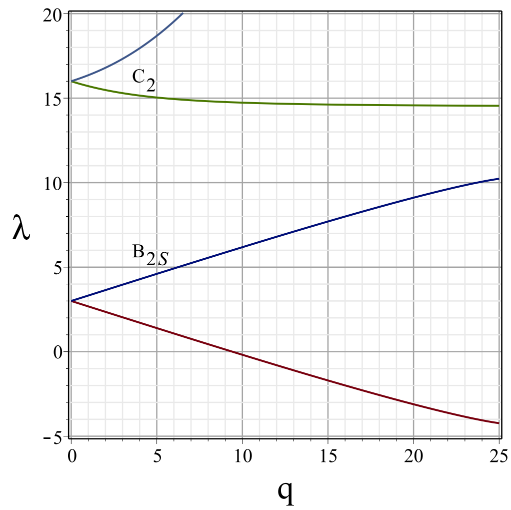

Figure 1: Eigenvalues of C_2 and B_{2S} as functions of q .

For $9.416 < q < 12.00$ the *damping* matrix B_{2S} is indefinite and therefore in this case we call the system *gyroscopically stabilized*. ■

Theorem 2 also opens up for constructing other classes of systems with special properties e.g. systems, which are *marginally stable*. We shall consider this particular problem in the next section.

4 Marginally stable systems

4.1 Marginal stability

As mentioned above, a system is called *marginally stable* if all the eigenvalues are *purely imaginary* and the system is non-degenerate, that means that it must have a full set of eigenvectors (please see note on terminology in Introduction). For a general real marginally stable system (I, B_1, C_1) all the eigenvalues and eigenvectors are pairwise complex conjugate and can therefore be represented by the two diagonal matrices Λ and $\overline{\Lambda}$ and the two eigenvector matrices $X$and $\overline{X}$ respectively, where $\Lambda = i\Lambda_0$, $\Lambda_0 \in \mathbb{R}^{n\times n}$. Then the eigenvector equation for L_1 given by

Equation (6) is

$$\begin{bmatrix} 0 & I \\ -C_1 & -B_1 \end{bmatrix} \begin{bmatrix} X & \overline{X} \\ X\Lambda & -\overline{X}\Lambda \end{bmatrix} = \begin{bmatrix} X & \overline{X} \\ X\Lambda & -\overline{X}\Lambda \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} \quad . \tag{17}$$

By premultiplying (17) by L_1 we receive

$$\begin{bmatrix} 0 & I \\ -C_1 & -B_1 \end{bmatrix}^2 \begin{bmatrix} X & \overline{X} \\ X\Lambda & -\overline{X}\Lambda \end{bmatrix} = \begin{bmatrix} X & \overline{X} \\ X\Lambda & -\overline{X}\Lambda \end{bmatrix} \begin{bmatrix} -\Lambda_0^2 & 0 \\ 0 & -\Lambda_0^2 \end{bmatrix} \quad , \tag{18}$$

which shows

Lemma 2 *The eigenvalues of L_1^2 of a marginally stable system are all of even multiplicity and negative.*

Now consider a system $(I,0,C)$ with $B=0$. Then by using the similarity transformation given by (6) , we can find a new system (I,B_1,C_1) . By premultiplying Equation (6) by L_1 and rearranging we obtain

$$\begin{bmatrix} 0 & I \\ -C_1 & -B_1 \end{bmatrix}^2 \begin{bmatrix} T_1 & T_2 \\ -T_2C & T_1 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ -T_2C & T_1 \end{bmatrix} \begin{bmatrix} -C & 0 \\ 0 & -C \end{bmatrix} \quad . \tag{19}$$

If we further assume that C is *real symmetrizable* there exists a real eigenvector matrix T_C and a real diagonal matrix Λ_C for which

$$T_C^{-1}CT_C = \Lambda_C \in \mathbb{R}^{n\times n} \quad , \quad T_C \in \mathbb{R}^{n\times n} \quad .$$

Then (19) can be rewritten as

$$\begin{aligned} \begin{bmatrix} 0 & I \\ -C_1 & -B_1 \end{bmatrix}^2 \begin{bmatrix} T_1T_C & T_2T_C \\ -T_2T_C\Lambda_C & T_1T_C \end{bmatrix} &= \\ \begin{bmatrix} T_1T_C & T_2T_C \\ -T_2T_C\Lambda_C & T_1T_C \end{bmatrix} \begin{bmatrix} -\Lambda_C & 0 \\ 0 & -\Lambda_C \end{bmatrix} \quad . \end{aligned} \tag{20}$$

If $\Lambda_C > 0$ then by Lemma 2 the system is marginally stable. Comparing Equation (18) with (20) we see that $\Lambda_C = \Lambda_0^2$ and that the transformation matrix can be chosen real . For T we have

$$T = \begin{bmatrix} T_1T_C & T_2T_C \\ -T_2T_C\Lambda_0^2 & T_1T_C \end{bmatrix} \quad , \quad \det(T) \neq 0 \quad , \quad T \in \mathbb{R}^{n\times n} \quad . \tag{21}$$

One can show that there is a unique correspondence between the complex eigenvectors X and $\overline{X}$ defining the transformation matrix given in (18) and the real entities T_1T_C and T_2T_C defining the real transformation matrix given in (21) . The proof is left to the reader. The above gives rise to

Theorem 4 *A system (I, B_1, C_1) is marginally stable if and only if it can be transformed into a system $(I, 0, C)$, where C is real symmetrizable with positive eigenvalues.*

As an example we consider a system with a *positive definite* stiffness matrix $C_1 = C_1^T > 0$ and a *skew-symmetric* damping matrix $B_1 = -B_1^T$. It is well known that such a system is marginally stable. A much more interesting case appears if we deal with a *positive definite* stiffness matrix C_1 and a symmetric *indefinite* damping matrix $B_1 = B_1^T$. Then the system (I, B_1, C_1) may be stable or unstable depending on the properties of B_1. For a certain class of indefinite matrices B_1 which possess a *Hamiltonian* spectrum, see [10], there exists a positive number ε_0 such that

$$I\ddot{x} + \varepsilon B_1 \dot{x} + C_1 x = 0 \tag{22}$$

is marginally stable for all $|\varepsilon| \leq \varepsilon_0$. This means that (22) is equivalent to a system

$$I\ddot{q} + C(\varepsilon)q = 0 \quad , \quad C(\varepsilon) > 0 \quad .$$

In general a non-degenerate system (I, B_1, C_1) with a *Hamiltonian* spectrum can always be transformed into a system of the form $(I, 0, C)$. It can be shown that $C \in \mathbb{R}^{n \times n}$ can be determined by the quadratic matrix equation shown below

$$C^2 + (B_1^2 - B_1 C_1 B_1^{-1} - C_1)C + B_1 C_1 B_1^{-1} C_1 = 0 \quad .$$

The stability of the system is entirely determined by the eigenvalues of the matrix C. If C is real symmetrizable with positive eigenvalues then according to Theorem 4 the system is *marginally stable* and otherwise *unstable*.

We remind the reader that a test for positive eigenvalues can be done using linear matrix inequalities, i.e. without explicitly computing any eigenvalues and with a lower algorithmic complexity.

Example 5 Given a diagonal matrix D and a unitary skew-symmetric matrix U by

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad , \quad U = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad , \quad UU^T = I \quad .$$

The system (I, B_1, C_1) given by

$$C_1 = D^2 = C_1^T \quad , \quad B_1 = DU - UD \quad ,$$

has a *Hamiltonian* spectrum. The system $(I, \varepsilon B_1, C_1)$ is non-degenerate for $\varepsilon \neq 1$. It can therefore be transformed into a system $(I, 0, C(\varepsilon))$. Figure 2 shows

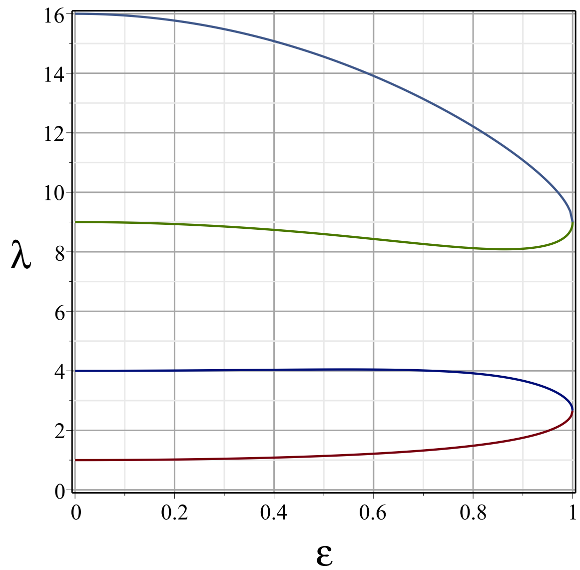

Figure 2: Eigenvalues of $C(\varepsilon)$ for $0 < \varepsilon < 1$.

the eigenvalues of $C(\varepsilon)$ as functions of ε. In [4] it is shown that the system is *marginally stable* for $|\varepsilon| < 1$. For $\varepsilon = 0.5$ we find for $C(0.5)$ and its eigenvalues Λ (compare with Figure 2)

$$C(0.5) = \begin{bmatrix} 9.220 & 0.631 & -798.583 & -0.201 \\ -2.962 & 3.793 & 292.664 & -1.2345 \\ 0.000 & 0.000 & 1.138 & 0.000 \\ -4.715 & -1.379 & 465.944 & 14.182 \end{bmatrix} \quad , \quad \Lambda = \begin{bmatrix} 14.556 \\ 8.595 \\ 4.045 \\ 1.138 \end{bmatrix} \quad .$$

■

4.2 Construction of marginally stable systems

We are now able to construct real marginally stable systems (I, B_1, C_1) with a set of prescribed imaginary eigenvalues and which possess *a full set of eigenvectors*.

This can be done by specifying a system $(I, 0, C_0)$ where C_0 is real symmetrizable with positive eigenvalues. From Equation (13) and (14) we get

$$
\begin{aligned}
C_1 &= (Z^2 + C_0)(I + ZC_0^{-1}Z)^{-1} \quad , & (23)\\
B_1 &= (C_0 Z - ZC_0)(Z^2 + C_0)^{-1} \quad . & (24)
\end{aligned}
$$

The use of Equations (23) and (24) is shown in the following example.

Example 6 Using the definition of D and U given in Example 5, we now define the matrix C_0 by

$$C_0 = -DUDU \quad ,$$

which has two positive eigenvalues $\{8/3, 9\}$ both with multiplicity two and a full set of eigenvectors. Using

$$Z = D \quad ,$$

Equations (23) and (24) determine a marginally stable system (I, B_1, C_1) given by

$$
\begin{aligned}
B_1 &= \frac{1}{2520}\begin{bmatrix} -144 & 60 & -200 & 102 \\ -480 & -192 & -384 & -300 \\ 2520 & 756 & 192 & -90 \\ -2688 & 1680 & 320 & 144 \end{bmatrix} \\
C_1 &= \frac{1}{35}\begin{bmatrix} 107 & 10 & -15 & 6 \\ 30 & 192 & -54 & -30 \\ -105 & -126 & 237 & -30 \\ 84 & -140 & -60 & 272 \end{bmatrix} \quad .
\end{aligned}
$$

Each eigenvalue of the system has the multiplicity two and the squares of the eigenvalues are $\{-8/3, -9\}$ as expected. ■

4.3 Dissipative and gyroscopic marginally stable systems

The question is now if we can construct marginally stable systems where the system matrices possess symmetry. To do this we assume that C_0 is symmetric and positive definite and that Z is symmetric or skew-symmetric, which means that $C_0 = C_0^T > 0$ and $Z = \pm Z^T$. Z must be chosen in such a way that

$$WW^T = Z^2 + C_0 = C_0 \pm ZZ^T > 0 \quad .$$

If the above inequality is satisfied, then also $I \pm ZC_0^{-1}Z^T > 0$. Using Equations (23) and (24) we define the similar system $C_2 = W^{-1}C_1W$, $B_2 = W^{-1}B_1W$ for which we obtain

$$C_2 = W^T(I \pm ZC_0^{-1}Z^T)^{-1}W \quad , \quad B_2 = W^{-1}(C_0Z \mp Z^TC_0)W^{-T} \quad . \tag{25}$$

We have

Lemma 3 *If $Z = \pm Z^T$ and $WW^T = C_0 \pm ZZ^T > 0$ where $C_0 = C_0^T > 0$, then the system (I, B_2, C_2) given by (25) is marginally stable, and if*

i. $Z = +Z^T \Rightarrow C_2 = C_2^T > 0 \quad , \quad B_2 = -B_2^T \quad ,$ *the system is* gyroscopic.

ii. $Z = -Z^T \Rightarrow C_2 = C_2^T > 0 \quad , \quad B_2 = B_2^T \quad ,$ *the system is* dissipative.

Lemma 3 is demonstrated in the next example.

Example 7 Let $Z = 3^{-\frac{1}{2}}U = -Z^T$ where U is the skew-symmetric unitary matrix given in Example 5. If we e.g. choose $C_0 = \mathrm{diag}[1,4,9,16]$ then Lemma 3 Item *ii* results in a marginally stable system with the symmetric matrices C_2 and B_2

$$C_2 = \begin{bmatrix} 0.7001 & 0.0558 & -0.0090 & -0.0565 \\ 0.0558 & 10.4352 & 0.8995 & 1.5890 \\ -0.0090 & 0.8995 & 4.3508 & 1.1660 \\ -0.0565 & 1.5890 & 1.1660 & 18.9534 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} 0.0000 & -1.1094 & -0.6396 & -1.5471 \\ -1.1094 & 0.0000 & 0.2957 & 0.2002 \\ -0.6396 & 0.2957 & 0.0000 & -0.5277 \\ -1.5471 & 0.2002 & -0.5277 & 0.0000 \end{bmatrix} \quad .$$

■

5 Conclusions

We have investigated two kinds of a transformation which is based on all non-degenerate systems with the same Jordan structure. In the first place we introduced systems where a non-symmetric and non-symmetrizable stiffness matrix can be transformed into a symmetrizable matrix with the subsequent possibility to obtain a symmetric stiffness matrix by a similarity transformation. Of course it can happen that under this transformation the symmetric part of the new damping matrix becomes indefinite, see e.g. Example 4. However, in that case it is possible that the gyroscopic part of the new damping matrix can stabilize the system. This last facility has not been addressed in this paper, but can be found in [5]. Secondly we considered systems where a suitable transformation could remove the damping part, which means that in the stable case the system is marginally stable. Several examples are given and the general construction of marginally stable systems with prescribed eigenvalues and a full set of eigenvectors is carried out. Moreover we consider marginally systems where the system matrices possess symmetry as it is the case with dissipative and gyroscopic systems.

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